Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 23 (2024) 1-15 https://doi.org/10.56947/amcs.v23.297

SOME FIXED POINT THEOREMS IN C^*-ALGEBRA VALUED PARTIAL METRIC SPACES

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This paper is dedicated to Professor Said Fahlaoui

Abstract. In this present paper, we introduce the concept of $\psi-$fixed point and we give some fixed point results on C^*-algebra valued metric spaces. Then, we deduce analogous results on C^*-algebra valued partial metric spaces.

1. Introduction and preliminaries

In 1994, Matthews [5] introduced the notion of partial metric space and showed that the Banach contraction principle can be generalized to the partial metric. Later on, many researchers studied fixed point theorems in partial metric spaces, the reader can refer to [5, 3].

In 2014, Ma et al. [4] introduced the concept of C^*-algebra valued metric spaces as a generalization of metric spaces they proved certain fixed point theorems by giving the definition of C^*-algebra valued contractive mapping analogous to Banach contraction principle [1], many authors obtained fixed point results in C^*-algebra valued partial metric space which is more general than partial metric space.

Inspired by the work done in [3], we introduce the notion of $\psi-$fixed point results for various classes of operators defined on a C^*-algebra valued metric space $(U, \mathcal{A}, d)$ and we establish new fixed point theorems. We obtain some corollaries in the setting of C^*-algebra valued partial metric spaces $(U, \mathcal{A}, p)$.

Throughout this paper, we denote $\mathcal{A}$ by an unital C^*-algebra with linear involution $*$, such that for all $x, y \in \mathcal{A}$,

$$(xy)^* = y^*x^*, \text{ and } x^{**} = x.$$

We call an element $x \in \mathcal{A}$ a positive element, denoted by $x \succeq \theta$ if $x \in \mathcal{A}_h = \{x \in \mathcal{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x. Using positive element, we define a partial ordering $\preceq$ on $\mathcal{A}_h$ as follows :

$$x \preceq y \text{ if and only if } y - x \succeq \theta,$$

Date: Received: March 16, 2024; Accepted: Apr 2, 2024.
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2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. $\psi-$fixed point, C^*-algebra valued metric space, C^*-algebra valued partial metric space.

where θ means the zero element in $\mathcal{A}$. We denote the set $x \in \mathcal{A} : x \succeq \theta$ by $\mathcal{A}_+$ and $|x| = (x^*x)^{\frac{1}{2}}$.

Remark 1.1. When $\mathcal{A}$ is a unital C^*-algebra, then for any $x \in \mathcal{A}_+$ we have

$$x \preceq I \Longleftrightarrow \|x\| \leq 1.$$

Definition 1.2. [2] Let U be a non-empty set. A mapping $p : U \times U \to \mathcal{A}$ is called a C^*-algebra valued metric on U if the following conditions are satisfied:

(i) $\theta \preceq p(x,y)$ for all $x, y \in U$ and $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$.
(ii) $p(x,y) = p(y,x)$ for all $x, y \in U$.
(iii) $p(x,x) \preceq p(x,y)$ for all $x, y \in U$.
(iv) $p(x,y) \preceq p(x,z) + p(z,y) - p(z,z)$ for all $x, y, z \in U$.

Then $(U, \mathcal{A}, p)$ is called a C^*-algebra valued partial metric space.

If we take $\mathcal{A} = \mathbb{R}$, then the new notion of C^*-algebra valued partial metric space becomes equivalent to the definition of the real partial metric space.

Example 1.3. Let $U = [0,1]$ and $a \in \mathcal{A}$ be a nonzero element.
Define $p(x,y) = max\{1+x, 1+y\}aa^*$. Then we can easily show that
$p : U \times U \to \mathcal{A}$ is a C^*-algebra valued partial metric.

Example 1.4. Let $U = [0,1]$ and $\mathcal{A} = \mathbb{R}^2$ with a usual norm, be a real Banach space. Let $p : U \times U \to \mathbb{R}^2$ be given as follows:

$$p(x,y) = (|x-y|, |x-y|).$$

Then $(U, \mathbb{R}^2, p)$ is a complete C^*-algebra valued partial metric.

Definition 1.5. [2] Let $(U, \mathcal{A}, p)$ be a C^*-algebra valued partial metric space.

- A sequence $\{x_n\} \subset U$ converges to $x \in U$ whenever for every $\varepsilon > 0$ there is a natural number N such that for all $n > N$,
$$\|p(x_n, x) + p(x,x)\| \leq \varepsilon.$$
We denote it by
$$\lim_{n \longrightarrow \infty} p(x_n, x) - p(x,x) = \theta.$$
- $\{x_n\}$ is a partial Cauchy sequence respect to $\mathcal{A}$, whenever $\varepsilon > 0$ there is a natural number N such that
$$(p(x_n, x_m) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))((p(x_n, x_m) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))^* \preceq \varepsilon^2.$$
for all $n, m > N$
- $(U, \mathcal{A}_+, p)$ is said to be complete with respect to $\mathcal{A}$ if every partial Cauchy sequence with respect to $\mathcal{A}$ converges to $x \in U$, such that
$$\lim_{n \to \infty} (p(x_n, x) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x,x)) = \theta.$$

If we take

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

Then p^s is a C^*- algebra -valued metric.

Lemma 1.6. [2] *Let $(U, \mathcal{A}, p)$ be a C^*- algebra- valued partial metric space.*

- *$\{x_n\}$ is a partial Cauchy sequence in $(U, \mathcal{A}, p)$ if and only if it is Cauchy sequence in the C^*-algebra valued metric $(U, \mathcal{A}, p^s)$.*
- *A C^*-algebra valued partial metric space $(U, \mathcal{A}, p)$ is complete if and only if C^*- algebra- valued metric space $(U, \mathcal{A}, p^s)$ is complete.*
  Furthermore,
  $$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty}(2p(x_n, x) - p(x_n, x_n) - p(x, x)) = \theta$$
  or
  $$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow lim_{n\to\infty}(p(x_n, x) - p(x_n, x_n)) = \theta$$
  and
  $$\lim_{n\to\infty}(p(x_n, x) - p(x, x)) = \theta.$$

Lemma 1.7. [2] *If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a C^*-algebra valued partial metric space $(U, \mathcal{A}, p)$. Then*
$$\lim_{n\to\infty}(p(x_n, y_n) - p(x_n, x_n)) = p(x, y) - p(x, x)$$
and
$$\lim_{n\to\infty}(p(x_n, y_n) - p(y_n, y_n)) = p(x, y) - p(y, y).$$

2. $\psi-$FIXED POINT RESULTS IN C^*-ALGEBRA VALUED METRIC SPACES

In this section, we establish new fixed point theorems on complete C^*-algebra valued metric space and we study the existence and uniqueness of $\psi-$fixed point for differents operators.

Definition 2.1. Let $(U, \mathcal{A}, d)$ be a C^*-algebra valued metric space
$\psi : U \to \mathcal{A}^+$ is a function, and $T : U \to U$ be an operator such that,
$T^0 := 1_X$, $T^1 := T$, $T^{n+1} := T\text{o}T^n; n \in \mathbb{N}$
We denote $Fix(T) := \{x \in U; Tx = x\}$ the set of all fixed points of T.
and $N_\psi := \{x \in U; \psi(x) = \theta\}$ the set of all zeros of the function ψ .
An element $\rho \in U$ is a $\psi-$fixed point of T if and only if $\rho \in Fix(T) \cap N_\psi$.

Definition 2.2. We say that T is a $\psi-$ Picard operator if and only if

(i) $Fix(T) \cap N_\psi = \{\rho\}$.
(ii) $T^n x \to \rho$ as $n \to \infty$, $\forall x \in U$

Definition 2.3. T said to be a weakly $\psi-$Picard operator if and only if

(i) $Fix(T) \cap N_\psi \neq \emptyset$.
(ii) the sequence $T^n x \to z\rho$ converges for each $x \in U$ and the limit is a $\psi-$fixed point of T.

Definition 2.4. Let $\mathcal{G}$ the set of functions $G : \mathcal{A}_+^3 \to \mathcal{A}_+$ satisfying the following conditions:

(G1) $max\{x, y\} \preceq G(x, y, z)$ for all $x, y, z \in \mathcal{A}_+$
(G2) $G(\theta, \theta, \theta) = \theta$

(G3) G is continuous.

Example 2.5. Let $\mathcal{A}_+ = \mathbb{R}^2_+$ and $G : \mathcal{A}^3_+ \to \mathcal{A}_+$ such that
$G(x, y, z) = x + y + z$, we have $G \in \mathcal{G}$.

Definition 2.6. Let $(U, \mathcal{A}, d)$ be a C^*-algebra valued metric space,
$\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$.
The operator $T : U \to U$ is said an $(G, \psi)-$ contraction with respect to d if and only if

$$G(d(Tx, Ty), \psi(Tx), \psi(Ty)) \preceq \tau G(d(x, y), \psi(x), \psi(y)), \forall x, y \in U,\ \tau \in (0, 1). \tag{2.1}$$

Theorem 2.7. *Let $(U, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space,
$\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$. Suppose that the following conditions hold:*

(a) *ψ is lower semi-continous*
(b) *$T : U \to U$ is an $(G, \psi)-$ contraction with respect to the metric d.*

Then

(1) *$Fix(T) \subseteq N_\psi$.*
(2) *T is a $\psi-$Picard operator.*
(3) *$\forall x \in U$, $\forall n \in \mathbb{N}$ we have*

$$d(T^n x, \rho) \preceq \frac{k^n}{1-k} G(d(Tx, x), \psi(Tx), \psi(x))$$

where $\{\rho\} \in Fix(T) \cap N_\psi = Fix(T)$.

Proof. Let $u \in U$ be a fixed point of T. By 2.1 with $x = y = u$ we obtain

$$G(\theta, \psi(u), \psi(u)) \preceq \tau G(\theta, \psi(u), \psi(u)) \tag{2.2}$$

$$\Rightarrow G(\theta, \psi(u), \psi(u)) = \theta \tag{2.3}$$

where $\tau \in (0, 1)$.
From $(G1)$ we have

$$\psi(u) \preceq G(\theta, \psi(u), \psi(u)). \tag{2.4}$$

By 2.2 and 2.4 we obtain $\psi(u) = \theta$, which proves (1).
Let $x \in U$ by 2.1 we have

$$G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) \preceq \tau G(d(T^n x, T^{n-1}y), \psi(T^n x), \psi(T^n x))$$

$$\Rightarrow G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) \preceq \tau^n G(d(Tx, x), \psi(Tx), \psi(x)), \forall n \in \mathbb{N} \cup \{0\}.$$

By $G1$ we have

$$max\{d(T^{n+1}x, T^n x), \psi(T^{n+1}x)\} \preceq \tau^n G(d(Tx, x), \psi(Tx), \psi(x)) \tag{2.5}$$

$$\Rightarrow d(T^{n+1}x, T^n x) \preceq \tau^n G(d(Tx, x), \psi(Tx), \psi(x)); n \in \mathbb{N} \cup \{0\} \tag{2.6}$$

Since $\tau \in (0, 1)$ wich implies that $\{T^n x\}$ is a Cauchy sequence. By the completeness of $(U, \mathcal{A}, d)$, there exists $\rho \in U$ such

$$\lim_{n\to\infty} d(T^n x, \rho) = \theta \tag{2.7}$$

From 2.7 we have

$$\lim_{n\to\infty} \psi(T^{n+1}x) = \theta \tag{2.8}$$

Using that ψ is semi -continuous, and from 2.7 and 2.8 we obtain $\psi(\rho) = \theta$
Then

$$G(d(T^{n+1}x, T\rho), \psi(T^{n+1}x), \psi(T\rho)) \preceq \tau G(d(T^n x, \rho), \psi(T^n x), \psi(\rho)), \forall n \in \mathbb{N} \cup \{0\}. \tag{2.9}$$

Letting $n \to \infty$ in 2.9, using 2.7, 2.8, $G2$ and the continuity of G we have

$$G(d(z, T\rho), \theta, \psi(T\rho)) \preceq \tau G(\theta, \theta, \theta) = \theta.$$

From $G1$ we obtain $d(\rho, T\rho) = \theta$ i.e ρ is a $\psi-$ fixed point of T.
Let $v \in U$ another $\psi-$fixed point of T.
Putting $x = \rho$ and $y = v$ in 2.1 we have

$$\begin{aligned} &G(d(\rho, v), \theta, \theta)) \preceq \tau G(d(\rho, v), \theta, \theta) \\ &\Rightarrow d(\rho, v) = \theta \\ &\Rightarrow \rho = v \end{aligned}$$

so we get (2).
Using 2.4 we get

$$d(T^n x, T^{n+m} x) \preceq \frac{\tau^n (1 - \tau^m)}{1 - \tau} G(d(Tx, x), \psi(Tx), \psi(x)),\ \forall n, m \in \mathbb{N} \cup \{0\}.$$

Letting $m \to \infty$ in the above inequality and using 2.7 we get

$$d(T^n x, \rho) \preceq \frac{\tau^n}{1 - \tau} G(d(Tx, x), \psi(Tx), \psi(x)),\ \forall n \in \mathbb{N} \cup \{0\}.$$

□

Example 2.8. Let $U = [0, 1]$ and $\mathcal{A} = \mathbb{R}^2$.
Define $d : U \times U \to \mathbb{R}^2$ such that $d(x, y) = (0, x + y)$ $\forall x, y \in U$, then $(U, \mathbb{R}^2, d)$ is a C^*-algebra valued metric space.
Let $\psi : U \to \mathbb{R}^2$ such that $\psi(x) = (x, x)$
and $G : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $G(a, b, c) = a + b + c$, $\forall a, b, c \in \mathbb{R}^2$.
Let $T : U \to U$ such that $Tx = \frac{x}{2}$.
we have

$$G(d(Tx, Ty), \psi(Tx), \psi(Ty)) = (\frac{x + y}{2}, x + y)$$

and

$$G(d(x, y), \psi(x), \psi(y)) = (x + y, 2(x + y)).$$

Then T has the required propreties montioned in Theorem 2.7 so we prove that 0 is a $\psi-$fixed point of T.

Remark 2.9. Taking $\psi = \theta$ and $G(a, b, c) = a + b + c$ in Theorem 2.7 we get the Banach contraction principle in C^*-algebra valued metric space.

Definition 2.10. Let $(U, \mathcal{A}, d)$ be a C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$.

$T : U \to U$ said to be a graphic $(G, \psi)-$contraction with respect to the metric d if and only if

$$G(d(T^2x, Tx), \psi(T^2x), \psi(Tx)) \preceq \tau G(d(Tx, x), \psi(Tx), \psi(x))x \in X, \ \tau \in (0, 1).$$

Theorem 2.11. *Let $(U, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$. Suppose that the following conditions hold:*

(a) *ψ is lower semi-continous.*
(b) *$T : U \to U$ is a graphic $(G, \psi)-$ contraction with respect to the metric d.*
(c) *T is continuous.*

Then

(i) *$Fix(T) \subseteq N_\psi$.*
(ii) *T is a weakly $\psi-$ Picard operator.*
(iii) *$\forall x \in U$, if $\lim_{n\to\infty} T^n x = \rho$.*
Then

$$d(T^n x, \rho) \preceq \frac{\tau^n}{1-\tau} G(d(Tx, x), \psi(Tx), \psi(x)), \ n \in \mathbb{N}, \tau \in (0, 1). \tag{2.10}$$

Proof. Let $u \in U$ be a fixed point of T. Using 2.10 with $x = u$, we obtain

$$G(\theta, \psi(u), \psi(u)) \preceq \tau G(\theta, \psi(u), \psi(u)) \tag{2.11}$$

$$\Rightarrow G(\theta, \psi(u), \psi(u)) = \theta \ (\tau \in (0, 1)). \tag{2.12}$$

From $(G1)$ we have

$$\psi(u) \preceq G(\theta, \psi(u), \psi(u)). \tag{2.13}$$

By 2.11 and 2.13 we obtain $\psi(u) = \theta$, which proves (i).

Let $x \in X$ by 2.10 we have

$$G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) \preceq \tau G(d(T^n x, T^{n-1}x), \psi(T^n x), \psi(T^n x)),$$

$$\Rightarrow G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) \preceq \tau^n G(d(Tx, x), \psi(Tx), \psi(x)) \ \forall n \in \mathbb{N} \cup \{0\}.$$

By $G1$ we have

$$max\{d(T^{n+1}x, T^n x), \psi(T^{n+1}x)\} \preceq \tau^n G(d(Tx, x), \psi(Tx), \psi(x)) \tag{2.14}$$

$$\Rightarrow d(T^{n+1}x, T^n x) \preceq \tau^n G(d(Tx, x), \psi(Tx), \psi(x)), \ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.15}$$

Since $\tau \in (0, 1)$ which implies that $\{T^n x\}$ is a Cauchy sequence. By the completeness of $(U, \mathcal{A}, d)$, there exists a $\rho \in U$ such

$$\lim_{n\to\infty} d(T^n x, \rho) = \theta. \tag{2.16}$$

From 2.16 we have

$$\lim_{n\to\infty} \psi(T^{n+1}x) = \theta. \tag{2.17}$$

Using that ψ is lower semi -continuous, and by 2.16 and 2.17 we obtain

$$\psi(\rho) = \theta.$$

Then

$$G(d(T^{n+1}x, Tz), \psi(T^{n+1}x), \psi(T\rho)) \preceq \tau G(d(T^n x, \rho), \psi(T^n x), \psi(\rho)), \ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.18}$$

Letting $n \to \infty$ in 2.18 ,by 2.16, 2.17 , $G2$ and the continuity of G we have

$$G(d(\rho, T\rho), \theta, \psi(T\rho)) \preceq \tau G(\theta, \theta, \theta) = \theta$$

From $G1$ we obtain $d(\rho, T\rho) = \theta$ i.e ρ is a $\psi-$ fixed point of T.
Let $v \in U$ another $\psi-$ fixed point of T.
Putting $x = \rho$ and $y = v$ in 2.10 we obtain

$$\begin{aligned}
&G(d(\rho, v), \theta, \theta)) \preceq \tau G(d(\rho, v), \theta, \theta) \\
&\Rightarrow d(\rho, v) = \theta \\
&\Rightarrow \rho = v,
\end{aligned}$$

so we get (ii). Using 2.14 we get

$$d(T^n x, T^{n+m} x) \preceq \frac{\tau^n(1-\tau^m)}{1-\tau} G(d(Tx, x), \psi(Tx), \psi(x)) \ \forall n, m \in \mathbb{N} \cup \{0\}.$$

letting $m \to \infty$ in the above inequality and using 2.16 we get

$$d(T^n x, \rho) \preceq \frac{\tau^n}{1-\tau} G(d(Tx, x), \psi(Tx), \psi(x)), \ \forall n \in \mathbb{N} \cup \{0\}.$$

□

Definition 2.12. Let $(U, \mathcal{A}, d)$ be a C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$.

The operator $T : U \to U$ said to be an $(G, \psi)-$ weak contraction with respect to the metric d if and only if

$$G(d(Tx, Ty), \psi(Tx), \psi(Ty)) \tag{2.19}$$
$$\preceq \tau G(d(x, y), \psi(x), \psi(y)) + \alpha(G(d(y, Tx), \psi(y), \psi(Tx)) \tag{2.20}$$
$$- G(\theta, \psi(y), \psi(Tx))) \ x, y \in X, \ \tau \in (0,1), \ \alpha \geq 0. \tag{2.21}$$

Theorem 2.13. *Let $(U, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$.*
Suppose that the following conditions hold:

(a) *ψ is lower semi-continous*
(b) *$T : U \to U$ is a $(G, \psi)-$ weak contraction with respect to the metric d.*

Then

(i) *$Fix(T) \subseteq N_\psi$.*
(ii) *T is a weakly $\psi-$ Picard operator.*

(iii) $\forall x \in U,$ *if* $\lim_{n\to\infty} T^n x = \rho$
then

$$d(T^n x, \rho) \preceq \frac{\tau^n}{1-\tau} G(d(Tx,x), \psi(x), \psi(Tx)),\ n \in \mathbb{N}.$$

Proof. Let $u \in U$ be a fixed point of T. By 2.19 with $x = y = u$ we obtain

$$G(\theta, \psi(u), \psi(u)) \preceq \tau G(\theta, \psi(u), \psi(u)) + \alpha(G(\theta, \psi(u), \psi(u)) - G(\theta, \psi(u), \psi(u)) \tag{2.22}$$

$$= \tau G(\theta, \psi(u), \psi(u)) \tag{2.23}$$

$$\Rightarrow G(\theta, \psi(u), \psi(u)) = \theta \ (\tau \in (0,1)). \tag{2.24}$$

From $(G1)$, we have

$$\psi(u) \preceq G(\theta, \psi(u), \psi(u)) \tag{2.25}$$

By 2.22 and 2.25 we obtain $\psi(u) = \theta$, which proves (i).

Let $x \in X$ by 2.19 we have

$$G(d(T^n x, T^{n+1}x), \psi(T^n x), \psi(T^{n+1}x)) \preceq kG(d(T^{n-1}x, T^n x), \psi(T^{n-1}x), \psi(T^n x)) \tag{2.26}$$

$$+ \alpha(G(\theta, \psi(T^n x), \psi(T^n x))) - G(\theta, \psi(T^n x), \psi(T^n x)) \tag{2.27}$$

$$\Rightarrow G(d(T^n x, T^{n+1}x), \psi(T^{n+1}x), \psi(T^n x)) \preceq \tau^n G(d(x, Tx), \psi(x), \psi(Tx))\ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.28}$$

By $G1$, we have

$$max\{d(T^{n+1}x, T^n x), \psi(T^{n+1}x)\} \preceq \tau^n G(d(Tx,x), \psi(Tx), \psi(x)) \tag{2.29}$$

$$\Rightarrow d(T^{n+1}x, T^n x) \preceq k^n G(d(Tx,x), \psi(Tx), \psi(x)),\ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.30}$$

Since $\tau \in (0,1)$ which implies that $\{T^n x\}$ is a Cauchy sequence.
By the completeness of $(U, \mathcal{A}, d)$, there exists $\rho \in U$ such

$$\lim_{n\to\infty} d(T^n x, \rho) = \theta \tag{2.31}$$

By 2.31 we have

$$\lim_{n\to\infty} \psi(T^{n+1}x) = \theta. \tag{2.32}$$

Using that ψ is lower semi -continuous, by 2.31 and 2.32 we get $\psi(\rho) = \theta$.
Moreover,

$$G(d(T^{n+1}x, T\rho), \psi(T^{n+1}x), \psi(T\rho)) \preceq \tau G(d(T^n x, \rho), \psi(T^n x), \psi(\rho)),\ \forall n \in \mathbb{N}\cup\{0\}. \tag{2.33}$$

Letting $n \to \infty$ in 2.34 , using 2.31, 2.32, $G2$ and the continuity of G we have $G(d(\rho, T\rho), \theta, \psi(T\rho)) \preceq \tau G(\theta, \theta, \theta) = \theta$.
From $G1$ we obtain $d(\rho, T\rho) = \theta$ i.e ρ is a $\psi-$ fixed point of T.
Let $v \in U$ another $\psi-$fixed point of T.
Putting $x = \rho$ and $y = v$ in 2.19 we have

$$G(d(\rho, v), \theta, \theta)) \preceq \tau G(d(\rho, v), \theta, \theta)$$
$$\Rightarrow d(\rho, v) = \theta$$
$$\Rightarrow \rho = v$$

so we get (ii) Using 2.14 we get

$$d(T^n x, T^{n+m} x) \preceq \frac{\tau^n(1-\tau^m)}{1-\tau} G(d(Tx,x), \psi(Tx), \psi(x));\ \forall n, m \in \mathbb{N} \cup \{0\}.$$

Letting $m \to \infty$ in the above inequality and by 2.16 we get

$$d(T^n x, \rho) \preceq \frac{\tau^n}{1-\tau} G(d(Tx,x), \psi(Tx), \psi(x)),\ n \in \mathbb{N} \cup \{0\}.$$

□

Example 2.14. Let $U = \mathbb{C}$ and $\mathcal{A} = M_2(\mathbb{C})$ the set of all $n \times n$ matrices with entries in $\mathbb{C}$.

We define

$$d(a,b) = \begin{pmatrix} |a_1 - b_1| & 0 \\ 0 & |a_2 - b_2| \end{pmatrix}$$

$G : M_2(\mathbb{C}) \times M_2(\mathbb{C}) \times M_2(\mathbb{C}) \to M_2(\mathbb{C})$ such $G(A,B,C) = A^2 + B + C$

and

$$\psi : X \to M_2(\mathbb{C})$$

with

$$\psi(x_1, x_2) = \begin{pmatrix} 2|x_1 - x_2| & 0 \\ 0 & 2|x_1 - x_2| \end{pmatrix}$$

and $T : U \to U$ defined by $T(x_1, x_2) = (\dfrac{x_1}{2}, \dfrac{x_2}{2})$

We have T satisfying the condition 2.19 with $\tau = \frac{1}{2}$ and $\alpha = 4$.

Therefore, T has a $\psi-$fixed point $\rho = (0,0)$.

Theorem 2.15. *Let $(U, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$.*

Suppose that the following conditions hold:

(a) *ψ is lower semi-continous*

(b) *$T : U \to U$ satisfying*

$$G(d(Tx,Ty), \psi(Tx), \psi(Ty)) \preceq \tau[G(d(Tx,x), \psi(Tx), \psi(x)) + G(d(Ty,y), \psi(Ty), \psi(y))] \tag{2.34}$$

$\tau \in (0, \frac{1}{2})\ \forall x, y \in U.$

Then

(i) *$Fix(T) \subseteq N_\psi$.*

(ii) *T is a $\psi-$Picard operator.*

Proof. Let $u \in X$ be a fixed point of T. By 2.34 with $x = y = u$ we obtain for $(\tau \in (0, \frac{1}{2}))$

$$G(\theta, \psi(u), \psi(u)) \preceq \tau[G(\theta, \psi(u), \psi(u)) + G(\theta, \psi(u), \psi(u))] \tag{2.35}$$

$$\Rightarrow G(\theta, \psi(u), \psi(u)) = \theta \tag{2.36}$$

From $(G1)$ we have

$$\psi(u) \preceq G(\theta, \psi(u), \psi(u)) \Rightarrow \psi(u) = \theta$$

which we prove (i).

Let $x \in U$ be an arbitrary point, we have

$$\begin{aligned} G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) &\preceq \tau[G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) \\ &+ G(d(T^n x, T^{n-1}x), \psi(T^n x), \psi(T^{n-1}x))] \\ &\preceq \frac{\tau}{1-\tau} G(d(T^n x, T^{n-1}x), \psi(T^n x), \psi(T^{n-1}x)) \\ &\preceq (\frac{\tau}{1-\tau})^n G(d(Tx, x), \psi(Tx), \psi(x)),\ \forall n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

By $G1$ we have

$$max\{d(T^{n+1}x, T^n x), \psi(T^{n+1}x)\} \preceq (\frac{\tau}{1-\tau})^n G(d(Tx, x), \psi(Tx), \psi(x)) \tag{2.37}$$

$$\Rightarrow d(T^{n+1}x, T^n x) \preceq (\frac{\tau}{1-\tau})^n G(d(Tx, x), \psi(Tx), \psi(x))\ n \in \mathbb{N} \cup \{0\} \tag{2.38}$$

Since $\tau \in (0, \frac{1}{2})$ which implies that $\{T^n x\}$ is a Cauchy sequence. By the completeness of $(U, \mathcal{A}, d)$, there exists $\rho \in U$ such

$$\lim_{n\to\infty} d(T^n x, \rho) = \theta \tag{2.39}$$

From 2.39 we have

$$\lim_{n\to\infty} \psi(T^{n+1}x) = \theta. \tag{2.40}$$

Using that ψ is lower semi-continuous, and from 2.39 and 2.40 we obtain

$$\psi(\rho) = \theta.$$

Then,

$$G(d(T^{n+1}x, T\rho), \psi(T^{n+1}x), \psi(T\rho)) \preceq \tau[G(d(T^{n+1}x, T^n x), \psi(T^{n+1}x), \psi(T^n x)) \tag{2.41}$$

$$+ G(d(T\rho, \rho), \psi(T\rho), \psi(\rho)), n \in \mathbb{N} \cup \{0\}. \tag{2.42}$$

Letting $n \to \infty$ in 2.41, by 2.39, 2.40, $G2$ and the continuity of G we have

$$(1-\tau)G(d(T\rho, \rho), \theta, \psi(T\rho)) \preceq \tau G(\theta, \theta, \theta) = \theta.$$

From $G1$ we obtain $d(\rho, T\rho) = \theta$ i.e ρ is a $\psi-$fixed point of T.

Let $v \in U$ another $\psi-$ fixed point of T.

Putting $x = \rho$ and $y = v$ in 2.34 we have

$$\begin{aligned} &G(d(\rho, v), \theta, \theta)) \preceq \tau G(\theta, \theta, \theta) \\ &\Rightarrow d(\rho, v) = \theta \\ &\Rightarrow \rho = v \end{aligned}$$

so we get (ii). □

Remark 2.16. If we take $\psi \equiv \theta$ and $G(a,b,c) = a+b+c$ in Theorem 2.15 we obtain Kannan's fixed point on C^* algebra valued metric space.

Theorem 2.17. *Let $(U, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$. Suppose that the following conditions hold:*

(a) *ψ is lower semi-continous*
(b) *$T : U \to U$ satisfying*

$$G(d(Tx,Ty),\psi(Tx),\psi(Ty)) \preceq \alpha G(d(x,y),\psi(x),\psi(y)) + \beta G(d(x,Tx),\psi(x),\psi(Tx)) \tag{2.43}$$

$$+\gamma G(d(y,Ty),\psi(y),\psi(Ty)) \tag{2.44}$$

where $\alpha,\beta,\gamma \in [0,\infty)$ such $\alpha+\beta+\gamma < 1$ and $\forall x,y \in U$.

Then

(i) *$Fix(T) \subseteq N_\psi$*
(ii) *T is a $\psi-$ Picard operator*

Proof. Let $u \in U$ be a fixed point of T. Using 2.43 with $x = y = u$ we obtain

$$G(\theta,\psi(u),\psi(u)) \preceq \alpha F(\theta,\psi(u),\psi(u)) + \beta(G(\theta,\psi(u),\psi(u)) + \gamma G(\theta,\psi(u),\psi(u)) \tag{2.45}$$

$$= (\alpha+\beta+\gamma)G(\theta,\psi(u),\psi(u)) \tag{2.46}$$

$$\prec G(\theta,\psi(u),\psi(u)) \tag{2.47}$$

$$\Rightarrow G(\theta,\psi(u),\psi(u)) = \theta \tag{2.48}$$

From $(G1)$ we have

$$\psi(u) \preceq G(\theta,\psi(u),\psi(u)). \tag{2.49}$$

By 2.45 and 2.49 we obtain $\psi(u) = \theta$, which proves (i)

Let $x \in U$ by 2.43 we have

$$G(d(T^nx,T^{n+1}x),\psi(T^nx),\psi(T^{n+1}x)) \preceq \alpha G(d(T^{n-1}x,T^nx),\psi(T^{n-1}x),\psi(T^nx))$$
$$+\beta(G(d(T^nx,T^{n+1}),\psi(T^nx),\psi(T^{n+1}x))) + \gamma G(d(T^{n-1}x,T^nx)),\psi(T^{n-1}x),\psi(T^nx))$$
$$\Rightarrow G(d(T^nx,T^{n+1}x),\psi(T^{n+1}x),\psi(T^nx)) \preceq (\frac{\alpha+\gamma}{1-\beta})^n G(d(x,Tx),\psi(x),\psi(Tx))\ \forall n \in \mathbb{N}\cup\{0\}.$$

By $G1$ we have

$$max\{d(T^{n+1}x,T^nx),\psi(T^{n+1}x)\} \preceq (\frac{\alpha+\gamma}{1-\beta})^n G(d(Tx,x),\psi(Tx),\psi(x)) \tag{2.50}$$

$$\Rightarrow d(T^{n+1}x,T^nx) \preceq (\frac{\alpha+\gamma}{1-\beta})^n G(d(Tx,x),\psi(Tx),\psi(x)) \forall n \in \mathbb{N}\cup\{0\}. \tag{2.51}$$

Since $\dfrac{\alpha+\gamma}{1-\beta} \in (0,1)$ which implies that $\{T^nx\}$ is a Cauchy sequence.
By the completeness of $(U,\mathcal{A},d)$, there exists $\rho \in U$ such

$$\lim_{n\to\infty} d(T^n x, \rho) = \theta \tag{2.52}$$

From 2.31 we have

$$\lim_{n\to\infty} \psi(T^{n+1}x) = \theta. \tag{2.53}$$

Using that ψ is lower semi -continuous, and from 2.52 and 2.53 we obtain $\psi(\rho) = \theta$.

Then

$$G(d(T^{n+1}x, T\rho), \psi(T^{n+1}x), \psi(T\rho)) \preceq \tau G(d(T^n x, \rho), \psi(T^n x), \psi(\rho)),\ \forall n \in \mathbb{N}\cup\{0\}. \tag{2.54}$$

Letting $n \to \infty$ in 2.54,by 2.52, 2.53, $G2$ and the continuity of G we obtain

$$G(d(\rho, T\rho), \theta, \psi(T\rho)) \preceq \frac{\alpha+\gamma}{1-\beta} G(\theta, \theta, \theta) = \theta.$$

From $G1$ we have $d(\rho, T\rho) = \theta$ i.e ρ is a $\psi-$ fixed point of T.

Let $v \in U$ another $\psi-$fixed point of T.

Putting $x = \rho$ and $y = v$ in 2.43 we have

$$\begin{aligned} &G(d(\rho, v), \theta, \theta)) \preceq \alpha G(d(\rho, v), \theta, \theta) \\ &\Rightarrow d(\rho, v) = \theta \\ &\Rightarrow \rho = v \end{aligned}$$

so we get (ii). □

Remark 2.18. Taking $\psi \equiv \theta$ and $G(a, b, c) = a + b + c$ in Theorem 2.17 we obtain Reich's fixed point theorem.

Theorem 2.19. *Let $(U, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space, $\psi : U \to \mathcal{A}^+$ be a function, and $G \in \mathcal{G}$.*
Suppose that the following conditions hold:

(a) *ψ is lower semi-continous*
(b) *$T : U \to U$ satisfying*

$$G(d(Tx, Ty), \psi(Tx), \psi(Ty)) \preceq \tau[G(d(x, Ty), \psi(x), \psi(Ty)) - G(\theta, \psi(x), \psi(Ty)) \tag{2.55}$$

$$+ G(d(y, Tx), \psi(y), \psi(Tx))], \tag{2.56}$$

where $\tau \in (0, \frac{1}{2})$; $x, y \in U$.

Then

(i) *$Fix(T) \subseteq N_\psi$.*
(ii) *T is $\psi-$Picard operator.*

Proof. Let $u \in U$ be a fixed point of T. Applying 2.55 with $x = y = u$, we obtain

$$\begin{aligned} G(\theta,\psi(u),\psi(u)) &\preceq \tau[G(\theta,\psi(u),\psi(u)) - G(\theta,\psi(u),\psi(u)) + G(\theta,\psi(u),\psi(u))] \\ &\Rightarrow G(\theta,\psi(u),\psi(u)) = \theta \end{aligned}$$

From $(G1)$ we have $\psi(u) \preceq G(\theta,\psi(u),\psi(u))$ then $\psi(u) = \theta$.

Let $x \in U$ be an arbitrary point, we have

$$\begin{aligned} &G(d(T^{n+1}x,T^nx),\psi(T^{n+1}x),\psi(T^nx)) \preceq \tau[G(d(T^nx,T^nx),\psi(T^nx),\psi(T^nx)) \\ &-G(\theta,\psi(T^nx),\psi(T^nx)) + G(d(T^{n-1}x,T^{n+1}x),\psi(T^{n-1}x),\psi(T^{n+1}x))] \\ &\preceq \tau[G(d(T^{n-1}x,T^{n+1}x),\psi(T^{n-1}x),\psi(T^{n+1}x))] \\ &\Rightarrow G(d(T^{n+1}x,T^nx),\psi(T^{n+1}x),\psi(T^nx)) \preceq \tau^n G(d(Tx,x),\psi(Tx),\psi(x)),\ \forall n \in \mathbb{N}\cup\{0\}. \end{aligned}$$

By $G1$ we have

$$max\{d(T^{n+1}x,T^nx),\psi(T^{n+1}x)\} \preceq \tau^n G(d(Tx,x),\psi(Tx) \tag{2.57}$$

$$\Rightarrow d(T^{n+1}x,T^nx) \preceq \tau^n G(d(Tx,x),\psi(Tx),\psi(x)),\ \forall n \in \mathbb{N}\cup\{0\}. \tag{2.58}$$

Since $\tau \in (0,\frac{1}{2})$ wich implies that $\{T^nx\}$ is a Cauchy sequence. By the completeness of $(U,\mathcal{A},d)$, there exists $\rho \in U$ such

$$\lim_{n\to\infty} d(T^nx,\rho) = \theta \tag{2.59}$$

From 2.59 we have

$$\lim_{n\to\infty} \psi(T^{n+1}x) = \theta \tag{2.60}$$

Using that ψ is lower semi-continuous, and from 2.59 and 2.60 we obtain

$$\psi(\rho) = \theta.$$

Then, $\forall n \in \mathbb{N}\cup\{0\}$

$$G(d(T^{n+1}x,T\rho),\psi(T^{n+1}x),\psi(T\rho)) \preceq k[G(d(T^{n+1}x,T^nx),\psi(T^{n+1}x),\psi(T^nx)) \tag{2.61}$$

$$+ G(d(T\rho,\rho),\psi(T\rho),\psi(\rho)). \tag{2.62}$$

Letting $n \to \infty$ in 2.61, using 2.59, 2.60, $G2$ and the continuity of G we have

$$(1-\tau)G(d(T\rho,\rho),\theta,\psi(Tz)) \preceq \tau G(\theta,\theta,\theta) = \theta.$$

From $G1$ we obtain $d(\rho,T\rho) = \theta$ i.e ρ is a $\psi-$fixed point of T.

Let $v \in U$ another $\psi-$fixed point of T.

Putting $x = z$ and $y = v$ in 2.55 we have

$$\begin{aligned} G(d(\rho,v),\theta,\theta)) &\preceq \tau G(\theta,\theta,\theta) \\ &\Rightarrow d(\rho,v) = \theta \\ &\Rightarrow \rho = v \end{aligned}$$

which proves (ii). □

Remark 2.20. Taking $\psi \equiv \theta$ and $G(a,b,c) = a+b+c$ in Theorem 2.19 we obtain Chatterjea's fixed point theorem.

3. $\psi-$FIXED POINT IN C^* ALGEBRA VALUED PARTIAL METRIC SPACES

In this section we deduce some fixed point theorems in C^*-algebra valued partial metric spaces from the previous obtained results in C^* algebra valued metric spaces.

If we consider the C^*-algebra valued metric $p^s : U \times U \to \mathcal{A}$ defined by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and the function $\psi : U \to \mathcal{A}$ defined by $\psi(x) = p(x,x)$ with $G(a,b,c) = a+b+c$, $\forall a,b,c \in \mathcal{A}$.

From Theorem 2.7 we obtain the following result which generalizes the Matthews fixed point theorem [5].

Corollary 3.1. *Let $(U, \mathcal{A}, p)$ be a complete C^*-algebra valued partial metric space and $T : U \to U$ be a mapping such that*

$$p(Tx,Ty) \preceq \tau p(x,y)\ ,\ \forall x,y \in U,\ \tau \in (0,1).$$

Then T has a unique fixed point $u \in U$, such that $p(u,u) = \theta$.

Proof. Consider the metric p^s on U defined by 1.1 and the function $\psi : U \to \mathcal{A}$ defined by $\psi(x) = p(x,x)$. Applying Theorem 2.7 with $G(a,b,c) = a+b+c$ and using Lemma 1.7, we obtain the result.

□

From Theorem 2.11 we have the following result.

Corollary 3.2. *Let $(U, \mathcal{A}, p)$ be a complete C^*-algebra valued partial metric space and $T : U \to U$ be a mapping such that*

$$p(T^2x,Tx) \preceq \tau p(Tx,x),\ \forall x \in U,\ \tau \in (0,1).$$

Then T has a unique fixed point $u \in U$, such that $p(u,u) = \theta$.

From Theorem 2.13 we get the following result.

Corollary 3.3. *Let $(U, \mathcal{A}, p)$ be a complete C^*-algebra valued partial metric space and $T : U \to U$ be a mapping such that*

$$p(Tx,Ty) \preceq \tau p(x,y) + \alpha[p(y,Tx) - \frac{p(y,y)+p(Tx,Tx)}{2}]\ ,\ \forall x,y \in U,\ \tau \in (0,1)$$
and $\alpha \geq 0$.

Then T has a unique fixed point $u \in U$, such that $p(u,u) = \theta$.

From Theorem 2.15 we get the following Kannan's fixed point theorem on C^*-algebra valued partial metric space.

Corollary 3.4. *Let $(U, \mathcal{A}, p)$ be a complete C^*-algebra valued partial metric space and $T : U \to U$ be a mapping such that*

$$p(Tx,Ty) \preceq \tau[p(x,Tx) + (p(y,Ty)]\ ,\ \forall x,y \in U\ ,\ \tau \in (0,\tfrac{1}{2}).$$

Then T has a unique fixed point $u \in U$, who checks $p(u,u) = \theta$.

From Theorem 2.17 we get the following Reich's fixed point theorem on C^*-algebra valued partial metric space.

Corollary 3.5. *Let $(U, \mathcal{A}, p)$ be a complete C^*-algebra valued partial metric space. Suppose the mapping $T : U \to U$ satisfies:*

$$p(Tx, Ty) \preceq \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty)\ ,\ \forall x, y \in U\ ,\ \alpha, \beta, \gamma \in [0, \infty)\ \text{with}\ \alpha + \beta + \gamma < 1.$$

Then T has a unique fixed point $u \in U$, with $p(u, u) = \theta$.

By Theorem 2.19, we get the following Chatterjea's fixed point theorem on C^*-algebra valued partial metric space.

Corollary 3.6. *If $(U, \mathcal{A}, p)$ be a complete C^*-algebra valued partial metric space and $T : U \to U$ is a mapping such that*

$$p(Tx, Ty) \preceq \tau[p(x, Ty) + (p(y, Tx)]\ ,\ \forall x, y \in U\ ,\ \tau \in (0, \tfrac{1}{2})$$

then T admit a unique fixed point $u \in U$.

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