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THE NON-HOMOGENEOUS CAUCHY PROBLEM OF SEMIGROUP OF LINEAR OPERATORS

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Abstract. This paper presents the results of ω-order reversing partial contraction mapping generated by the non-homogeneous Cauchy problem. We investigated several concepts of solution for non-homogeneous Cauchy problem where an analysis was arrived out on the relationship between C' strong and respectively C_0-solution. We showed that Z is the infinitesimal generator of a C_0-semigroup of contractions. Finally, we obtained that the problem has a unique classical solution and f, a member of $L^1(a, b; X)$ is continuous on (a, b).

1. Introduction

Suppose $Z : D(Z) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{J(t); t \geq 0\}$, then for each $a \geq 0$ and $\xi_\rho \in D(Z)$, the function $u : [a, +\infty) \to X$, defined by $u(t) = J(t-a)\xi$ for each $t \geq 0$, is the unique solution of the homogeneous Cauchy problem

$$\begin{cases} u' = Zu \\ u(a) = \xi. \end{cases} \tag{1.1}$$

From this reason, it is quite natural to consider that, for each $\xi \in X$, that function $u' = Zu + f$, $u(a) = \xi$ is a solution for (1.1), in a generalized sense. We then consider the non-homogeneous problem

$$\begin{cases} u' = Zu + f \\ u(a) = \xi, \end{cases} \tag{1.2}$$

where Z is the infinitesimal generator of a C_0-semigroup, $\xi \in X$, and $f \in L'(a, b; X)$.

Assume X is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - ORCP_n$ the ω-order reversing partial contraction mapping, M_m be a matrix, $\mathcal{L}(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A. This paper consist of results of the non-homogeneous Cauchy problem of semigroup of linear operators. In [1] and [2], Akinyele *et al.* established differentiable and analytical conclusions on ω-order preserving partial contraction mapping in semigroup of linear operator. They also made a description of ω-order

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reversing partial contraction mapping as a compact semigroup of linear operator. Akinyele *et al.* [3], obtained results of semigroup of linear operators in extrapolation spaces. An operator calculus for infinitesimal semigroup generators was showed by Balakrishnan [4]. Banach [5] introduced and first proposed the idea of Banach spaces. Caki and Or [6], presented asymptotically lacunary statistical equivalent sequences in partial metric spaces. The nonlinear Schrödinger evolution equation was established by Brezis and Gallouet [7]. A resolvent method to the stability operator semigroup was generated by Chill and Tomilov [8]. Davies [9] discovered the spectrum of linear operators. For equations of linear evolution, Engel and Nagel presented the one-parameter semigroup in their paper [10]. Massit and Rossafi [11], obtained results equal-norm parseval continuous k-frames in Hilbert spaces. Omosowon *et al.* [12] derived the outcomes of a semigroup of linear equations that produced a wave equation. Rauf *et al.* [13], established some results of stability and spectra properties on semigroup of linear operator. Vrabie [14] demonstrated a few applications of the C_0-semigroup's findings. Yosida [15] derived several conclusions on the differentiability and representation of a linear operator one-parameter semigroup.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [14]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (ω-$ORCP_n$) [13]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \mathrm{Dom}\alpha : x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (classical or C'-solution) [10]
The function $u : [a, b] \to X$ is called classical, or C'-solution of the problem (1.2), if u is continuous on $[a, b]$, continuously differentiable on $(a, b]$, $u(t) \in D(Z)$ for each $t \in (a, b]$ and it satisfied $u(t) = Zu(t) + f(t)$ for each $t \in [a, b]$ and $u(a) = \xi$.

Definition 2.4 (Absolutely Continuous) [14]
The function $u : [a, b] \to X$ is called absolutely continuous, or strong solution, of the problem (1.2), if u is absolutely continuous on $[a, b]$, $u' \in L^1(a, b; X)$, $u(t) \in D(Z)$, that is for $t \in (a,)$, and it satisfies $u'(t) = Zu(t) + f(t)$ almost everywhere for $t \in (a, b)$ and $u(a) = \xi$.

Definition 2.5 (C^0 or mild solution) [10]
The function $u : [a, b] \to X$, defined by

$$u(t) = J(t-a)\xi + \int_a^t J(t-s)f(s)ds \tag{2.1}$$

is called C^0 or mild solution of problem (1.2).

Example 1
For every 2×2 matrix in $[M_m(\mathbb{R}^n)]$.
Suppose

$$Z = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $J(t) = e^{tZ}$, then we have

$$e^{tZ} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

Example 2
For every 3×3 matrix in $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(Z)$ where $\rho(Z)$ is a resolvent set on X.
Suppose we have

$$Z = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $J(t) = e^{tZ_\lambda}$, then we have

$$e^{tZ_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 3
Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\| \cdot \|_\infty$ and let $\{J(t); t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[J(t)f](s) = f(t+s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{J(t); t \in \mathbb{R}_+\}$ satisfies Examples 1 and 2 above.
Example 4
Let $Z : D(Z) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup $\{J(t);\ t \geq 0\}$, for which there exists $\eta \in X$ such that $J(t)\eta \notin D(Z)$ for each $t \geq 0$. Let us define $f(s) = J(s)\eta$ for each $s \in [0, T]$ and let us observe that f is continuous. On the other hand, the problem (1.2) with $\xi = 0$ has no solution. Indeed, if u is a strong solution of the problem (1.2), in view that each classical solution of (1.2) is a strong solution of the same problem, but not conversely, it is then given by

$$u(t) = \int_0^t J(t-s)J(s)\eta ds = tJ(t)\eta$$

for each $t \in [0, T]$, function which clearly is not almost everywhere differentiable on $[0, T]$.
Example 5 Let H be a real Hilbert space, whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and let $Z : D(Z) \subseteq H \to H$ where $Z \in \omega - ORCP_n$ such that $-Z$ generates a C_0-semigroup of contractions on H. Assume that Z is self-adjoint and invertible with compact inverse Z^{-1}. then by a theorem of Hilbert, there exists a sequence

of positive numbers $\mu_k > 0$, $\mu_{k+1} \leq \mu_k$ for $k \in \mathbb{N}$ and an orthonormal basis $\{e_k; k \in \mathbb{N}\}$ of H such that

$$Z^{-1}e_k = \mu_k e_k \tag{2.2}$$

for $k \in \mathbb{N}$ and $Z \in \omega - ORCP_n$. Let $\lambda_k = \mu_k^{-1}$, and observe that $e_k \in D(Z)$ and

$$Ze_k = \lambda_k e_k$$

for $k \in \mathbb{N}$ and $Z \in \omega - ORCP_n$.

We also have that $\lim\limits_{k\to\infty} \lambda_k = +\infty$. Assume $\alpha > 0$ and defined $Z_\alpha : D(Z\alpha) \subseteq H \to H$ by

$$D(Z_\alpha) = \left\{ u \in H; u = \sum_{k=1}^{\infty} u_k e_k,\ \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < +\infty \right\}$$

$$Z_\alpha u = \sum_{k=1}^{\infty} \lambda_k^{\alpha} u_k e \quad for \quad u \in D(Z_\alpha) \quad and \quad Z_\alpha \in \omega - ORCP_n.$$

A simple calculation shows that $Z_\alpha = Z^\alpha$. We notice that $D(Z^\alpha)$, endowed with the natural inner product $\langle \cdot, \cdot \rangle_\alpha : D(Z^\alpha) \times D(Z^\alpha) \to \mathbb{R}$, defined by

$$\langle u, v \rangle_\alpha = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \langle u_k, v_k \rangle$$

for each $u, v \in D(Z^\alpha)$, $u = \sum_{k=1}^{\infty} u_k e_k$, $v = \sum_{k=1}^{\infty} v_k e_k$ and $Z \in \omega - ORCP_n$ is a real Hilbert space. In addition, with respect to this inner product, the family $\{\lambda_k^{-\alpha} e_k; k \in \mathbb{N}\}$ is an orthonormal basis in $D(Z^\alpha)$.

3. Main Results

This section present results of the non-homogeneous Cauchy problem of semi-group of linear operators generated by ω-$ORCP_n$:

Theorem 3.1. *Assume $Z : D(Z) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{J(t); t \geq 0\}$, let $f \in L^1(a, b; X)$ be continuous on (a, b), and let*

$$v(t) = \int_a^t J(t-s) f(s) ds$$

for $t \in [a, b]$. If at least one of the conditions below is satisfied, then

(i) *v is continuously differentiable on (a, b);*

(ii) *$v(t) \in D(Z)$, $Z \in \omega - ORCP_n$ for each $t \in (a, b)$ and $t \to Zv(t)$ is continuous on (a, b), then for each $\xi \in D(Z)$, (1.2) has a unique classical solution. If there exists $\xi \in D(Z)$ such that (1.2) has a classical solution, then v satisfies both (i) and (ii).*

Proof. Let us observe that for each $t \in (a, b)$ and $h > 0$, we have

$$\frac{1}{h}(J(h) - I)v(t) = \frac{v(t+h) - v(t)}{h} - \frac{1}{h} \int_t^{t+h} J(t+h-s) f(s) ds. \tag{3.1}$$

Assume that (i) holds, since f is continuous, and v is continuously differentiable on (a, b), it follows that the right-hand side of (3.1) converges for h tending to zero. Hence the left-hand side converges too, and consequently, $v(t) \in D(Z)$, $Z \in \omega - ORCP_n$ and

$$v'(t) = Zv(t) + f(t) \tag{3.2}$$

for each $t \in (a, b)$. If (ii) holds, then v is differentiable from the right on (a, b), and its right derivative is continuous on (a, b). Since v is clearly continuous, it follows that v is continuously differentiable on (a, b), and satisfies (3.2). Hence, in both cases, (i) and (ii), v satisfies (3.2). Since $v(a) = 0$, it follows that, for each $\xi \in D(Z)$, $Z \in \omega - ORCP_n$, $t \mapsto u(t) = J(t-a)\xi + v(t)$ for $t \in [a, b]$ is a classical solution of (1.2). Assume now that there exists $\xi \in D(Z)$, such that (1.2) has a classical solution u which in view that if Z generates a uniformly continuous semigroup and f is continuous from $[a, b]$ to $\mathbb{R}^n$, then function $u : [a, b] \to X$ is a classical solution of the nonhomogeneous problem (1.2) if and only if it is given by the so-called variation of constants formula

$$u(t) = J(t-a)\xi + \int_a^t J(t-s)f(s)ds \tag{3.3}$$

for each $t \in [a, b]$. Then given by (3.3), the function $t \mapsto v(t) = u(t) - J(t-a)\xi$ is differentiable on (a, b), and in addition $t \mapsto v(t) = u(t) - J(t-a)\xi$ is differentiable on (a, b), and in addition $v'(t) = u'(t) - J(t-a)Z\xi$ is continuous on (a, b). Thus, v satisfies (i). If $\xi \in D(Z)$, we have $J(t-a) \in D(Z)$ for each $t \in [a, b]$, $Z \in \omega - ORCP_n$ and therefore it follows that $v(t) = u(t) - J(t-a)\xi D(Z)$ for each $t \in (a, b)$, $Z \in \omega - ORCP_n$, and $t \mapsto Zv(t) = Zu(t) - ZJ(t-a)\xi = u'(t) - f(t) - J(t-a)Z\xi$ is continuous on (a, b). So, v satisfies (ii), and the proof is completed

□

Theorem 3.2. *Let $Z : D(Z) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup $\{J(t); t \geq 0\}$, then we have*

(i) *It f is of class C' on $[a, b]$, then for each $\xi \in D(Z)$ and $Z \in \omega - ORCP_n$, the problem (1.2) has a unique classical solution.*
(ii) *If $f \in L^1(a, b; X)$ is continuous on (a, b), $f(s) \in D(Z)$, $Z \in \omega - ORCP_n$, for each $s \in (a, b)$, and $Zf(\cdot) \in L^1(a, b; X)$, then for each $\xi \in D(Z)$, the problem (1.2) has a unique classical solution.*

Proof. Let us observe that

$$t \to v(t) = \int_a^t J(t-s)f(s)ds = \int_0^{t-a} J(s)f(t-s)ds$$

is continuously differentiable on (a, b). Indeed, a simple calculation shows that

$$v'(t) = J(t-a)f(a) + \int_0^{t-a} J(s)f'(t-s)ds = J(t-a)f(a) + \int_a^t J(t-s)f'(s)ds$$

or each $t \in (a, b)$. The conclusion follows from (i) in Theorem 3.1 and this achieves the proof of (i).

To prove (ii), we have for each $t \in (a,b)$ and $s \in (a,b]$, we have $J(t-s)f(s) \in D(Z)$ and $Z \in \omega - ORCP_n$. Moreover, the function $t \mapsto ZJ(t-s)f(s) = J(t-s)Zf(s)$ is integrable and thus v defined as in Theorem 3.1, satisfies $v(t) \in D(Z)$, $Z \in \omega - ORCP_n$ for each $t \in (a,b)$.

In addition, the function

$$t \mapsto Zv(t) = Z\int_a^t J(t-s)f(s)ds = \int_0^t J(t-s)Zf(s)ds$$

is continuous on $[a,b]$. We conclude the proof with the help of (ii) in Theorem 3.1.

$\square$

Theorem 3.3. *Suppose $Z : D(Z) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{J(t); t \geq 0\}$ and let $f \in C([a,b];X)$. If at least one of the two conditions below is satisfies.*

(i) *$f \in L^1(a,b;D(Z))$;*
(ii) *$f \in W^{1,1}(a,b;X)$,*

then for each $\xi \in D(Z)$ and $Z \in \omega - ORCP_n$, the problem (1.2) has a unique classical solution.

Proof. It is sufficient to consider only the case $x = 0$. So, we will prove first that whenever f satisfies either (i) or (ii), the function

$$v(t) = \int_a^t J(t-s)f(s)ds$$

belongs to $C'([a,b];X)$. Let $t \in [a,b)$ and $h \in (0,b-a)$. We assume first that f satisfies (i). We then have

$$\frac{v(t+h)-v(t)}{h} = \int_a^t J(t-s)\frac{J(h)-I}{h}f(s)ds + \frac{1}{h}\int_t^{t+h} J(t+h-s)f(s)ds$$

since $\|J(h)x - x\| \leq \|Zx\|h$ for each $x \in D(Z)$, $Z \in \omega - ORCP_n$ and $h > 0$ by Lebesgue dominated convergence theorem, it follows that

$$\lim_{h\to 0}\frac{J(h)-I}{h}f = Zf$$

in $L^1(a,b;X)$. So

$$\frac{d^+v}{dt}(t) = \int_a^t J(t-s)Zf(s)ds + f(t)$$

for each $t \in [a,b)$. Now, if we assume that $f \in W^{1,1}(a,b;X)$ for each $t \in [a,b)$ and $h \in (0,b-a)$, we have

$$\frac{v(t+h)-v(t)}{h} = \int_a^t J(s)\frac{f(t+h-s)-f(t-s)}{h}ds + J(h)\left(\frac{1}{h}\int_a^{a+h} J(t-s)f(s)f(s)ds\right).$$

Since

$$\lim_{h\to 0}\frac{f(t+h-*)-f(t-*)}{h} = f'(*)$$

in $L^1(a,b;X)$ and

$$\lim_{h\to 0} J(h)\left(\frac{1}{h}\int_a^{a+h} J(t-s)f(s)ds\right) = J(t-a)f(a),$$

we have

$$\frac{d^+v}{dt} = \int_a^t J(s)f'(t-s)ds + J(t-a)f(a)$$

for each $t \in [a,b)$. So in both cases (i) and (ii),

$$\frac{d^+v}{dt} \in C([a,b);X).$$

Similar arguments show that

$$\frac{d^-v}{dt} \in C((a,b];X).$$

and thus $v \in C'([a,b];X)$. Hence the proof is completed.

□

Conclusion

It has been demonstrated in this study that results of the non-homogeneous Cauchy problem of semigroup of linear operators by using partial contraction mapping with ω-order reversing has been established.

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