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SEMISIMPLE MODULES AND FACTS ON $Z(M)$, $Z_2(M)$ AND $\mathrm{Soc}(M)$

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ABSTRACT. In this work, we show that the set of (isotopical) homogeneous components of semisimple modules is exactly the set of eliminator submodules [2]. Also we show that every semisimple module M is $\mathbb{EM}$-semi-clever [5] and we show that $M = Z(M) \oplus Z(M)^{\times}$.

1. Introduction

Let M ba a semisimple module. Decompose M into a direct sum of simple submodules. Grouping these according to their isomorphism type, we can write $M = \bigoplus_{N \in A} N$, where each $N \in A$ is a sum of isomorphic simple submodules. We show that each $N \in A$ is the sum of the set of isomorphic simple submodules and A is exactly the set of minimal eliminator submodules. Also we present some facts on $Z(M)$, $Z_2(M)$ and $\mathrm{Soc}(M)$ regarding eliminator submodules in every modules.
Through the paper we heavily apply the notations terminologies introduced in [1], [2] and [3].

2. $Z(M), Z_2(M), \mathrm{Z}_{-2}(M)$

Definition 2.1. For every R-module M ad $K \subseteq M$ we set $\mathrm{cl}(K) = \{m \in M \mid (K:m) \subseteq_{\mathrm{e}} R\}$.

Lemma 2.2. *Let R be a ring and M be an R-module. For any eliminator submodule (refer to [2] for the definition) K, $\mathrm{cl}(K)$ is also an eliminator submodule.*

Proof. Let $A \subseteq K$ be a finite set, $m \in M$ and $\mathrm{ann}_R(A)m = 0$. For every $r \in (K:A)$, $\mathrm{ann}_R(Ar) \subseteq \mathrm{ann}_R(mr)$, implying $mr \in K$. So, $(K:A) \subseteq (K:m)$, thus $(K:m) \subseteq_{\mathrm{e}} R$, implying $m \in \mathrm{cl}(K)$. □

Lemma 2.3. *Let R be a right mininjective ring [6] with unit. If J is a minimal right ideal and $c \in R$ such that $cJ \neq 0$, then $R = Rc + \mathrm{ann}_{\mathrm{l}}(J)$.*

Proof. $J \cap \mathrm{ann}_{\mathrm{r}}(c) = 0$, so $cJ \longrightarrow J$ given by $cx \longrightarrow x$ is a well defined homomorphism. Thus, there exists $s \in R$ such that for every $x \in J$, $scx = x$, Consequently, $1 - sc \in \mathrm{ann}_{\mathrm{l}}(J)$, implying $1 \in Rc + \mathrm{ann}_{\mathrm{l}}(J)$. □

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Lemma 2.4. *Let M a R-module and $\{J_1, J_2, \cdots, J_n\}$ be an independent set of annihilator-like submodules (refer to [2] for the definition). For every $b_i, c_i \in J_i$ where $1 \leq i \leq n$, $\mathrm{ann}_R(b_1 + b_2 + \cdots + b_n) \subseteq \mathrm{ann}_R(c_1 + c_2 + \cdots + c_n)$ implies $\mathrm{ann}_R(b_i) \subseteq \mathrm{ann}_R(c_i)$ for all $1 \leq i \leq n$.*

Proof. By the induction on n. Let $r \in \mathrm{ann}_R(b_1 + b_2 + \cdots + b_{n-1})$. Then,

$$\mathrm{ann}_R(b_n r) = \mathrm{ann}_R(b_1 r + b_2 r + \cdots + b_n r) \subseteq \mathrm{ann}_R(c_1 r + c_2 r + \cdots + c_n r) =$$

$$\bigcap_{i=1}^{n} \mathrm{ann}_R(c_i r) \subseteq \mathrm{ann}_R(c_k r)$$

So, $\mathrm{ann}_R(b_n r) \subseteq \mathrm{ann}_R(c_n r)$, implying $cr_k = 0$ for each $1 \leq k \leq n-1$. Thus,

$$\mathrm{ann}_R(b_1 + b_2 + \cdots + b_{n-1}) \subseteq \bigcap_{i=1}^{n-1} \mathrm{ann}_R(c_i r) = \mathrm{ann}_R(c_1 + c_2 + \cdots + c_{n-1})$$

Consequently, $\mathrm{ann}_R(b_i) \subseteq \mathrm{ann}_R(c_i)$ for all $1 \leq i \leq n-1$ by the induction. Repeating the same argument but in different order results in $\mathrm{ann}_R(b_i) \subseteq \mathrm{ann}_R(c_i)$ for all $2 \leq i \leq n$. □

Please pay attention that for every right R-module M, $\mathrm{Z}_{-2}(M)$ is defined in [4] as follow.

$$\mathrm{Z}_{-2}(M) = \{m \in M \mid \exists c \in M \text{ with } \mathrm{ann}_R(c) \subseteq \mathrm{ann}_R(m) \ \&$$
$$\forall r, s \in R (\mathrm{ann}_R(ms) \subseteq \mathrm{ann}_R(cr) \Rightarrow crR = 0)\}$$

Lemma 2.5. *Let M a R-module and $\{J_1, J_2, \cdots, J_n\}$ be an independent set of annihilator-like submodules. If $m_i \in \mathrm{Z}_{-2}(J_i)$ for each $1 \leq i \leq n$, then $\sum_{i=1}^{n} m_i \in \mathrm{Z}_{-2}(M)$.*

Proof. for each $1 \leq i \leq n$, there exists $c_i \in J_i$ with $\mathrm{ann}_R(c_i) \subseteq \mathrm{ann}_R(m_i)$ such that for every $r, s \in R$, $\mathrm{ann}_R(m_i s) \subseteq \mathrm{ann}_R(c_i r)$ implies $c_i rR = 0$. Set $m = \sum_{i=1}^{n} m_i$ and $m = \sum_{i=1}^{n} c_i$. We have

$$\mathrm{ann}_R(c) = \cap_{i=1}^{n} \mathrm{ann}_R(c_i) \subseteq \cap_{i=1}^{n} \mathrm{ann}_R(m_i) = \mathrm{ann}_R(m)$$

Now suppose $r, s \in R$ and $\mathrm{ann}_R(ms) \subseteq \mathrm{ann}_R(cr)$. Then, for each $1 \leq i \leq n$, $\mathrm{ann}_R(m_i s) \subseteq \mathrm{ann}_R(c_i r)$ by Lemma 1.4, implying $c_i rR = 0$. Thus $crR = 0$. □

Lemma 2.6. *Let M a R-module. If A is an independent set of annihilator-like submodules, then $\bigoplus(\{\mathrm{Z}_{-2}(J) \mid J \in A\}) \subseteq \mathrm{Z}_{-2}(M)$.*

Proof. Follows from Lemma 1.5. □

Lemma 2.7. *Let M be an R-module.*

(1) *$Z(M)$ is an eliminator submodules.*
(2) *$Z_2(M)$ is an $\mathbb{EM}$-closed eliminator submodules.*
(3) *$Z(M)^{\times\times} = Z_2(M)$ and $Z(M)^{\times} = Z_2(M)^{\times}$.*
(4) *Any minimal submodule is contained in either $Z(M)$ or $Z(M)^{\times}$.*
(5) $\mathrm{Soc}(M) = \mathrm{Soc}(Z(M)) \oplus \mathrm{Soc}(Z(M)^{\times})$.

Proof. (1) Let $A \subseteq Z(M)$ be a finite set, $m \in M$ and $\mathrm{ann}_R(A)m = 0$. We have $\mathrm{ann}_R(A) \subseteq_{\mathrm{e}} R$, so $\mathrm{ann}_R(m) \subseteq_{\mathrm{e}} R$, thus $m \in Z(M)$.
(2) Since $Z_2(M) = \mathrm{cl}(Z(M))$, $Z_2(M)$ is an eliminator submodules by Lemma 1.2. Since $Z_2(M)$ is essentially closed, it is enough to show that $Z_2(M) \subseteq_{\mathrm{e}} Z_2(M)^{\times\times}$. Let I be a submodule with $Z_2(M) \cap I = 0$. I is $\mathbb{EM}$-contained by [4, (1-2)], so $I \subseteq Z_2(M)^{\times}$, thus $Z_2(M)^{\times\times} \cap I = 0$.
(3) Since $Z(M) \subseteq_{\mathrm{e}} Z_2(M)$, $Z(M)^{\times} = Z_2(M)^{\times}$, so $Z(M)^{\times\times} = Z_2(M)^{\times\times} = Z_2(M)$.
(4) Let I be a minimal submodule with $I \not\subseteq Z(M)$. Then, $I \cap Z(M) = 0$, implying $I \subseteq Z(M)^{\times}$ by [4, (3-11) or (1-2)].
(5) Follows from (4). ☐

3. Semisimple Modules

Proposition 3.1. *Let M be a $\mathbb{B}$-ind.finite (or $\mathbb{B}$-Artinian or $\mathbb{B}$-Noetherian) cofaithful module. Every minimal $\mathbb{EM}$-closed eliminator submodule is contained in a minimal $\mathbb{B}$-submodule.*

Proof. Let J be a minimal $\mathbb{EM}$-closed eliminator submodule. There exists a maximal $\mathbb{B}$-submodule N such that $J \not\subseteq N$ by [2, Proposition 2.13 and Proposition 2.14], [5, (1-19)] and [1, Proposition 2.13], so $J \cap N = 0$ by [3, Lemma 3.1 and Proposition 1.13] and [5, (1-8)], implying $J \subseteq N^{\bullet}$ by [2, Lemma 2.9]. On the other hand, $N^{\bullet}$ is a minimal $\mathbb{B}$-submodule by [1, Lemma 2.5]. ☐

Proposition 3.2. *Let M be a $\mathbb{C}$-ind.finite (or $\mathbb{C}$-Artinian or $\mathbb{C}$-Noetherian) cofaithful module. Every minimal $\mathbb{B}$-submodule is contained in a minimal $\mathbb{C}$-submodule.*

Proof. Let J be a minimal $\mathbb{B}$-submodule. There exists a maximal $\mathbb{C}$-submodule N such that $J \not\subseteq N$ by [1, Proposition 3.8 and Proposition 2.13] and [5, (1-19)], so $J \cap N = 0$ by [5, (1-7)] and [2, Proposition 2.13], implying $J \subseteq N^{\circ}$ by [1, Lemma 3.3]. On the other hand, N° is a minimal $\mathbb{C}$-submodule by [1, Lemma 2.5]. ☐

Proposition 3.3. *Let M be a $\mathbb{AM}$-ind.finite, or $\mathbb{AM}$-Artinian or $\mathbb{AM}$-Noetherian cofaithful module. Every minimal $\mathbb{AM}$-closed annihilator submodule is contained in a minimal $\mathbb{C}$-submodule.*

Proof. Let J be a minimal $\mathbb{AM}$-closed annihilator submodule. There exists a maximal $\mathbb{C}$-submodule N such that $J \not\subseteq N$ by [1, Proposition 3.8 and Proposition 2.13] and [5, (1-19)], so $J \cap N = 0$ by [3, Proposition 1.13], [5, (1-15)] and [5, (1-8)], implying $J \subseteq N^{\circ}$ by [1, Lemma 3.3]. On the other hand, N° is a minimal $\mathbb{C}$-submodule by [1, Lemma 2.5]. ☐

Proposition 3.4. *Let M be a $\mathbb{B}$-ind.finite, or $\mathbb{B}$-Artinian or $\mathbb{B}$-Noetherian cofaithful module.*

(1) *Every simple submodule is contained in a member of $\langle \mathbb{B}^{\mathrm{mn}} \colon M \rangle$.*
(2) $\mathrm{Soc}(M) = \bigoplus\{\mathrm{Soc}(K) \mid K \in \langle \mathbb{B}^{\mathrm{mn}} \colon M \rangle\}$.

Proof. (1) Follows from [3, Proposition 3.12] and Proposition 2.1.
(2) Follows from (1). ☐

Proposition 3.5. *Let M be a $\mathbb{C}$-ind.finite, or $\mathbb{C}$-Artinian or $\mathbb{C}$-Noetherian cofaithful module .*

(1) *Every simple submodule is contained in a member of $\langle \mathbb{C}^{\mathrm{mn}} \colon M\rangle$.*
(2) $\mathrm{Soc}(M) = \bigoplus\{\mathrm{Soc}(K) \mid K \in \langle \mathbb{C}^{\mathrm{mn}} \colon M\rangle\}$.

Proof. (1) Follows from [3, Proposition 3.12], Proposition 2.1 and Proposition 2.2.
(2) Follows from (1). □

Theorem 3.6. *Let M be a cofaithful semisimple module.*

(1) *Every eliminator submodule is a $\mathbb{B}$-summand $\mathbb{B}$-submodule.*
(2) *Every $\mathbb{B}$-submodule is a $\mathbb{B}$-summand and every eliminator submodule is $\mathbb{EM}$-closed.*
(3) *Every $\mathbb{C}$-submodule is a $\mathbb{C}$-summand.*
(4) *M is $\mathbb{EM}$-semisimple.*
(5) *$\langle \mathbb{EM}^{\mathrm{mn}} \colon M\rangle$ is the set of (isotopical) homogeneous components of M.*
(6) *$M = Z(M) \oplus Z(M)^{\times}$.*
(7) *$Z_2(M) = Z(M)$.*
(8) *M is the direct sum of racial orthogonal modules N and S, where N is nonsingular and S is singular. Also this representation is unique.*
(9) *M is the direct sum of a set of pairwise racial orthogonal homogeneous modules. Also this representation is unique.*
(10) *M is polyform.*
(11) *M is $\mathbb{EM}$-semi-clever*

Proof. (1) Let K be an eliminator submodule. $M = K \oplus K^{\times}$ by [3, Proposition 3.11], so K is an $\mathbb{EM}$-summand. Applying [2, Lemma 2.16] completes the proof.
(2) Follows from (1).
(3) Let K be a $\mathbb{C}$-submodule. For every simple submodule J, $J \subseteq K$ or $J \subseteq K^{\circ}$ by [1, Lemma 3.3]. Thus, $M = K \oplus K^{\circ}$.
(4 and 5) We have $\langle \mathbb{EM} \cap \mathbb{EM}^{\mathrm{c}} : M\rangle = \langle \mathbb{EM} : M\rangle$. The rest follows from [3, Proposition 3.12].
(6) Follows from Lemma 1.7.
(7) Follows from Lemma 1.7 and (6).
(8) Follows from (6).
(9) Follows from (4) and (5).
(10) M is the only essential submodule, so is a dense submodule.
(11) Follows from [5, (3-4)] and (1). □

Lemma 3.7. *Let R be a left faithful ring. Every non zero square minimal right ideal is contained in $Z(R_R)^{\times}$.*

Proof. Let J be a non zero square minimal right ideal. There exists $c \in J$ with $cJ \neq 0$. Then $J \cap \mathrm{ann}_{\mathrm{r}}(c) = 0$, so $c \notin Z(R_R)$. Thus, $J \not\subseteq Z(R_R)$, implying $J \subseteq Z(R_R)^{\times}$ by Lemma 1.7. □

Proposition 3.8. *Let R be a right mininjective ring [6] with unit. Every zero square minimal right ideal is contained in $Z(R_R)$.*

Proof. Temporarily suppose that there exists a zero square minimal right ideal K with $K \nsubseteq Z(R_R)$. Consider $c \in K - Z(R_R)$. There exists a minimal right ideal J with $J \cap \mathrm{ann_l}(c) = 0$. Then, $K = cJ$ and $R = Rc \oplus \mathrm{ann_l}(J)$, so $RJ = RcJ = RK$ by Lemma 1.3, thus $KRJ = 0$, implying $cJ = 0$ which is a contradiction. □

Proposition 3.9. *Let R be an I-finite (conatining no infinite set of orthogonal idempotent [6]) right mininjective* r$\mathbb{I}$*-mini ring [6] with unit. Then, $R = Z(R_R)^{\times} \oplus Z_2(R_R)$.*

Proof. $R = Q \oplus T$ where Q is semisimple ring and every minimal right ideal of T is nilpotent by [6, Theorem 1.12] . Set S as the sum of nilpotent minimal right ideals of R. Then $S \subseteq Z(T)$ by Proposition 2.8 and obviously, $S \subseteq_{\mathrm{e}} T$ and $Z(R_R) \subseteq T$. So, $Q = T^{\times} = S^{\times} = Z(R_R)^{\times}$, thus $Z_2(R_R) = Z(R_R)^{\times\times} = T$ by Lemma 1.7. □

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