

SOME PROPERTIES OF A COMPLEX SYMMETRIC OPERATOR

RITU KATHURIA^{1*}

ABSTRACT. A square matrix with complex entries which is equal to its transpose is called a complex symmetric matrix. Such matrices are found in many areas of pure and applied mathematics. A conjugation on a Hilbert space H is defined as a conjugate linear isometric involution C on H . A bounded linear operator $T : H \rightarrow H$ has a complex symmetric matrix with regard to an orthonormal basis if, for every conjugation C on H , $T = CT^*C$. Such an operator is called a complex symmetric operator. Several characteristics of complex symmetric matrices and associated operators are discussed in this article.

1. INTRODUCTION

Let H denote a complex Hilbert space and $\mathcal{L}(H)$ denote the space of all bounded linear operators on H . An operator $C : H \rightarrow H$ which is anti-linear and satisfies $\langle Cf, Cg \rangle = \langle g, f \rangle \quad \forall f, g \in H$ and $C^2 = I$ is said to be a conjugation on H .

A linear operator $T : H \rightarrow H$ is said to be bounded if $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$ is finite. If such a bounded linear operator T satisfies the equation $T = CT^*C$ then it is called C -Symmetric. If T is C -Symmetric with respect to a certain conjugation on H , then T is said to be complex symmetric. The matrix representation of such an operator is symmetric for some orthonormal basis.

The class of complex symmetric operators was initially studied by Garcia and Putinar [4],[5]. These operators are well behaved and tractable operators and they include the Hankel operators, normal operators and the Volterra Integration operators. In Linear Algebra these operators generalize the notion of symmetric matrices. Certain applications of the complex symmetric operators have also been discussed in [5].

It is known that for a given conjugation C , one can determine an orthonormal basis $\{e_n\}$ of H for which $Ce_n = e_n \quad \forall n \in \mathbb{N}$.

Date: Received: Mar 4, 2024; Accepted: Apr 8, 2024.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47B99,; Secondary 46C99.

Key words and phrases. Hardy-Hilbert space, Complex symmetric operators, Symmetric matrix, Toeplitz operators.

Now let α_{ij} denote the entries of the matrix of a C -Symmetric operator T with respect to the basis $\{e_n\}$. Then

$$\begin{aligned}\alpha_{ij} &= \langle Te_j, e_i \rangle \\ &= \langle CT^*Ce_j, e_i \rangle = \langle CT^*e_j, Ce_i \rangle \\ &= \langle e_i, T^*e_j \rangle \\ &= \langle Te_i, e_j \rangle \\ &= \alpha_{ji}\end{aligned}$$

Thus T has a symmetric matrix representation.

Next, let $\mathcal{L}^2$ denote the Hilbert space defined on the unit circle $\mathbb{T}$ and $\mathcal{L}^\infty$ denote the Banach space of all the essentially bounded functions on $\mathbb{T}$. Then $\{e^{in\theta} : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{L}^2$. The space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{T}$ for which $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ is called the Hardy-Hilbert space and is denoted by H^2 . Also then $\{z^n : n = 0, 1, 2, \dots\}$ is an orthonormal basis for H^2 .

Let P be the orthogonal projection from $\mathcal{L}^2$ onto H^2 . Then, for a given $\phi \in \mathcal{L}^\infty$, the induced Toeplitz operator $T_\phi : H^2 \rightarrow H^2$ is defined as $T_\phi f = P(\phi f) \forall f \in H^2$. The class of Toeplitz operators is one of the most important classes of operators on Hardy spaces. In fact, due to their applications in prediction theory, boundary-value problems for analytic functions, singular integral equations and quantum mechanics, study of Toeplitz operators is very popular amongst operator theorists. So the motivation to investigate the conditions when a Toeplitz operator becomes complex symmetric is well justified. It is found that the study of complex symmetric Toeplitz operators helps to connect various problems in the field of quantum mechanics. This paper is organized as follows. The second section defines and studies basic properties of some conjugations on H^2 . In the third section we investigate the complex symmetry of normal operators, Volterra operators and some Toeplitz operators.

2. CONJUGATIONS

Consider the canonical conjugation $Jf(z) = \overline{f(\bar{z})}$ on H^2 .

Theorem 2.1. *Let C be any conjugation on H^2 then $TC = JT$ for some unitary operator $T : H^2 \rightarrow H^2$.*

Proof. As C is a conjugation on H^2 , we have $Ce_n = e_n \forall n \in \mathbb{N}$ for some basis $\{e_n\}$. Further, let $T : H^2 \rightarrow H^2$ be given by

$$T\left(\sum_{n=0}^{\infty} a_n e_n\right) = \sum_{n=0}^{\infty} a_n z^n,$$

then T is an isomorphism and $Te_n = z^n$ for all n .

So,

$$TT^*\left(\sum a_n z^n\right) = T\left(\sum a_n e_n\right) = \sum a_n z^n$$

and

$$T^*T(\sum a_n e_n) = T^*(\sum a_n z^n) = \sum a_n e_n.$$

Thus T is a Unitary operator.

Further, for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$,

$$\begin{aligned} Jf(z) &= \sum_{n=0}^{\infty} \overline{a_n z^n} \\ &= \sum_{n=0}^{\infty} \bar{a}_n z^n = \sum_{n=0}^{\infty} \bar{a}_n T e_n \\ &= T \sum_{n=0}^{\infty} \bar{a}_n e_n = T \sum_{n=0}^{\infty} \bar{a}_n C e_n \\ &= T \sum_{n=0}^{\infty} \bar{a}_n C T^* z^n = T C T^* \sum_{n=0}^{\infty} a_n z^n \\ &= T C T^* f(z) \end{aligned}$$

Thus, $J = T C T^*$ or $JT = TC$ as $T^*T = I$

□

Corollary 2.2. $J^*T = TC^*$

Proof. Let us consider the same unitary T as in Theorem 2.1.
Then ,

$$J = T C T^*$$

which gives

$$\begin{aligned} J^* &= (T C T^*)^* \\ &= T C^* T^* \end{aligned}$$

which implies

$$J^*T = T C^*$$

□

Theorem 2.3. Let C and J be any two conjugations defined on H^2 . Then

- (i) CJ is unitary
- (ii) CJ is C -Symmetric
- (iii) CJ is J -Symmetric

Proof. Since the operators C and J are conjugations on the space H^2 , $C^2 = I$ and $J^2 = I$

(i) Let $U = CJ$

Then

$$\begin{aligned}
 \langle f, U^*g \rangle &= \langle Uf, g \rangle \\
 &= \langle CJf, g \rangle \\
 &= \langle CJf, C^2g \rangle \\
 &= \langle Cg, Jf \rangle \\
 &= \langle f, JCg \rangle \text{ for all } f, g \text{ in } H^2
 \end{aligned}$$

Thus, $U^* = JC$.

Then ,

$$U^*U = JCCJ = JC^2J = J^2 = I$$

and

$$UU^* = UJC = CJJC = CJ^2C = C^2 = I$$

So, U is unitary.

(ii) Now,

$$CU = CCJ = J = JC^2 = JCC = U^*C$$

Thus,

$$CU = U^*C$$

So, U is C-Symmetric.

(iii) Again,

$$JU^* = JJC = C = CJ^2 = CJJ = UJ$$

Thus,

$$JU^* = UJ$$

So, U is J-Symmetric. □

Lemma 2.4. *If a Unitary operator $U : H^2 \rightarrow H^2$ is C-symmetric, then UC is a Conjugation.*

Theorem 2.5. *If $T : H^2 \rightarrow H^2$ is a linear transformation which is both J-symmetric and C-symmetric, then*

(i) T is CJC- symmetric.

(ii) T is JCJ- symmetric.

Proof. Let C and J be conjugations on H^2 .

Then from theorem 2.3 , $U = CJ$ is a unitary operator which is both J-symmetric and C-symmetric. Also from lemma 2.4 , UC is a conjugation. But $U = CJ$. Therefore CJC is a conjugation on H^2 . Further, T is both J-symmetric and C-symmetric. Hence $CT = T^*C$ and $JT = T^*J$. Therefore we get that,

(i)

$$\begin{aligned}
 (CJC)T &= CJT^*C \\
 &= CTJC \\
 &= T^*CJC \\
 &= T^*(CJC)
 \end{aligned}$$

(ii) Similarly,

$$\begin{aligned}
 (JCJ)T &= JCT^*J \\
 &= JTCJ \\
 &= T^*JCJ \\
 &= T^*(JCJ).
 \end{aligned}$$

□

Remark 2.6. From theorem 2.3, it is clear that if J is the canonical conjugation and C is any conjugation on H^2 , then the operator $U = CJ$ is unitary and C-symmetric as well as J-symmetric.

Then $UJ = CJJ = CJ^2 = C$ or that $C = UJ$. Thus any conjugation on H^2 can be expressed as the product of J-symmetric unitary operator and the canonical conjugation J . This also gives us a way to construct more conjugations with the help of the canonical conjugation J .

Further, it has been proved in [2] that each unitary operator on Hilbert space can be factored as $U = CJ$ where C and J are conjugations. Geometric interpretation of this would be that every planar rotation can be thought of as the product of two reflections.

3. SOME COMPLEX SYMMETRIC OPERATORS

In this section we try to find out which well known operators are complex symmetric and which are not. To understand this here are some examples

3.1. Normal Operators: Consider a Lebesgue space $L^2(\mu)$ of a planar, positive Borel measure μ with compact support. Let M_z denote the multiplication operator on $L^2(\mu)$, and C the complex Conjugation given by

$$Cf(z) = f(\bar{z})$$

Clearly $C^2 = I$. Further we have

$$\begin{aligned}
 CM_zf(z) &= C(zf(z)) \\
 &= \bar{z}f(\bar{z}) \\
 &= M_{\bar{z}}f(\bar{z}) \\
 &= M_z^*Cf(z)
 \end{aligned}$$

Thus $CM_z = M_z^*C$ or that $M_z = CM_z^*C$. Hence the multiplication operator M_z on $L^2(\mu)$ is C-symmetric. Moreover, as multiplication operators M_z are the orthogonal summands of any normal operator. Hence normal operators belong to the class of complex symmetric operators.

It may be noted here that in general subnormal operators are not C-symmetric [2], as on some part of their spectrum they have nonzero Fredholm index. Consequently the multiplication operator M_z on the usual Hardy space H^2 which is also the unilateral shift on H^2 , is not C-symmetric.

3.2. Volterra operators: Let V denote the Volterra operator

$$Vf(x) = \int_0^x f(t)dt \text{ on } L^2[0, 1]$$

and let the involution $Cf(t) = \overline{f(1-t)}$ be the conjugation operator.

Then it is well known that the adjoint of Volterra operator is given by

$$V^*f(x) = \int_x^1 f(t)dt .$$

Now

$$\begin{aligned} CV^*Cf(x) &= CV^*\overline{f(1-x)} \\ &= C \int_x^1 \overline{f(1-x)}dt \\ &= C \int_x^1 f(1-x)dt \\ &= - \int_0^x f(t)dt = \int_x^0 f(t)dt = Vf(x) \end{aligned}$$

Thus, V is C-symmetric.

3.3. Compression of a Toeplitz operator: Let $\phi \in L^\infty$. Then the Toeplitz operator with symbol ϕ is the operator $T_\phi : H^2 \rightarrow H^2$ given by $T_\phi f = (P\phi f)$ for all $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 . Then it is known that $T_\phi^* = T_{\bar{\phi}}$ for all $\phi \in L^\infty[1]$.

Further, let h be a non constant inner function. Then the compression of T_ϕ to H_h is the operator

$$T_\phi = P_h T_\phi P_h$$

We show that this compression of T_ϕ is C-symmetric with respect to the conjugation C on H^2 . For this, let $f, g \in H_h$. Then,

$$\begin{aligned} \langle CT_\phi f, g \rangle &= \langle Cg, T_\phi f \rangle \\ &= \langle Cg, P_h T_\phi P_h f \rangle = \langle P_h Cg, T_\phi f \rangle = \langle Cg, P(\phi f) \rangle \\ &= \langle PCg, \phi f \rangle = \langle Cg, \phi f \rangle = \langle \overline{g}z h, \phi f \rangle = \langle \overline{f}z h, \phi g \rangle \\ &= \langle Cf, \phi g \rangle = \langle PP_h Cf, \phi g \rangle = \langle P_h Cf, T_\phi f \rangle = \langle Cf, P_h T_\phi P_h g \rangle \\ &= \langle Cf, T_\phi g \rangle = \langle T_\phi^* Cf, g \rangle \end{aligned}$$

Thus, $CT_\phi = T_\phi^* C$.

This proves that T_ϕ is C-symmetric.

4. CONCLUSION

The complex symmetric operators have diverse applications. The multiplication operator M_z on $L^2(\mu)$ is C-symmetric with respect to some conjugation. All normal operators are also complex symmetric but in general subnormal operators are not so. However, the multiplication operator M_z on the usual Hardy space H^2 is

not C-symmetric. The well known Volterra operator on $L^2[0, 1]$ with the conjugation given by $Cf(t) = \overline{f(1-t)}$ is complex symmetric. In general, a Toeplitz operator may not be complex symmetric.

Acknowledgement. The author is grateful to the editor for the valuable suggestions that have improved the quality of this paper.

REFERENCES

1. Batra R. (2023). Weighted rationalized Toeplitz Hankel operators. *Annals of Mathematics and Computer Science*, 13 (2023) 12-24.
2. Brown A., Halmos P. R., (1963). Algebraic Properties of Toeplitz operators, *J. Reine Angew. Math.* 213 (1963/1964) 89-102. <https://doi.org/10.1515/crll.1964.213.89>
3. Fatehi M., Complex Symmetric weighted Composition Operators, *Complex Variables and elliptic equations*, 64 (2019), 710-720. <https://doi.org/10.1080/17476933.2018.1498087>
4. Garcia S.R., Putinar M., (2006), Complex Symmetric operators and applications. *Trans. Amer. Math. Soc.* 358 (2006) 1285-1315. <https://doi.org/10.1090/S0002-9947-05-03742-6>
5. Garcia S.R., Putinar M., (2007), Complex Symmetric operators and applications II. *Trans. Amer. Math. Soc.* 359 (2007) 3913-3931. <https://doi.org/10.1090/S0002-9947-07-04213-4>
6. Guo K.Y., Zhu S., A Canonical decomposition of Complex Symmetric operators, *J. Oper. Theory* 72 (2) (2014) 529-547. <https://doi.org/10.7900/jot.2013aug15>
7. Ko E., Lee J., On Complex Symmetric Toeplitz operators, *J. Math. Anal. Appl.* 434 (1) (2016) 20-34. <https://doi.org/10.1016/j.jmaa.2015.09.004>
8. Li R., Yang Y., Lu Y., A class of Complex Symmetric Toeplitz operators on Hardy and Bergman spaces, *J. Math. Anal. App.* 489 (2020) 124173. <https://doi.org/10.56947/amcs.v19.212>
9. Moraa P., Ambogo D., Nyamwala F., Numerical Range of Inner Product Type Integral Transformers on Hilbert Spaces, *Annals of Mathematics and Computer Science*, 19 (2023), 1-9. <https://doi.org/10.56947/amcs.v19.212>

¹ DEPARTMENT OF MATHEMATICS, MOTILAL NEHRU COLLEGE, UNIVERSITY OF DELHI, DELHI, INDIA.

Email address: ritu.kathuria@mln.du.ac.in