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EXPLORING NINE DIFFERENT PROOFS OF A FAMOUS LOGARITHMIC INEQUALITY

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Abstract. This article explores and analyzes nine different proofs of a fundamental logarithmic inequality, say "$\log(1+x)/x \geq 1/(1+x/2)$ for $x > -1$". Some proofs are already published; others are new or consider new approaches or new angles of research. They are based on various techniques, such as differentiation, series expansion, and the application of well-known inequalities such as the primitive, Cauchy-Schwarz, logarithmic mean, hyperbolic tangent function, Jensen, and Hermite-Hadamard inequalities. A graphical work illustrates some results. Therefore, our study clarifies the different mathematical foundations of this seemingly straightforward but important inequality and goes beyond it with some new material.

1. Introduction and preliminaries

Understanding and proving (mathematical) inequalities is a fundamental aspect of analysis. Indeed, inequalities provide powerful tools for comparing quantities (functions, measures, etc.). They aim to describe relationships between them, which is essential in various branches of mathematics and real-world applications. From elementary arithmetic to advanced calculus, mastering the principles of inequalities allows us to rigorously analyze functions, optimize solutions, and establish critical limits. Whether exploring geometric properties, studying probability, or modeling physical phenomena, a solid understanding of inequalities is essential to advancing mathematical comprehension and problem solving in a variety of fields. As a matter of fact, a wide range of inequalities already exist. However, much more needs to be done to address emerging challenges, explore complex relationships, and provide comprehensive tools for analyzing diverse theoretical and practical scenarios. Thus, the continued development of new inequalities plays a crucial role in the advancement of mathematical research. This is particularly true for logarithmic-type inequalities, which have been of renewed interest in recent years. See, for instance, [5], [9], [14], and [3].

In this article, we focus on one of the most famous logarithmic inequalities expressed in the lemma below.

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Lemma 1.1. *For any $x > -1$, we have*

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

It finds its source in the work of [11, Item 2.56] (for $x \geq 0$) and [10], with the aim of improving the following basic logarithmic inequality: For any $x > -1$, we have

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x}. \tag{1.1}$$

Thus, the inequality in Lemma 1.1 improves it thanks to a better constant choice in the denominator term. This seemingly simple inequality is, however, of significant importance in various fields, including calculus, probability theory, and optimization. In particular, it can be utilized in problems related to growth rates, optimization of functions, and modeling various phenomena where logarithmic functions are involved, such as compound interest calculations or population growth models. The aim of this article is mainly theoretical; we propose to review and analyze nine different proofs of this inequality, each highlighting different aspects of its mathematical basis. Below, we provide a brief overview of the nature of these proofs, categorized by their levels of novelty and originality according to the following criteria: already published ($\square$), new angles with a relative originality ($\bigcirc$), innovative viewpoints with a certain originality ($\vee$), or thorough applications of existing general results ($\triangle$).

Proof 1 "Differentiation": $\square$. This is the fundamental proof, which was previously published in [10].

Proof 2 "Series Expansion I": $\bigcirc$. This proof introduces a simple but original approach utilizing standard series expansions for $\log(1+x)$ and geometric series. However, it is valid for $x \in (-1, 0)$ only. Although its interest lies in its simplicity, its applicability is limited.

Proof 3 "Series Expansion II": $\vee$. By taking advantage of series expansion techniques, this proof extends beyond the constraint of $x \in (-1, 0)$ by establishing a natural connection between the key functions $\log(1+x)/x$ and $1/(1+x/2)$. It introduces a new mathematical perspective and potential improvements to the main inequality, as we will see later.

Proof 4 "Primitive": $\bigcirc$. This proof shows the use of primitives to achieve the objective, using standard integral techniques.

Proof 5 "Cauchy-Schwarz Inequality": $\square\,\triangle$. By applying the Cauchy-Schwarz inequality and standard integral calculus, this proof offers an alternative approach to derive the desired inequality. This approach was inspired by a result established in [6].

Proof 6 "Logarithmic Mean Inequality": $\triangle$. By applying the logarithmic mean inequality, this proof provides a succinct derivation of the main inequality.

Proof 7 "Hyperbolic Tangent Function": $\vee$. This proof focuses around an inequality of the hyperbolic tangent function and a logarithmic-type composition, thus presenting a new approach.

Proof 8 "Jensen Inequality": $\square$ $\triangle$. By exploiting the inherent convexity of a central function through the Jensen inequality, this proof gives the desired result. It is inspired by an integral inequality in [6].

Proof 9 "Left-side Hermite-Hadamard Inequality": $\triangle$. Again using the convexity property as the main argument, this proof applies the Hermite-Hadamard inequality on the left side to establish the desired result.

These proofs are self-contained, as far as possible, to make them accessible to the reader. On the other hand, thanks to their overall creativity, they can be a source of inspiration for other purposes. Thus, by exploring diverse options, we aim to better understand the underlying principles of inequality and its implications in analysis.

The following sections form the article: Section 2 presents the nine proofs in a row. Section 3 provides additional precison on some technical inequalities used. A conclusion is given in Section 4.

2. Nine proofs of Lemma 1.1

The nine proofs of Lemma 1.1 are presented below. As mentioned previously, they aim to be as self-contained as possible; if an intermediary inequality is necessary, it is demonstrated. Some general inequalities are also used and recalled in the next section. As described in the introduction, we mainly use differentiation, series expansion, and the application of inequalities such as the primitive, Cauchy-Schwarz, logarithmic mean, hyperbolic tangent function, Jensen, and Hermite-Hadamard inequalities.

Proof 1 "Differentiation". This proof is the former one; in fact, it is a slightly modified presentation of that of [10, page 55]. For any $x > -1$, let us set

$$f(x) = \log(1+x) - \frac{x}{1+x/2}.$$

By differentiating with respect to x, we have

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x/2)^2} = \frac{x^2}{(x+2)^2(1+x)}.$$

Since $1+x > 0$, we obviously have $f'(x) \geq 0$. Therefore, $f(x)$ is a non-decreasing function. Let us distinguish between the case $x \geq 0$ and the case $x \in (-1, 0)$. For the case $x \geq 0$, we have $f(x) \geq f(0) = 0$, so

$$\log(1+x) \geq \frac{x}{1+x/2},$$

implying that

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

For the case $x \in (-1, 0]$, we have $f(x) \leq f(0) = 0$, so

$$\log(1+x) \leq \frac{x}{1+x/2}.$$

By dividing both sides by x, which is negative, we get

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

By putting the two cases together, the desired inequality is established.

Proof 2 "Series Expansion I". This proof only deals with the case $x \in (-1,0)$. It is based on two types of series expansions. As such, it is basic but seems to offer an original angle in view of the literature on the subject. It is, however, of limited interest because of the constraint on x. A more elegant series expansion solution will be given in the next proof.

For any $t \in (-1,1)$, the following classical logarithmic and geometric series expansions hold:

$$\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k, \qquad \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k.$$

By setting $y = -x$, the series expansion of the logarithmic function above with $t = -y \in (0,1)$ implies that

$$\frac{\log(1+x)}{x} = \frac{-\log(1-y)}{y} = \frac{1}{y}\sum_{k=1}^{\infty} \frac{y^k}{k} = \sum_{k=1}^{\infty} \frac{y^{k-1}}{k} = \sum_{k=0}^{\infty} \frac{y^k}{k+1}.$$

For any integer $k \geq 0$, we have $2^k \geq k+1$. By using this, the fact that $y \in (0,1)$ and the geometric series expansion above with $t = y/2 \in (0,1)$, we obtain

$$\frac{\log(1+x)}{x} \geq \sum_{k=0}^{\infty} \frac{y^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{y}{2}\right)^k = \frac{1}{1-y/2} = \frac{1}{1+x/2}.$$

The desired inequality is established (but only for $x \in (-1,0)$).

Proof 3 "Series Expansion II". This proof is more complete than the previous one in the sense that it considers the general case $x > -1$, but with the price of more technical series developments. Classically, for any $t \in (-1,1)$, we have

$$\log\left(\frac{1+t}{1-t}\right) = \log(1+t) - \log(1-t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-t)^k$$
$$= \sum_{k=1}^{\infty} \frac{t^k}{k}[1-(-1)^k] = 2\sum_{k=1}^{\infty} \frac{t^{2k-1}}{2k-1}.$$

For any $x > -1$, we have $x/(x+2) \in (-1,1)$, and for $t = x/(x+2)$, we get $(1+t)/(1-t) = 1+x$. As a result, we can write

$$\log(1+x) = \log\left(\frac{1+t}{1-t}\right) = 2\sum_{k=1}^{\infty} \frac{t^{2k-1}}{2k-1} = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x}{2+x}\right)^{2k-1}.$$

This formula also appears in [1, Item 4.1.29] by taking $a = 1$, but only in an expanded form (without the summation symbol sigma), starting with the three first terms, which can make the relation between the functions of interest not immediate. Therefore, by dividing by x, we obtain the following series expansion:

$$\frac{\log(1+x)}{x} = 2\sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2(k-1)} \left(\frac{1}{2+x}\right)^{2k-1}. \tag{2.1}$$

As a key remark, all the terms in the series are non-negative for $x > -1$; we have $x^{2(k-1)} = (x^{k-1})^2 \geq 0$, $2+x \geq 0$ and $2k-1 \geq 0$ for any $k \geq 1$. Therefore, by isolating the first term in the series expansion, we have

$$\frac{\log(1+x)}{x} = \frac{2}{x+2} + 2\sum_{k=2}^{\infty} \frac{1}{2k-1} x^{2(k-1)} \left(\frac{1}{2+x}\right)^{2k-1} \geq \frac{2}{x+2} = \frac{1}{1+x/2}.$$

This ends the proof.

Remark 2.1. Since the series expansion in Equation (2.1) is without restriction on x except $x > -1$, we can derive other simple lower bounds for $\log(1+x)/x$ that are more precise than the one in Lemma 1.1. In particular, by keeping the first two terms of the series expansion only, we establish that

$$\begin{aligned}\frac{\log(1+x)}{x} &= \frac{2}{x+2} + \frac{2}{3}x^2\left(\frac{1}{2+x}\right)^3 + 2\sum_{k=3}^{\infty} \frac{1}{2k-1} x^{2(k-1)} \left(\frac{1}{2+x}\right)^{2k-1} \\ &\geq \frac{2}{x+2} + \frac{2}{3}x^2\left(\frac{1}{2+x}\right)^3 = \frac{8}{3}\left[\frac{x^2+3x+3}{(x+2)^3}\right]. \end{aligned} \tag{2.2}$$

Similarly, by isolating only the first three terms of the series expansion, we obtain

$$\begin{aligned}\frac{\log(1+x)}{x} &= \frac{2}{x+2} + \frac{2}{3}x^2\left(\frac{1}{2+x}\right)^3 + \frac{2}{5}x^4\left(\frac{1}{2+x}\right)^5 \\ &\quad + 2\sum_{k=4}^{\infty} \frac{1}{2k-1} x^{2(k-1)} \left(\frac{1}{2+x}\right)^{2k-1} \\ &\geq \frac{2}{x+2} + \frac{2}{3}x^2\left(\frac{1}{2+x}\right)^3 + \frac{2}{5}x^4\left(\frac{1}{2+x}\right)^5 \\ &= \frac{2}{15}\left[\frac{23x^4+140x^3+380x^2+480x+240}{(x+2)^5}\right]. \end{aligned} \tag{2.3}$$

These inequalities are logically sharp since they improve the one in Lemma 1.1, and they are new in the literature, to the best of our knowledge. Some illustrations of them will be given in the next section.

Proof 4 "Primitive". This proof is based on primitive and basic bounds. We find no trace of this approach in the literature. Let us distinguish between the case $x \geq 0$ and the case $x \in (-1, 0)$. For the case $x \geq 0$, we have

$$\begin{aligned}(1+x)\log(1+x) - x &= [(1+t)\log(1+t) - t]\big|_{t=0}^{t=x} = \int_0^x \log(1+t)dt \\ &= \int_0^x \left(\int_0^t \frac{1}{1+s}ds\right) dt.\end{aligned}$$

On the other hand, it is clear that, for any $s \in [0, t]$, we have $1/(1+s) \geq 1/(1+t)$, implying that

$$\int_0^x \left(\int_0^t \frac{1}{1+s} ds \right) dt \geq \int_0^x \left(\int_0^t \frac{1}{1+t} ds \right) dt = \int_0^x \frac{t}{1+t} dt$$
$$= \int_0^x \left(1 - \frac{1}{1+t} \right) dt = [t - \log(1+t)]|_{t=0}^{t=x} = x - \log(1+x).$$

(Alternatively, some of the previous steps can be omitted by directly using the logarithmic inequality in Equation (1.1)).

As a result, by putting the two above equations together, we get

$$(1+x)\log(1+x) - x \geq x - \log(1+x),$$

which implies that $(2+x)\log(1+x) \geq 2x$, and, after a rearrangement,

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

For the case $x \in (-1, 0)$, we follow the same lines, but with other bounds of integration. We have

$$-[(1+x)\log(1+x) - x] = [(1+t)\log(1+t) - t]|_{t=x}^{t=0} = \int_x^0 \log(1+t) dt$$
$$= -\int_x^0 \left(\int_t^0 \frac{1}{1+s} ds \right) dt.$$

On the other hand, it is clear that, for any $s \in [t, 0]$, we have $1/(1+s) \leq 1/(1+t)$. As a result, we obtain

$$-\int_x^0 \left(\int_t^0 \frac{1}{1+s} ds \right) dt \geq -\int_x^0 \left(\int_t^0 \frac{1}{1+t} ds \right) dt = \int_x^0 \frac{t}{1+t} dt$$
$$= \int_x^0 \left(1 - \frac{1}{1+t} \right) dt = [t - \log(1+t)]|_{t=x}^{t=0} = -[x - \log(1+x)].$$

(Again, some of the previous steps can be reduced by directly using the logarithmic inequality in Equation (1.1)). Therefore, by putting the two above inequalities together, we get

$$-[(1+x)\log(1+x) - x] \geq -[x - \log(1+x)],$$

which implies that $(2+x)\log(1+x) \leq 2x$, and, by taking into account the negative sign of x, we find that

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

This ends this proof.

Proof 5 "Cauchy-Schwarz Inequality". This proof is centered around the Cauchy-Schwarz inequality. It is inspired by a technique developed in [6]. For any $x > -1$, the following decomposition is true:

$$1 = 1^2 = \left(\int_0^1 dt \right)^2 = \left(\int_0^1 \frac{\sqrt{1+xt}}{\sqrt{1+xt}} dt \right)^2.$$

By applying the Cauchy-Schwarz inequality to the integral term, we get

$$\begin{aligned} 1 &= \left(\int_0^1 \frac{\sqrt{1+xt}}{\sqrt{1+xt}} dt \right)^2 \leq \left\{ \int_0^1 [\sqrt{1+xt}]^2 dt \right\} \left\{ \int_0^1 \frac{1}{[\sqrt{1+xt}]^2} dt \right\} \\ &= \left(1 + x \int_0^1 t dt \right) \left(\int_0^1 \frac{1}{1+xt} dt \right) = \left(1 + x \times \frac{1}{2} \right) \left(\frac{\log(1+xt)}{x} \Bigg|_{t=0}^{t=1} \right) \\ &= \left(1 + \frac{x}{2} \right) \frac{\log(1+x)}{x}. \end{aligned}$$

Since $1 + x/2 > 0$, this inequality implies that

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

We obtain the claimed inequality.

Proof 6 "Logarithmic Mean Inequality". This proof is based on a direct application of the logarithmic mean inequality (see [4] and the subsequent section). It also appears in [11, Item 11.27]. For any $x > -1$, by taking $a = 1$ and $b = 1 + x$, we get

$$\frac{b-a}{\log(b) - \log(a)} \leq \frac{a+b}{2},$$

which is equivalent to

$$\frac{x}{\log(1+x)} = \frac{b-a}{\log(b) - \log(a)} \leq \frac{a+b}{2} = \frac{x+2}{2}.$$

It can be rewritten as the desired inequality, i.e.,

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

Proof 7 "Hyperbolic Tangent Function". The proof is centered around the basic properties of the hyperbolic tangent function. We recall that, for any $t \in \mathbb{R}$, the hyperbolic tangent function is defined as

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}.$$

Then we have

$$[\tanh(t)]' = \frac{[\cosh(t)]^2 - [\sinh(t)]^2}{[\cosh(t)]^2} = 1 - [\tanh(t)]^2 \leq 1.$$

Hence, for any $u \geq 0$, we have

$$\tanh(u) = \int_0^u [\tanh(t)]' dt \leq \int_0^u dt = u.$$

This inequality is not new. See [11, 2.17] with another approach, for instance.

Let us distinguish between the case $x \geq 0$ and the case $x \in (-1, 0)$. For the case $x \geq 0$, by taking $u = (1/2)\log(1 + x) \geq 0$, a direct application of the inequality above gives

$$\tanh\left[\frac{1}{2}\log(1+x)\right] = \tanh(u) \leq u = \frac{1}{2}\log(1+x),$$

with

$$\begin{aligned}\tanh\left[\frac{1}{2}\log(1+x)\right] &= \frac{e^{(1/2)\log(1+x)} - e^{-(1/2)\log(1+x)}}{e^{(1/2)\log(1+x)} + e^{-(1/2)\log(1+x)}}\\ &= \frac{\sqrt{1+x} - 1/\sqrt{1+x}}{\sqrt{1+x} + 1/\sqrt{1+x}} = \frac{x}{2+x}.\end{aligned}$$

Hence, we have

$$\frac{x}{2+x} \leq \frac{1}{2}\log(1+x),$$

which can be rearranged as

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

For the case $x \in (-1, 0)$, we need to modify the central hyperbolic tangent inequality. For any $u \leq 0$, since the hyperbolic tangent function is odd, we have

$$\tanh(u) = -\tanh(-u) \geq -(-u) = u.$$

Thus, by taking $u = (1/2)\log(1 + x) \leq 0$, we obtain

$$\tanh\left[\frac{1}{2}\log(1+x)\right] = \tanh(u) \geq u = \frac{1}{2}\log(1+x),$$

with again

$$\tanh\left[\frac{1}{2}\log(1+x)\right] = \frac{x}{2+x}.$$

Thus, by dividing both sides by x, which is negative, after a rearrangement, we get

$$\frac{\log(1+x)}{x} \geq \frac{1}{1+x/2}.$$

This completes the proof.

Proof 8 "Jensen Inequality". This proof uses the convexity of a well-chosen function and the Jensen inequality (see [12] and the subsequent section). It is also inspired by a technique in [6]. For any $x > -1$ and $t \in [0, 1]$, let us consider the function

$$g(t) = \frac{1}{1+xt}.$$

Then, by differentiating two times, we get

$$g'(t) = -\frac{x}{(1+xt)^2}, \quad g''(t) = \frac{2x^2}{(1+xt)^3}.$$

Since $1+xt \geq 0$, we have $g''(t) \geq 0$, implying that $g(t)$ is convex. It follows from the Jensen inequality applied with the functions $g(t)$, t and the measure $\nu(A) = \int_A dt$, with $A \subseteq [0,1]$ that

$$g\left(\int_0^1 t dt\right) \leq \int_0^1 g(t)dt.$$

By calculating the involved integrals, we establish that

$$\begin{aligned}\frac{1}{1+x/2} &= g\left(\frac{1}{2}\right) = g\left(\int_0^1 t dt\right) \leq \int_0^1 g(t)dt \\ &= \int_0^1 \frac{1}{1+xt}dt = \left.\frac{\log(1+xt)}{x}\right|_{t=0}^{t=1} = \frac{\log(1+x)}{x}.\end{aligned}$$

The stated result is obtained.

Proof 9 "Left-side Hermite-Hadamard Inequality". This proof also uses the convexity of a well-chosen function and the left-side Hermite-Hadamard inequality (see [7] and the subsequent section). For any $t \in (0,1]$, let us consider the simple function

$$h(t) = \frac{1}{t}.$$

Then it is immediate that

$$h'(t) = -\frac{1}{t^2}, \quad h''(t) = \frac{2}{t^3}.$$

We have $h''(t) \geq 0$, implying that $h(t)$ is convex. It follows from the left-side Hermite-Hadamard inequality applied with this function, $a = 1$ and $b = 1+x$, with $x > -1$, that

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b h(t)dt,$$

which after calculations gives

$$\begin{aligned}\frac{1}{1+x/2} &= h\left(\frac{2+x}{2}\right) = h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b h(t)dt = \frac{1}{x}\int_1^{1+x} \frac{1}{t}dt \\ &= \frac{1}{x}\log(t)\Big|_{t=1}^{t=1+x} = \frac{\log(1+x)}{x}.\end{aligned}$$

Discussion. Therefore, these nine different proofs provide insight into the mathematical foundations underlying a seemingly simple but meaningful logarithmic inequality. Through them, we aim to improve the understanding of logarithmic inequalities and their various applications in analysis. If a choice must be made, thanks to its deep mathematical connection between $\log(1+x)$ and $1/(1+x/2)$ and the new inequalities highlighted in Remark 2.1 immediately derived from it, the "best of the nine proofs" can be attributed to Proof 3 "Series Expansion II".

3. Complements

This section provides some additional elements to the previous findings, beginning with a graphical work.

3.1. **Graphical work.** In order to give an illustration of the inequality in Lemma 1.1, Figure 1 displays the main involved functions.

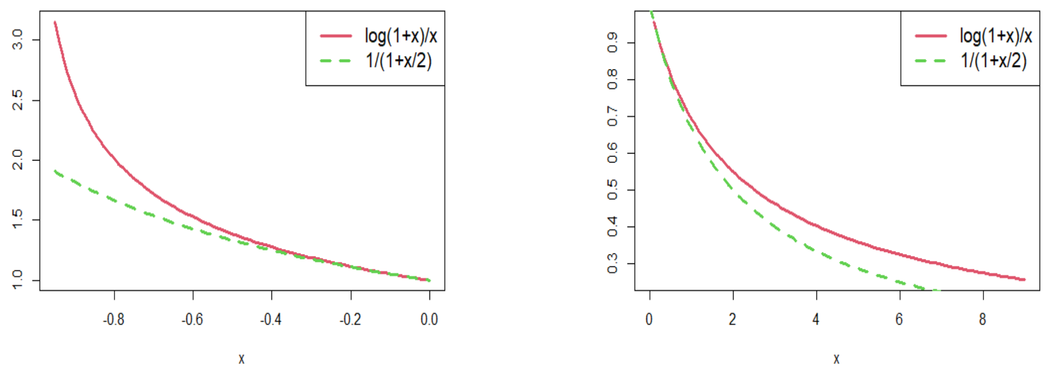

FIGURE 1. Plots of the functions $\log(1+x)/x$ and $1/(1+x/2)$ for $x \in (-1, 0]$ (left) and $x \in [0, 9]$ (right)

As expected, we clearly see that $\log(1 + x)/x$, represented by the plain red curve, is always greater than $1/(1+x/2)$, represented by the green dashed curve.

Complementary to that, the bounds in Equations (2.2) and (2.3) are new. We illustrate them in Figure 2 by considering the two following difference functions for the sake of visibility:

$$u(x) = \frac{\log(1+x)}{x} - \frac{8}{3}\left[\frac{x^2+3x+3}{(x+2)^3}\right] \tag{3.1}$$

and

$$v(x) = \frac{\log(1+x)}{x} - \frac{2}{15}\left[\frac{23x^4+140x^3+380x^2+480x+240}{(x+2)^5}\right]. \tag{3.2}$$

Thus, based on Remark 2.1, we know that $u(x) \geq 0$ and $v(x) \geq 0$ for any $x > -1$.

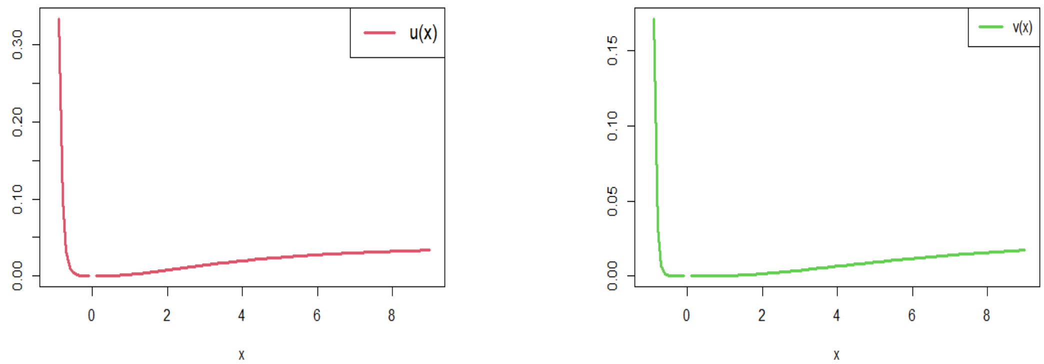

FIGURE 2. Plots of the functions $u(x)$ in Equation (3.1) for $x \in (-1, 9]$ (left) and $v(x)$ in Equation (3.2) for $x \in (-1, 9]$ (right)

From this figure, by analyzing the y-axis, the sharpness of the obtained inequalities is flagrant, with the expected preference for the one implied by $v(x)$.

3.2. **Used general results.** We complete this part by recalling the used versions of the logarithmic mean, Cauchy-Schwarz, Jensen, and Hermite-Hadamard inequalities.

The logarithmic mean inequality is formulated in the theorem below.

Theorem 3.1. *(The logarithmic mean inequality, see [4]) For any $a > 0$, $b > 0$ and $a \neq b$, we have*

$$\frac{b-a}{\log(b)-\log(a)} \leq \frac{a+b}{2}.$$

Further information and recent developments about this inequality can be found in [2] and [8].

Basically, the integral version of the Cauchy-Schwarz inequality is presented below.

Theorem 3.2. *(The Cauchy-Schwarz inequality) Let $(a, b) \in \mathbb{R}^2$ with $a < b$ and $i(x)$ and $j(x)$, $x \in [a, b]$, be two real-valued square-integrable functions. Then we have*

$$\left[\int_a^b i(x)j(x)dx\right]^2 \leq \int_a^b |i(x)|^2 dx \int_a^b |j(x)|^2 dx.$$

By bounding the integral of the product of two functions, this result provides a means of establishing inequalities and quantifying relationships between functions in a wider range of mathematical contexts.

The coming results are centered around the concept of convexity. The theorem below is about the considered Jensen inequality.

Theorem 3.3. *(The Jensen inequality, see [12]). Let $(a, b) \in \mathbb{R}^2$ with $a < b$, and ν be a positive measure such that $\nu([a, b]) = 1$. Let $m(x)$, $x \in [a, b]$, be a real* *function integrable with respect to the measure ν, $I \subseteq \mathbb{R}$ such that $m(x) \in I$ for any $x \in [a,b]$, and $\ell(x)$, $x \in I$, be a convex function. Then we have*

$$\ell\left[\int_a^b m(x)d\nu(x)\right] \leq \int_a^b \ell[m(x)]d\nu(x).$$

The inequality holds in the reverse direction if $\ell(x)$, $x \in I$, is concave.

The Jensen inequality is a fundamental concept in analysis, providing a tool for understanding the behavior of convex functions. It plays a crucial role in probability theory, optimization, and statistics.

The theorem below describes the considered Hermite-Hadamard inequalities.

Theorem 3.4. *(The Hermite-Hadamard inequalities, see [7]). Let $(a,b) \in \mathbb{R}^2$ with $a < b$ and $\ell(x)$, $x \in [a,b]$ be a real-valued convex function. Then we have*

$$\ell\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \ell(x)dx,$$

which is the left-side Hermite-Hadamard inequality, and

$$\frac{1}{b-a}\int_a^b \ell(x)dx \leq \frac{\ell(a)+\ell(b)}{2},$$

which is the right-side Hermite-Hadamard inequality, Both the inequalities hold in the reverse direction if $\ell(x)$, $x \in [a,b]$, is concave.

The Hermite-Hadamard inequalities serve as fundamental tools in analysis, providing limits on the integral mean value of a function over an interval. Their importance extends to various fields such as convex analysis, probability theory, and optimization.

Recent advances on the Jensen and Hermite-Hadamard inequalities can be found in [13], and the references therein.

4. Conclusion

In conclusion, this article examines nine diverse proofs of a fundamental logarithmic inequality, providing insight into various mathematical techniques and principles. These proofs were "self-contained as much as possible". We provided a deeper comprehension of the notion of inequality through differentiation, series expansion, and the application of inequalities such as the primitive, Cauchy-Schwarz, logarithmic mean, hyperbolic tangent function, Jensen, and Hermite-Hadamard inequalities. A graphical work illustrates some obtained inequalities, and new developments and results were also given. Overall, this article contributes to a broader understanding of logarithmic inequalities and their relevance in mathematical research. We hope that it can be inspirational for other kinds of inequalities beyond the logarithmic nature.

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