

ON SOME OPERATOR NORM INEQUALITIES

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ABSTRACT. In this paper, we use known classical inequalities to establish some operator norm inequalities for a pair of norm-attainable operators on a complex Hilbert space.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the operator norm of A is defined by

$$\|A\| := \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| \leq 1\}. \quad (1.1)$$

Following (1.1), $\|Ax\| \leq \|A\|$ for each $x \in \mathcal{H}$. Moreover, for any two operators $A, B \in \mathcal{B}(\mathcal{H})$, it is easily proved that $\|A + B\| \leq \|A\| + \|B\|$ while $\|AB\| \leq \|A\|\|B\|$. There continues to be a lot of interest in the upper norm bounds for the sum and product of operators in $\mathcal{B}(\mathcal{H})$ for different classes of operators. For instance, Davidson and Power [2] proved that for positive operators $A, B \in \mathcal{B}(\mathcal{H})$,

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}, \quad (1.2)$$

with the following refinements later given by Kittaneh in [8] and [9] respectively as:

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|A^{\frac{1}{2}}B^{\frac{1}{2}}\|, \quad (1.3)$$

and

$$\|A + B\| \leq \frac{1}{2}(\|A\| + \|B\|) + \frac{1}{2}\sqrt{(\|A\| - \|B\|)^2 + 4\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|^2}. \quad (1.4)$$

For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$, it is shown in [9] that the inequality (1.4) is satisfied. However, the class of normal operators $A, B \in \mathcal{B}(\mathcal{H})$ has been shown in [11] to satisfy

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \||A|^{\frac{1}{2}}|B|^{\frac{1}{2}}\|. \quad (1.5)$$

Further refinements and generalizations of inequalities (1.2)-(1.5) have recently been given in [12]. On the other hand, norm inequalities related to products of operators in $\mathcal{B}(\mathcal{H})$ have also been very recently considered in [1], while numerical

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radii as well as some integral inequalities in [6, 13].

In this work, we focus on a special class of operators that was introduced in [5]. Following [5], an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be norm-attainable if there exists a unit vector $x \in \mathcal{H}$ such that $\|Ax\| = \|A\|$. In this work we introduce norm-attainability of a pair of operators $A, B \in \mathcal{B}(\mathcal{H})$.

Definition 1.1. A pair of operators $A, B \in \mathcal{B}(\mathcal{H})$ is said to be norm-attainable if there exists a unit vector $x \in \mathcal{H}$ such that $\|Ax\| = \|A\|$ and $\|Bx\| = \|B\|$.

We shall be interested in the operator norm product $\|A\|\|B\|$ for a pair of norm-attainable operators $A, B \in \mathcal{B}(\mathcal{H})$ and give a number of refinements of the upper bounds of such operators. In order to give our results in detail, we shall need the following well-known lemmas on inequalities. The first Lemma, which is called Holder-McCarthy inequality, is a well-known result that follows from spectral theorem for positive operators and Jensen's inequality (see [7]).

Lemma 1.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be positive operator and let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned} \langle Ax, x \rangle^r &\leq \langle A^r x, x \rangle \text{ for all } r \geq 1, \text{ and} \\ \langle A^r x, x \rangle &\leq \langle Ax, x \rangle^r \text{ for all } 0 < r \leq 1. \end{aligned}$$

The next Lemma is concerned with positive real numbers and is a simple consequence of the classical Jensen's inequality concerning the convexity of the function $f(t) = t^r, r \geq 1$.

Lemma 1.3. *If a and b are nonnegative real numbers, then*

$$(a + b)^r \leq 2^{r-1}(a^r + b^r), \text{ for } r \geq 1.$$

The third Lemma which is the last in this series is the famous Clarkson's inequality.

Lemma 1.4. *If $a, b \in \mathbb{C}$, then*

$$2(|a|^r + |b|^r) \leq |a + b|^r + |a - b|^r, \text{ for } r \geq 2.$$

We mention here that some other inequalities which shall be used in this study but their statements are not provided in this section due to their triviality include: Arithmetic-Geometric mean inequality, Mixed Cauchy-Schwarz inequality, and the Young's inequality. We proceed to give our results in the next section.

2. OPERATOR NORM INEQUALITIES

Our first result is the following

Theorem 2.1. *Let the pair of operators $A, B \in \mathcal{B}(\mathcal{H})$ be norm-attainable. Then*

$$\|A\|\|B\| = \|(A^*A)(B^*B)\|^{\frac{1}{2}}.$$

Proof. Let $x \in \mathcal{H}$ be the unit vector for which the pair A, B is norm attainable, then we have

$$\begin{aligned}
 \|A\|\|B\| &= \|Ax\|\|Bx\| \\
 &= (\|Ax\|^2\|Bx\|^2)^{\frac{1}{2}} \\
 &= (\langle Ax, Ax \rangle \langle Bx, Bx \rangle)^{\frac{1}{2}} \\
 &= (\langle A^*Ax, x \rangle \langle B^*Bx, x \rangle)^{\frac{1}{2}} \\
 &\leq \langle ((A^*A)(B^*B))x, x \rangle^{\frac{1}{2}}.
 \end{aligned}$$

It therefore follows immediately that

$$\|A\|\|B\| \leq \|(A^*A)(B^*B)\|^{\frac{1}{2}}.$$

On the other hand

$$\begin{aligned}
 \|(A^*A)(B^*B)\|^{\frac{1}{2}} &\leq (\|A^*A\|\|B^*B\|)^{\frac{1}{2}} \\
 &\leq (\|A^*\|\|A\|\|B^*\|\|B\|)^{\frac{1}{2}} \\
 &= (\|A\|^2\|B\|^2)^{\frac{1}{2}} \\
 &= \|A\|\|B\|,
 \end{aligned}$$

which completes the proof. $\square$

In our next result, we use the arithmetic-geometric mean inequality to obtain an upper bound for $\|A\|\|B\|$.

Theorem 2.2. *Let the pair of operators $A, B \in \mathcal{B}(\mathcal{H})$ be norm-attainable. Then*

$$\|A\|\|B\| \leq \frac{1}{2}\|A^*A + B^*B\|.$$

Proof. For a unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
 \|A\|\|B\| &= (\langle A^*Ax, x \rangle \langle B^*Bx, x \rangle)^{\frac{1}{2}} \\
 &\leq \left(\frac{1}{4} (\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle)^2 \right)^{\frac{1}{2}} \\
 &\quad (\text{by the arithmetic-geometric mean inequality}) \\
 &= \frac{1}{2} (\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle) \\
 &= \frac{1}{2} \langle (A^*A + B^*B)x, x \rangle.
 \end{aligned}$$

The result then follows by taking supremum over $x \in \mathcal{H}$. $\square$

We give another inequality in the same line but this inequality incorporates a number of classical inequalities and as a consequence we shall obtain some operator and commutator inequalities as special cases.

Theorem 2.3. *Let the pair of operators $A, B \in \mathcal{B}(\mathcal{H})$ be norm-attainable. Then for $r \geq 2$,*

$$\|A\|\|B\| \leq 2^{-\frac{1}{r}} \| |A^*A|^r + |B^*B|^r \|^{\frac{1}{r}}. \quad (2.1)$$

Proof. For a unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned} (\|B\|\|A\|)^r &= \left(\langle A^*Ax, x \rangle^{\frac{1}{2}} \langle B^*Bx, x \rangle^{\frac{1}{2}} \right)^r \\ &= (\langle A^*Ax, x \rangle \langle B^*Bx, x \rangle)^{\frac{r}{2}} \\ &\leq \left(\frac{1}{4} (\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle)^2 \right)^{\frac{r}{2}} \\ &\quad (\text{by the arithmetic-geometric mean inequality}) \\ &= \left(\frac{1}{2} (\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle) \right)^r \\ &= 2^{-r} (\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle)^r \\ &\leq 2^{-r} (|\langle A^*Ax, x \rangle| + |\langle B^*Bx, x \rangle|)^r \quad (\text{by the basic properties of } |\cdot|) \\ &\leq 2^{-1} (|\langle A^*Ax, x \rangle|^r + |\langle B^*Bx, x \rangle|^r) \quad (\text{by Lemma 1.3}) \\ &\leq 2^{-1} (\langle |A^*A|^{2\alpha}x, x \rangle^{\frac{r}{2}} \langle |A^*A|^{2(1-\alpha)}x, x \rangle^{\frac{r}{2}} \\ &\quad + \langle |B^*B|^{2\alpha}x, x \rangle^{\frac{r}{2}} \langle |B^*B|^{2(1-\alpha)}x, x \rangle^{\frac{r}{2}}) \\ &\quad (\text{by mixed Cauchy-Schwarz inequality, where, } 0 < \alpha < 1) \\ &\leq 2^{-1} (\langle |A^*A|^rx, x \rangle^\alpha \langle |A^*A|^{1-\alpha}x, x \rangle^{1-\alpha} + \\ &\quad \langle |B^*B|^rx, x \rangle^\alpha \langle |B^*B|^{1-\alpha}x, x \rangle^{1-\alpha}) \quad (\text{by Lemma 1.2}) \\ &\leq 2^{-1} (\alpha \langle |A^*A|^rx, x \rangle + (1-\alpha) \langle |A^*A|^{1-\alpha}x, x \rangle + \alpha \langle |B^*B|^rx, x \rangle \\ &\quad + (1-\alpha) \langle |B^*B|^{1-\alpha}x, x \rangle) \quad (\text{by the Young's inequality}) \\ &= 2^{-1} (\alpha \langle (|A^*A|^r + |B^*B|^r)x, x \rangle + (1-\alpha) \langle (|A^*A|^{1-\alpha} + |B^*B|^{1-\alpha})x, x \rangle) \\ &= 2^{-1} (\langle \alpha(|A^*A|^r + |B^*B|^r)x, x \rangle + \langle (1-\alpha)(|A^*A|^{1-\alpha} + |B^*B|^{1-\alpha})x, x \rangle) \\ &= 2^{-1} (\langle (\alpha(|A^*A|^r + |B^*B|^r) + (1-\alpha)(|A^*A|^{1-\alpha} + |B^*B|^{1-\alpha}))x, x \rangle) \\ &= 2^{-1} (\langle (\alpha(|A^*A|^r + |B^*B|^r) + (1-\alpha)(|A^*A|^{1-\alpha} + |B^*B|^{1-\alpha}))x, x \rangle) \\ &= 2^{-1} \langle (|A^*A|^r + |B^*B|^r)x, x \rangle. \end{aligned}$$

The remaining part of the proof is trivial. $\square$

Letting $r = 2$ (Note: $r \geq 2$) in (2.1), we obtain the following operator norm inequality which shall hold for any pair of operators which is norm-attainable

$$\|A\|\|B\| \leq 2^{-\frac{1}{2}} \| |A^*A|^2 + |B^*B|^2 \|^{\frac{1}{2}}. \quad (2.2)$$

On the other hand, inequality (2.2) lead us to the following inequality which is simplified,

$$\|A\|^2\|B\|^2 \leq \frac{1}{2}(\|A\|^4 + \|B\|^4).$$

Specifically,

$$(\|A\|\|B\|)^r \leq \frac{1}{\sqrt{2}}(\|A\|^4 + \|B\|^4)^{\frac{1}{2}}.$$

The commutator of A and B is the operator $AB - BA$. Commutators play an important role in operator theory. It follows by triangle inequality and submultiplicity of the norm that if $A, B \in \mathcal{B}(\mathcal{H})$, then $\|AB - BA\| \leq 2\|A\|\|B\|$. In [4], it has also been obtained that $\|AB - BA\| \leq 2\min\{\|A\|, \|B\|\} \min\{\|A - B\|, \|A + B\|\}$. Another upper bound for the commutator has been obtained in [10]. Numeric simulations mainly based on operator theory techniques have become mandatory for all research sectors connected to applied physics. The most recent statistics show their use at all levels and in all applications like instrumentation, dosimetry, radiation protection, medical physics, regulation and in various engineering specialties. They often provide very precise information when experiments are impossible, difficult or expensive. They may constitute an alternate research line for several laboratories for science and technology. We can get some new upper bounds of the commutator which shall now apply to a pair of operators which is norm-attainable. Now from inequality (2.2), we get

$$\|AB - BA\| \leq \frac{1}{\sqrt{2}}\| |A^*A|^2 + |B^*B|^2 \|^{\frac{1}{2}}.$$

Simplified further, we obtain

$$\|AB - BA\| \leq \frac{1}{\sqrt{2}}(\|A\|^4 + \|B\|^4)^{\frac{1}{2}}.$$

By utilizing the steps involved in the proof of Theorem 2.3, one can derive the following result for the commutator inequality which coincides with the result obtained by Dragomir in [3, Proposition 1].

Corollary 2.4. *Let the pair $A, B \in \mathcal{B}(\mathcal{H})$ be norm-attainable. Then for $r \geq 2$,*

$$\|AB - BA\| \leq 2^{-r^{-1}+1} \|(A^*A)^r + (B^*B)^r\|^{\frac{1}{2r}} \|(AA^*)^r + (BB^*)^r\|^{\frac{1}{2r}}.$$

We proceed to conclude this paper by giving another result which utilizes the clarkson's inequality and elementary consequences of spectral theorem.

Theorem 2.5. *Let the pair of operators $A, B \in \mathcal{B}(\mathcal{H})$ be norm-attainable. Then for $r \geq 2$,*

$$\|A\|\|B\| \leq 2^{-2r-1} \| |A^*A + B^*B|^r + |A^*A - B^*B|^r \|^\frac{1}{r}.$$

Proof. It is now clear that for a unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned} \|A\|\|B\| &\leq 2^{-1} (|\langle A^*Ax, x \rangle|^r + |B^*Bx, x|^r) \\ &\leq 2^{-1} \left(\frac{1}{2} (|\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle|^r + |\langle A^*Ax, x \rangle - \langle B^*Bx, x \rangle|^r) \right) \\ &\quad (\text{by the Clarkson's inequality}) \\ &= \frac{1}{4} (|\langle (A^*A + B^*B)x, x \rangle|^r + |\langle (A^*A - B^*B)x, x \rangle|^r) \\ &\leq \frac{1}{4} (\langle |A^*A + B^*B|x, x \rangle^r + \langle |A^*A - B^*B|x, x \rangle^r) \\ &\quad (\text{by a consequence of spectral theorem}) \\ &\leq \frac{1}{4} (\langle |A^*A + B^*B|^r x, x \rangle + \langle |A^*A - B^*B|^r x, x \rangle) \quad (\text{by Lemma 1.2}) \\ &= \frac{1}{4} (\langle |A^*A + B^*B|^r \rangle + \langle |A^*A - B^*B|^r \rangle). \end{aligned}$$

Whence

$$\|A\|\|B\| \leq 2^{-2r-1} \| |A^*A + B^*B|^r + |A^*A - B^*B|^r \|^\frac{1}{r}.$$

□

Remark 2.6. The norm $\|A\|\|B\|$ is related to the norm of a basic elementary operator $M_{A,B} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, $M_{A,B}(X) = AXB \ \forall X \in \mathcal{B}(\mathcal{H})$, in the sense that $\|M_{A,B}\| = \|A\|\|B\|$. It therefore follows that the established inequalities in this work actually constitute upper bounds for the norm of a basic elementary operator induced by a pair of norm attainable bounded operators on a Hilbert space $\mathcal{H}$. It would be an interesting task to investigate the lower bounds as well.

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