

EXISTENCE OF ENTROPY SOLUTIONS TO NONLINEAR DEGENERATE WEIGHTED ELLIPTIC $p(\cdot)$ -LAPLACIAN PROBLEMS AND L^1 -DATA

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ABSTRACT. Our aim in this paper is to study the existence result of entropy solution for a specific type of nonlinear degenerate weighted elliptic $p(\cdot)$ -Laplacian problems with Dirichlet-type boundary condition and L^1 data. In the framework of the theory of weighted Sobolev spaces with variable exponents, we use the regularization approach combined with a priori estimates.

1. INTRODUCTION AND PRELIMINARIES

Let $\Omega \subset \mathbb{R}^N$, ($N \geq 2$) be a open bounded domain with a connected Lipschitz boundary $\partial\Omega$, and $p(x) \in (1, \infty)$ for all $x \in (0, \infty)$. In this work we investigate the existence question of entropy solution for the nonlinear degenerate elliptic following problem with variable exponent:

$$\begin{cases} -\operatorname{div}(\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))) + \omega|u|^{p(u)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $-\operatorname{div}(\omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u)))$ is a Leray-Lions operator on $W_0^{1,p(\cdot)}(\Omega, \omega)$ ($1 < p(\cdot) \leq N$, the interesting and difficult cases are those of $1 < p \leq N$, since the variational methods of Leray-Lions (see [13]) can be easily applied for $p > N$) and $p(\cdot)$ is a continuous function defined on $\bar{\Omega}$ such that $p(x) > 1$ for all $x \in \bar{\Omega}$.

Additionally, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$, such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$) and the datum f is in L^1 . The absorption function θ is continuous and defined from $\mathbb{R}$ to $\mathbb{R}^N$ which satisfy suitable condition (see assumption (H_3) below).

Over the past few years, the study of PDEs and variational problems with variable exponent has received considerable attention in many models coming from various areas of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics and image processing, etc. The notion of entropy solutions was introduced by Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J.L. Vazquez in [7], this notion was then adapted by many authors to investigate

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some nonlinear elliptic and parabolic problems with a constant or variable exponent and Dirichlet or Neumann boundary conditions (see for example [1], [9], [10], [12], and [19]). For instances illustrating the applications of problem (1.1), we present the following two models:

- **Model 1. The thermal model** The analysis of induction heating processes entails the model of heat transfer, which is predominantly influenced by three factors: the dissipation of heat due to eddy currents within the work piece, the diffusion of heat within the material, and the exchange of heat at the interface between the work piece and the surrounding air.

$$\rho c \frac{\partial T}{\partial t} - \operatorname{div}(k \nabla T) = Q_{em},$$

In terms of temperature evolution within the work piece, it is governed by the classical heat transfer equation. Where ρ is density of the work piece, the specific heat c , and the thermal conductivity k are all temperature dependent parameters in this context. Additionally, there is a heat source term Q_{em} caused by eddy currents. It is worth noting that the heat capacity can undergo significant changes in value as the temperature varies, particularly during metallurgical phase transitions and in close proximity to the Curie temperature. On the other hand, the thermal conductivity exhibits a gradual decrease in value as the temperature increases.

- **Model 2. Filtration in a porous medium.** The filtration phenomena of fluids in porous media can be represented the following equation,

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)],$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.

Our problem in particular cases it have been considered by many authors, for example, when $\omega \equiv 1$, Sanchon and Urbano in [11] established the existence and uniqueness of entropy solutions for the $p(x)$ -Laplacian equation with L^1 data. In the case when $\theta \equiv 0$ and especially Chao Zhang in [10] study the questions of existence and uniqueness of entropy solutions to the problem (1.1). In this work and by using the regularization approach, we prove in the first step existence of a sequence of weak solutions to approximate problems (1.1), we apply here the variational method combined with a special type of operators (operator of type (M), see definition 1.11 below). In the second step, we will prove that the sequence of weak solutions converges to some function u and by using some a priori estimates, we will show that this function u is an entropy solution of nonlinear elliptic problem (1.1).

The plan of our paper is as follows: In Section 2, we give some preliminaries and notations. In Section 3, the existence of entropy solution of (1.1) is obtained. In the present section, we will introduce an adequate functional space, we give some definitions, notations and results which well be used in this work where problems of type (1.1) can be established.

Let $\overline{\Omega}$ be a bounded open domain in $\mathbb{R}^N$, we consider the following set

$$C^+(\overline{\Omega}) = \{p : \overline{\Omega} \rightarrow \mathbb{R}^+ \text{ continuous such that } 1 < p^- < p^+ < \infty\},$$

where $p^- = \min_{x \in \overline{\Omega}} (p(x))$ and $p^+ = \max_{x \in \overline{\Omega}} (p(x))$.

Let ω be a measurable positive and a.e finite function defined in $\mathbb{R}^N$. Moreover, in all this paper, we suppose that the following integrability conditions are satisfied:

$$(H_1) \quad \omega \in L^1_{Loc}(\Omega) \text{ and } \omega^{\frac{-1}{p(x)-1}} \in L^1_{Loc}(\Omega). \\ (H_2) \quad \omega^{-s(x)} \in L^1_{Loc}(\Omega), \text{ where } s(x) \in (\frac{N}{p(x)}, \infty) \cap (\frac{1}{p(x)-1}, \infty].$$

For $p(\cdot) \in C^+(\overline{\Omega})$, we define the weighted Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega, \omega)$ by

$$L^{p(\cdot)}(\Omega, \omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \int_{\Omega} |u|^{p(x)} \omega(x) dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega, \omega)} = \inf \left\{ \lambda > 0, \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \omega(x) dx \leq 1 \right\}.$$

We denote by $L^{p'(\cdot)}(\Omega, \omega)$ the conjugate space of $L^{p(\cdot)}(\Omega, \omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

and where

$$\omega^*(x) = \omega^{1-p'(x)} \text{ for all } x \in \Omega.$$

On the space $L^{p(\cdot)}(\Omega, \omega)$, we consider the function $\varrho_{p(\cdot), \omega} : L^{p(\cdot)}(\Omega, \omega) \rightarrow \mathbb{R}^+$ defined by

$$\varrho_{p(\cdot), \omega}(u) = \varrho_{L^{p(\cdot)}(\Omega, \omega)}(u) = \int_{\Omega} |u(x)|^{p(x)} \omega(x) dx.$$

The connection between $\varrho_{p(\cdot), \omega}$ and $\|\cdot\|_{L^{p(\cdot)}(\Omega, \omega)}$ is established by the next result in [19].

Proposition 1.1. *Let u be an element of $L^{p(\cdot)}(\Omega, \omega)$, and hypothesis (H_1) be satisfied, the following assertions holds:*

- i) $\|u\|_{p(\cdot), \omega} < 1$ (respectively $>, = 1$) $\Leftrightarrow \varrho_{p(\cdot), \omega}(u) < 1$ (respectively $>, = 1$).
- ii) If $\|u\|_{p(\cdot), \omega} < 1$ then $\|u\|_{p(\cdot), \omega}^{p^+} \leq \varrho_{p(\cdot), \omega}(u) \leq \|u\|_{p(\cdot), \omega}^{p^-}$.
- iii) If $\|u\|_{p(\cdot), \omega} > 1$ then $\|u\|_{p(\cdot), \omega}^{p^-} \leq \varrho_{p(\cdot), \omega}(u) \leq \|u\|_{p(\cdot), \omega}^{p^+}$.
- iv) $\|u\|_{p(\cdot), \omega} \rightarrow 0 \Leftrightarrow \varrho_{p(\cdot), \omega}(u) \rightarrow 0$ and $\|u\|_{p(\cdot), \omega} \rightarrow \infty \Leftrightarrow \varrho_{p(\cdot), \omega}(u) \rightarrow \infty$.

For other properties of $L^{p(\cdot)}(\Omega, \omega)$ spaces we refer to [20].

Proposition 1.2. *Let $u \in L^{p(\cdot)}(\Omega, \omega)$, $v \in L^{p'(\cdot)}(\Omega, \omega)$ and hypothesis (H_1) be satisfied, we have :*

$$\int_{\Omega} \omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot), \omega} \|v\|_{p'(\cdot), \omega} \leq 2 \|u\|_{p(\cdot), \omega} \|v\|_{p'(\cdot), \omega}.$$

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(\cdot)}(\Omega, \omega) = \{u \in L^{p(\cdot)}(\Omega, \omega) \text{ such that } \nabla u \in L^{p(\cdot)}(\Omega, \omega)\}.$$

With the norm

$$\|u\|_{1,p(\cdot),\omega} = \|u\|_{p(\cdot),\omega} + \|\nabla u\|_{p(\cdot),\omega} \text{ for all } u \in W^{1,p(\cdot)}(\Omega, \omega).$$

In the following of this paper, the space $W_0^{1,p(\cdot)}(\Omega, \omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega, \omega)$ with respect the norm $\|\cdot\|_{1,p(\cdot),\omega}$.

Let $p(\cdot), s(\cdot)$ are two element of space $C^+(\overline{\Omega})$ where the function $s(\cdot)$ satisfies the hypothesis (H_2) , we define the following function

$$\begin{aligned} p^*(x) &= \frac{Np(x)}{N-p(x)} \text{ for } p(x) < N, \\ p_s(x) &= \frac{p(x)s(x)}{1+s(x)} < p(x), \\ p_s^*(x) &= \begin{cases} \frac{p(x)s(x)}{(1+s(x))N-p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \leq p_s(x), \end{cases} \end{aligned}$$

for all most all $x \in \Omega$.

The following statement is known (see [8]).

Proposition 1.3. *Let $\Omega \subset \mathbb{R}^N$ a open set of $\mathbb{R}^N$, $p(\cdot) \in C^+(\overline{\Omega})$ and let hypothesis (H_1) be satisfied, we have*

$$L^{p(\cdot)}(\Omega, \omega) \hookrightarrow L_{Loc}^1(\Omega)$$

The following statement holds(see [8]).

Proposition 1.4. *Let hypotheses (H_1) , (H_2) be satisfied and $p(\cdot) \in C^+(\overline{\Omega})$, the space $(W^{1,p(\cdot)}(\Omega, \omega), \|\cdot\|_{1,p(\cdot),\omega})$ is a separable and reflexive Banach space.*

For the next statement we refer to [8].

Proposition 1.5. *Assume that hypotheses (H_1) and (H_2) holds and $p(\cdot), s(\cdot) \in C^+(\overline{\Omega})$, then we have the continuous embedding*

$$W^{1,p(\cdot)}(\Omega, \omega) \hookrightarrow W^{1,p_s(\cdot)}(\Omega, \omega)$$

Moreover, we have the compact embedding

$$W^{1,p(\cdot)}(\Omega, \omega) \hookrightarrow W^{1,r(\cdot)}(\Omega, \omega)$$

provided that $r \in C^+(\overline{\Omega})$, $1 \leq r < p^*$ for all $x \in \Omega$.

Definition 1.6. Given a constant $k > 0$, we define the cut function $T_k : \mathbb{R} \rightarrow \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| < k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω , we define the truncated function $T_k(u)$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$. By Ph. B  nilan, L. Boccardo, T. Gallouet [7] we define also the space

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega) =$$

$$\{u : \Omega \rightarrow \mathbb{R} ; u \text{ is a measurable function and } T_k(u) \in W_0^{1,p(\cdot)}(\Omega, \omega) \forall k > 0\}.$$

the weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega)$ is defined as

Proposition 1.7. *For every $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega)$, there exists a unique measurable function $v : \Omega \rightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = v \chi_{\{|u| < k\}}$, a.e. in Ω for every $k > 0$, where χ_E denotes the characteristic function of a measurable set E . Moreover, if u belongs to $W_0^{1,p(\cdot)}(\Omega, \omega)$, then v coincides with the standard distributional gradient of u , and we will denote it by $v = \nabla u$.*

The next lemma was proved in [2]

Lemma 1.8. *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p(\cdot) < \infty$, we have:*

$$\frac{1}{p(\cdot)} |\xi|^{p(\cdot)} - \frac{1}{p(\cdot)} |\eta|^{p(\cdot)} \leq |\xi|^{p(\cdot)-2} \xi \cdot (\xi - \eta),$$

where a dot symbol denote the Euclidean scalar product in $\mathbb{R}^N$.

Lemma 1.9. *For $a > 0, b > 0$ and $1 \leq p(\cdot), q < \infty$ we have*

$$(a + b)^{p(\cdot)} \leq 2^{p(\cdot)-1} (a^{p(\cdot)} + b^{p(\cdot)}).$$

$$a^q + b^q \leq (a + b)^q.$$

We give the following definition of a monotonic operator introduced in [12].

Definition 1.10. Let Y be a reflexive Banach space and let A be an operator from Y to its dual Y' . We say that A is *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0 \quad \text{for all } u, v \in Y.$$

We give the following definition of an operator of type (M) introduced in [12].

Definition 1.11. Let Y be a reflexive Banach space and let A be an operator from Y to its dual Y' . We say that A is of type (M) if and only if

$$\left. \begin{array}{l} u_n \rightharpoonup u \text{ weakly in } Y, \\ Au_n \rightharpoonup \chi \text{ weakly in } Y', \\ \limsup_{n \rightarrow +\infty} \langle Au_n, u_n \rangle \leq \langle \chi, u \rangle, \end{array} \right\} \text{Then } Au = \chi.$$

The next theorem was proved in [12].

Theorem 1.12. *Let Y be a reflexive real Banach space and $A : Y \rightarrow Y'$ be a bounded operator, hemi-continuous, coercive and monotone on space Y . Then the equation $Au = v$ has at least one solution $u \in Y$ for each $v \in Y'$.*

The following statement is well-known (see [21]).

Theorem 1.13. *Suppose that $\Omega \subset \mathbb{R}^N$ be a bounded set. Let $p(\cdot) \in C^+(\overline{\Omega})$. Then, for all $u \in W_0^{1,p(\cdot)}(\Omega, \omega)$, the inequality*

$$\|u\|_{p(\cdot), \omega} \leq C_0 \|\nabla u\|_{p(\cdot), \omega}$$

is satisfied where the constant C_0 depends on the exponent $p(\cdot)$, $\text{diam}(\Omega)$ and the dimension N .

Remark 1.14. Further down, C_i ; $i \in \{1; 2; \dots\}$ is a positive constant and $\text{meas}\{B\}$ denotes the measure of the measurable set $B \subset \mathbb{R}^N$.

2. MAIN RESULTS

In this section, we will introduce the concept of the entropy solution of our problem (1.1). Thus, we will add the following necessary assumptions for the rest of the proof:

(H_3) θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0) = 0$ and for all real numbers x, y we have $|\theta(x) - \theta(y)| < \lambda_0 |x - y|$ where λ_0 is a real constant such that $0 < \lambda_0 < \min \left\{ (p^-/2)^{1/p^-}; (p^-/2)^{1/p^+} \right\}$.

(H_4) $f \in L^1(\Omega)$.

In the present section, the proof of our main result is divided into two steps. The first goal, we use the regularization approach by applying Minty-Browder theorem to create a sequence of weak solutions of the approximate problem (1.1). In the second step, we give some a priori estimates which will be used to prove the existence of an entropy solution for problem (1.1). All the a priori estimates in this section rely exclusively on the proof established by Boccardo and Gallouet.

Definition 2.1. A function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega, \omega)$ is an entropy solution of degenerate elliptic problem (1.1) if and only if

$$\int_{\Omega} \omega \Phi(\nabla u - \theta(u)) \nabla T_k(u - \varphi) + \int_{\Omega} \omega |u|^{p(u)-2} u T_k(u - \varphi) \leq \int_{\Omega} f T_k(u - \varphi) \quad (2.1)$$

for all $k > 0$, $\varphi \in W_0^{1,p(\cdot)}(\Omega, \omega) \cap L^\infty(\Omega)$ and $\Phi(\xi) = |\xi|^{p(\xi)-2} \xi$, $\forall \xi \in \mathbb{R}^N$.

Our main result of this work is the following Theorem:

Theorem 2.2. *Let hypotheses (H_1), (H_2), (H_3) and (H_4) be satisfied, then the problem (1.1) has a entropy solution.*

Let the operator

$$A_n : W_0^{1,p(\cdot)}(\Omega, \omega) \longrightarrow (W_0^{1,p(\cdot)}(\Omega, \omega))'$$

where $(W_0^{1,p(\cdot)}(\Omega, \omega))'$ is the dual space of $W_0^{1,p(\cdot)}(\Omega, \omega)$ and let

$$A_n = A_n^1 + A_n^2 - L_n,$$

where for $u_n, v \in W_0^{1,p(\cdot)}(\Omega, \omega)$

$$\begin{aligned}\langle A_n^1 u_n, v \rangle &= \int_{\Omega} \omega \Phi(\nabla u_n - \theta(u_n)) \nabla v dx, \\ \langle A_n^2 u_n, v \rangle &= \int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) v dx, \\ \langle L_n, v \rangle &= \int_{\Omega} T_n(f) v dx.\end{aligned}$$

We will prove that A_n satisfies the assertions of Theorem (1.12). Firstly, we prove that A_n is of type (M) and bounded, for that, let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega, \omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly in } W_0^{1,p(\cdot)}(\Omega, \omega), \\ A_n u_k \rightharpoonup \chi \text{ weakly in } (W_0^{1,p(\cdot)}(\Omega, \omega))', \\ \limsup_{k \rightarrow +\infty} \langle A_n u_k, u_k \rangle \leq \langle \chi, u \rangle. \end{cases} \quad (2.2)$$

According to the equation (2.2) the sequence $(u_k)_{k \in \mathbb{N}}$ converges weakly to u in $W_0^{1,p(\cdot)}(\Omega, \omega)$. Then, there exists a subsequence denoted $(w_k)_{k \in \mathbb{N}}$ such that $w_k \rightarrow u$ in $L^{p(\cdot)}(\Omega, \omega)$. This implies that $(w_k)_{k \in \mathbb{N}}$ is a bounded sequence in $W_0^{1,p(\cdot)}(\Omega, \omega)$. Then by using (H_3) , we have

$$(|\nabla w_n - \theta(w_n)|^{p(w_n)-2} (\nabla w_n - \theta(w_n)))_{k \in \mathbb{N}} \text{ is bounded in } \left(L^{p'(\cdot)}(\Omega, \omega) \right)^N.$$

So, we get

$$|\nabla w_n - \theta(w_n)|^{p(w_n)-2} (\nabla w_n - \theta(w_n)) \rightharpoonup |\nabla u - \theta(u)|^{p(u)-2} (\nabla u - \theta(u))$$

in $\left(L^{p'(\cdot)}(\Omega, \omega) \right)^N$ as $k \rightarrow +\infty$.

Finally, using (H_3) and last results, we conclude that for all $v \in W_0^{1,p(\cdot)}(\Omega, \omega)$

$$\begin{aligned}\langle \chi, v \rangle &= \lim_{k \rightarrow +\infty} \langle A_n u_k, v \rangle, \\ &= \lim_{k \rightarrow +\infty} \langle A_n w_k, v \rangle, \\ &= \lim_{k \rightarrow +\infty} \left(\int_{\Omega} \omega \Phi(\nabla w_k - \theta(w_k)) \nabla v dx + \int_{\Omega} T_n(\omega |w_k|^{p(w_k)-2} w_k) v dx \right) \\ &\quad - \int_{\Omega} T_n(f) v dx, \\ &= \int_{\Omega} \omega \Phi(\nabla u - \theta(u)) \nabla v dx + \int_{\Omega} T_n(\omega |u|^{p(u)-2} u) v dx - \int_{\Omega} T_n(f) v dx, \\ &= \langle A_n u, v \rangle.\end{aligned}$$

Consequently $A_n u = \chi$. Therefore, the operator A_n is of type (M).

One hand, using proposition (1.1), lemma (1.9) and hypothesis (H_3) , we have

for any $u_n \in W_0^{1,p(\cdot)}(\Omega, \omega)$, $\varphi \in W_0^{1,p'(\cdot)}(\Omega, \omega)$

$$\begin{aligned}
|\langle A_n^1 u_n, \varphi \rangle| &\leq \int_{\Omega} \omega |\nabla u_n - \theta(u_n)|^{p(u_n)-1} |\nabla \varphi| dx \\
&\leq \int_{\Omega} 2^{p(u_n)-2} \omega (|\nabla u_n|^{p(u_n)-1} + |\theta(u_n)|^{p(u_n)-1}) |\nabla \varphi| dx \\
&\leq 2^{p^+-2} \int_{\Omega} \omega (|\nabla u_n|^{p(u_n)-1} + |\theta(u_n)|^{p(u_n)-1}) |\nabla \varphi| dx \\
&\leq 2^{p^+-2} \int_{\Omega} \omega (|\nabla u_n|^{p(u_n)-1} + \lambda_1^{p^*-1} |u_n|^{p(u_n)-1}) |\nabla \varphi| dx \\
&\leq 2^{p^+-1} \left(\|\nabla u_n\|_{p(\cdot),\omega}^{p_1-1} \|\nabla \varphi\|_{p'(\cdot),\omega} + \lambda_1^{p^*-1} \|u_n\|_{p(\cdot),\omega}^{p_1-1} \|\nabla \varphi\|_{p'(\cdot),\omega} \right) \\
&\leq 2^{p^+-1} (1 + \lambda_1^{p^*-1}) \|u_n\|_{1,p(\cdot),\omega}^{p_1-1} \|\nabla \varphi\|_{p'(\cdot),\omega} \\
&\leq C_1 \|u_n\|_{1,p(\cdot),\omega}^{p_1-1} \|\varphi\|_{1,p'(\cdot),\omega},
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1^{p^*} &= \max \left(\lambda_1^{p^-}; \lambda_1^{p^+} \right), C_1 = 2^{p^+-1} (1 + \lambda_1^{p^*-1}) \\
\|\nabla u\|_{p(\cdot),\omega}^{p_1-1} &= \max \{ \|\nabla u\|_{p(\cdot),\omega}^{p^+-1}; \|\nabla u\|_{p(\cdot),\omega}^{p^--1} \} \\
\|u\|_{p(\cdot),\omega}^{p_1-1} &= \max \{ \|u\|_{p(\cdot),\omega}^{p^+-1}; \|u\|_{p(\cdot),\omega}^{p^--1} \}.
\end{aligned}$$

This implies that A_n^1 is bounded.

In the other hand, using Holder's inequality, hypothesis (H_3) and Proposition (1.5), we get

$$\begin{aligned}
|\langle A_n^2 u_n, \varphi \rangle| &\leq \int_{\Omega} \omega |u_n|^{p(u_n)-1} |\varphi| dx \\
&\leq \|u_n\|_{p(\cdot),\omega}^{p^+-1} \|\varphi\|_{p'(\cdot),\omega} \\
&\leq C_2 C_3 \|u_n\|_{1,p(\cdot),\omega}^{p_1-1} \|\varphi\|_{1,p'(\cdot),\omega},
\end{aligned}$$

where C_2, C_3 are two constants of compact embedding given by Proposition (1.5).

This allows us to deduce that A_n^2 is bounded.

Finally, by Holder's inequality, we get immediately the boundedness of L_n .

Hence, A_n is bounded.

Secondly, we will be proved that A_n is coercive and hemi-continous operator.

Using Theorem (1.13) and hypothesis (H_4) , there exist a positive constant C_4 such that

$$\begin{aligned}
\int_{\Omega} T_n(f) u_n dx &\leq 2 \|f\|_1 \|u_n\|_{\infty}, \\
&\leq 2 C_0 \|f\|_1 \|\nabla u_n\|_{p(\cdot),\omega}, \\
&\leq C_4 \|u_n\|_{1,p(\cdot),\omega},
\end{aligned} \tag{2.3}$$

where

$$C_4 = 2 C_0 \|f\|_1$$

Now, using (2.3), lemma (1.8) and lemma (1.9), we obtain that

$$\begin{aligned}
\langle A_n u_n, u_n \rangle &= \int_{\Omega} \omega |\nabla u_n - \theta(u_n)|^{p(u_n)-2} (\nabla u_n - \theta(u_n)) \nabla u_n dx + \int_{\Omega} \omega |u_n|^{p(u_n)} dx \\
&\quad - \int_{\Omega} T_n(f) u_n dx, \\
&\geq \int_{\Omega} \frac{1}{p(u_n)} \omega |\nabla u_n - \theta(u_n)|^{p(u_n)} dx - \int_{\Omega} \frac{1}{p(u_n)} \omega |\theta(u_n)|^{p(u_n)} dx \\
&\quad + \int_{\Omega} \omega |u_n|^{p(u_n)} dx - C_4 \|u_n\|_{1,p(\cdot),\omega}, \\
&\geq \int_{\Omega} 1/2^{p^+-1} \frac{1}{p(u_n)} \omega |\nabla u_n|^{p(u_n)} dx + \int_{\Omega} \omega |u_n|^{p(u_n)} dx \\
&\quad - \int_{\Omega} \frac{2}{p(u_n)} \omega |\theta(u_n)|^{p(u_n)} dx - C_4 \|u_n\|_{1,p(\cdot),\omega}, \\
&\geq 1/p^+ 2^{p^+-1} \int_{\Omega} \omega |\nabla u_n|^{p(u_n)} dx + \int_{\Omega} \left(1 - 2\lambda_0^{p(u_n)}/p^-\right) \omega |u_n|^{p(u_n)} dx \\
&\quad - C_4 \|u_n\|_{1,p(\cdot),\omega}, \\
&\geq \min\left(1/p^+ 2^{p^+-1}; M_0\right) \left(\int_{\Omega} \omega |\nabla u_n|^{p(u_n)} dx + \int_{\Omega} \omega |u_n|^{p(u_n)} dx\right) \\
&\quad - C_4 \|u_n\|_{1,p(\cdot),\omega}, \\
&\geq \min\left(1/p^+ 2^{p^+-1}; M_0\right) \left(\|\nabla u_n\|_{p(\cdot),\omega}^{p_2} + \|u_n\|_{p(\cdot),\omega}^{p_2}\right) - C_4 \|u_n\|_{1,p(\cdot),\omega}, \\
&\geq 1/2^{p_2-1} \min\left(1/p^+ 2^{p^+-1}; M_0\right) \|u_n\|_{1,p(\cdot),\omega}^{p_2} - C_4 \|u_n\|_{1,p(\cdot),\omega}, \\
&\geq C_5 \|u_n\|_{1,p(\cdot),\omega}^{p_2} - C_4 \|u_n\|_{1,p(\cdot),\omega},
\end{aligned}$$

where

$$\begin{aligned}
\lambda_2^* &= \min(\lambda_0^{p^-}; \lambda_0^{p^+}), M_0 = \left(1 - 2\lambda_0^{p^*}/p^-\right) \\
C_5 &= 1/2^{p_2-1} \min\left(1/p^+ 2^{p^+-1}; M_0\right) \\
\|u\|_{p(\cdot),\omega}^{p_2} &= \min\{\|u\|_{p(\cdot),\omega}^{p^+}, \|u\|_{p(\cdot),\omega}^{p^-}\} \\
\|\nabla u\|_{p(\cdot),\omega}^{p_2} &= \min\{\|\nabla u\|_{p(\cdot),\omega}^{p^+}, \|\nabla u\|_{p(\cdot),\omega}^{p^-}\}
\end{aligned}$$

Then

$$\frac{\langle A_n u_n, u_n \rangle}{\|u_n\|_{1,p(\cdot),\omega}} \longrightarrow +\infty \quad \text{as} \quad \|u_n\|_{1,p(\cdot),\omega} \longrightarrow +\infty$$

This implies that A_n is coercive.

To prove that the operator A_n is hemi-continuous, it suffices to show that A_n^1 is hemi-continuous. For that, let $(u_k)_{k \in \mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega, \omega)$ and $u \in W_0^{1,p(\cdot)}(\Omega, \omega)$ such that $u_k \longrightarrow u$ strongly in $W_0^{1,p(\cdot)}(\Omega, \omega)$. This implies that $u_k \longrightarrow u$ is

a.e and $(u_k - u)_{k \in \mathbb{N}}$ is bounded. In addition, we have $\nabla u_k \rightarrow \nabla u$ strongly in $(L^{p(\cdot)}(\Omega, \omega))^N$. This implies that $\nabla u_k \rightarrow \nabla u$ is a.e and $(\nabla u_k - \nabla u)_{k \in \mathbb{N}}$ is bounded. Now, using (H_3) , we have $\theta(u_k) \rightarrow \theta(u)$ is a.e and $(\theta(u_k) - \theta(u))_{k \in \mathbb{N}}$ is bounded. Then, $\Phi(\nabla u_k - \theta(u_k)) \rightarrow \Phi(\nabla u - \theta(u))$ is a.e and $(\Phi(\nabla u_k - \nabla(u)))_{k \in \mathbb{N}}$ is bounded. This implies by using Lebesgue dominated convergence theorem that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \omega(\Phi(\nabla u_k - \theta(u_k)) - \Phi(\nabla u - \theta(u))) \varphi dx = 0,$$

for any $\varphi \in W_0^{1,p(\cdot)}(\Omega, \omega)$.

From where $A_n^1 u_k \rightarrow A_n^1 u$ in $(W_0^{1,p(\cdot)}(\Omega, \omega))'$. This implies that A_n^1 is hemi-continuous. Finally, we get A_n is hemi-continuous.

Consequently, using Theorem (1.12), there exists $u_n \in W_0^{1,p(\cdot)}(\Omega, \omega)$ such that

$$\int_{\Omega} \omega \Phi(\nabla u_n - \theta(u_n)) \nabla v dx + \int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) v dx = \int_{\Omega} T_n(f) v dx. \quad (2.4)$$

for all $v \in W_0^{1,p(\cdot)}(\Omega, \omega)$.

Lemma 2.3. *Let hypotheses (H_1) , (H_2) , (H_3) and (H_4) be satisfied, then $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ is bounded in $(L^{p(\cdot)}(\Omega, \omega))^N$.*

Proof. Remark that

$$\int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) T_k(u_n) dx \geq \int_{\Omega_k(n)} T_n(\omega |u_n|^{p(u_n)-2} u_n) u_n dx,$$

where

$$\Omega_k(n) = \{|u_n| \leq k\}.$$

Using the characterization of the cut function $T_n(\cdot)$, we have

$$\int_{\Omega_k(n)} T_n(\omega |u_n|^{p(u_n)-2} u_n) u_n dx \geq 0,$$

This implies that

$$\int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) T_k(u_n) dx \geq 0, \quad (2.5)$$

Now, taking $v = T_k(u_n)$ as a test function in 2.4 and by using 2.5, we have that

$$\int_{\Omega_k(n)} \omega \Phi(\nabla T_k(u_n) - \theta(u_n)) \nabla T_k(u_n) dx \leq k \|f\|_1, \quad (2.6)$$

Using (2.6), lemma (1.8), lemma (1.9) and (H_3) , we get that

$$\begin{aligned} M_1 \|T_k(u_n)\|_{1,p(\cdot),\omega}^{p_2} &\leq 1/p^+ \int_{\Omega_k(n)} \omega |\nabla T_k(u_n)|^{p(u_n)} dx \\ &+ \left(1 - \lambda_0^{p^*}/p^-\right) \int_{\Omega_k(n)} \omega |u_n|^{p(u_n)} dx \\ &\leq \int_{\Omega_k(n)} \omega \Phi(\nabla T_k(u_n) - \theta(u_n)) \nabla T_k(u_n) dx \\ &\leq k \|f\|_1. \end{aligned} \quad (2.7)$$

This implies that

$$\|T_k(u_n)\|_{1,p(\cdot),\omega} \leq C_6^{\frac{1}{p^-}}, \quad (2.8)$$

where

$$M_1 = \frac{\min\left(1/p^+, \left(1 - \lambda_0^{p^*}/p^-\right)\right)}{2^{p^2-1}}$$

$$C_6 = \left(2^{p^2-1}k/M_1\right) \|f\|_1$$

. Then, for any $k > 0$, $(T_k(u_n))_{n \in \mathbb{N}}$ is uniformly bounded in $W_0^{1,p(\cdot)}(\Omega, \omega)$. $\square$

Lemma 2.4. *Let hypotheses (H_1) , (H_2) , (H_3) and (H_4) be satisfied, the sequence $(u_n)_{n \in \mathbb{N}}$ converges in measure to some measurable function u .*

Proof. To prove this, we show that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in measure. Let $k > 0$ be large enough positive number. We take for $T_k(u_n)$ as a test function in (2.4), we obtain that

$$\begin{aligned} \min_{x \in \Omega} (|\omega(x)|) k^{p^-} \text{meas}(|u_n| \geq k) &\leq \int_{\Omega} |\omega| k^{p(u_n)} dx, \\ &\leq \int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) T_k(u_n) dx \\ &\leq \int_{\Omega} \omega \Phi(\nabla u_n - \theta(u_n)) \nabla T_k(u_n) dx \\ &+ \int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) T_k(u_n) dx \\ &\leq \int_{\Omega} T_n(f) T_k(u_n) dx \\ &\leq k \|f\|_1 \end{aligned}$$

this implies that

$$\text{meas}(|u_n| \geq k) \leq \frac{C_7}{k^{p^- - 1}}$$

where

$$C_7 = \frac{\|f\|_1}{\min_{x \in \Omega} (|\omega(x)|)}.$$

Therefore

$$\text{meas}(|u_n| > k) \rightarrow 0 \text{ as } k \rightarrow +\infty, \text{ uniformly with respect to } n. \quad (2.9)$$

Moreover, for every fixed $t > 0$, every real positive k , let use $n, m \in \mathbb{N}$ and $\varepsilon > 0$, we know that

$$\{|u_n - u_m| > t\} \subset \{|u_n| > t\} \cup \{|u_m| > t\} \cup \{|T(u_n) - T(u_m)| > t\}. \quad (2.10)$$

Since, $T_k(u_n)$ converges strongly in $L^{p(\cdot)}(\Omega, \omega)$, then it is a Cauchy sequence in $L^{p(\cdot)}(\Omega, \omega)$.

This implies by Markov inequality that

$$\begin{aligned} \text{meas}(|T_k(u_n) - T_k(u_m)| > t) &\leq \text{meas}(|T_k(u_n) - T_k(u_m)|^{p(u_n)} > t^{p(u_n)}), \\ &\leq \int_{\Omega} \frac{\omega |T_k(u_n) - T_k(u_m)|^{p(u_n)}}{k^{p(u_n)}} dx, \\ &\leq 1/k^{p^-} \int_{\Omega} \omega |T_k(u_n) - T_k(u_m)|^{p(u_n)} dx, \\ &\leq \frac{\varepsilon}{3}. \end{aligned} \quad (2.11)$$

Then, using (2.9) - (2.11), we get that

$$\begin{aligned} \text{meas}\{|u_n - u_m| > t\} &\leq \text{meas}\{|u_n| > t\} + \text{meas}\{|u_m| > t\} \\ &\quad + \text{meas}\{|T(u_n) - T(u_m)| > t\} \\ &\leq \varepsilon, \end{aligned} \quad (2.12)$$

for all $n, m \geq n_0(t, \varepsilon)$.

This proves that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in measure and then it converges almost everywhere to some measurable function u .

Therefore

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \text{ in } W_0^{1,p(\cdot)}(\Omega, \omega), \\ T_k(u_n) &\rightarrow T_k(u) \text{ in } L^{p(\cdot)}(\Omega, \omega), \text{ and a.e. in } \Omega. \end{aligned} \quad (2.13)$$

□

Lemma 2.5. *Let hypotheses $(H_1), (H_2), (H_3)$ and (H_4) be satisfied, the sequence $(\nabla u_n)_{n \in \mathbb{N}}$ converges in measure to ∇u .*

Proof. Let ε, t, k, μ are positive real numbers and let $n \in \mathbb{N}$, we have the following inclusion

$$\begin{aligned} \{|\nabla u_n - \nabla u| > t\} &\subset \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \\ &\cup \{|\nabla T_k(u)| > k\} \cup \{|u_n - u| > \mu\} \cup J, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} J &= \\ &\{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k, |\nabla T_k(u)| \leq k, |u_n - u| \leq \mu\}. \end{aligned}$$

Using the same method as for (2.9) and lemma (2.4), there exists an $n_1 \in \mathbb{N}$ and for k sufficiently large such that

$$\text{meas}(\{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\}) \leq \frac{\varepsilon}{4}, \quad (2.15)$$

and

$$\text{meas}(\{|u_n - u| > \mu\}) \leq \frac{\varepsilon}{4}, \quad (2.16)$$

for all $n \geq n_1$.

Remark by using (2.13) that we get

$$T_{k+\mu}(u_n) \rightharpoonup T_{k+\mu}(u) \text{ in } W_0^{1,p(\cdot)}(\Omega, \omega), \quad (2.17)$$

then, using (H_3) , we have that

$$\theta(T_{k+\mu}(u_n)) \rightarrow \theta(T_{k+\mu}(u)) \text{ in } (L^{p(\cdot)}(\Omega, \omega))^N, \quad (2.18)$$

and

$$\nabla T_\mu(T_{k+\mu}(u_n)) \rightarrow \nabla T_\mu(T_{k+\mu}(u)) \text{ in } (L^{p(\cdot)}(\Omega, \omega))^N. \quad (2.19)$$

Then, using (2.17) - (2.19), we get that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} \omega \Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u_n))) \nabla T_\mu(T_{k+\mu}(u_n) - T_k(u)) dx &= \\ \int_{\Omega} \omega \Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u))) \nabla T_\mu(T_{k+\mu}(u) - T_k(u)) dx &. \end{aligned} \quad (2.20)$$

Since the

$$\lim_{\mu \rightarrow 0} \nabla T_\mu(T_{k+\mu}(u) - T_k(u)) = 0. \quad (2.21)$$

Let $\mu < 1$, from (H_3) there exist a constant positive C_8 such that

$$\begin{aligned} &\Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u_n))) \nabla T_\mu(T_{k+\mu}(u) - T_k(u)) \leq \\ &C_8(|T_{k+1}(u)|^{p_1-1} + |\nabla T_k(u_n)|^{p_1-1}) |\nabla T_1(T_{k+1}(u) - T_k(u))| \in L^1(\Omega). \end{aligned} \quad (2.22)$$

Then, we get by using the Dominated Convergence Theorem and (2.21) that

$$\lim_{\mu \rightarrow 0} \int_{\Omega} \omega \Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u))) \nabla T_\mu(T_{k+\mu}(u) - T_k(u)) dx = 0. \quad (2.23)$$

Let δ be a strictly positive number such that $\delta < \frac{\varepsilon}{8}$, there exist $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$- \int_{\Omega} \omega \Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u_n))) \nabla T_\mu(T_{k+\mu}(u_n) - T_k(u)) dx \leq \frac{\delta}{2}. \quad (2.24)$$

Now, the application

$$\mathcal{B} : (s, \xi_1, \xi_2) \rightarrow \omega(\Phi(\xi_1 - \theta(s)) - \Phi(\xi_2 - \theta(s)))(\xi_1 - \xi_2). \quad (2.25)$$

is continuous and the set

$$\mathbf{L} := \{(s, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N : |s| \leq k, |\xi_1| \leq k, |\xi_2| \leq k, |\xi_1 - \xi_2| > t\} \quad (2.26)$$

is a compact.

Moreover, we have

$$\omega(\Phi(\xi_1 - \theta(s)) - \Phi(\xi_2 - \theta(s)))(\xi_1 - \xi_2) > 0 \text{ for all } \xi_1 \neq \xi_2 \quad (2.27)$$

Then, the application $\mathcal{B}$ attains its minimum on $\mathbf{L}$, we shall note it by β , we can prove that $\beta > 0$.

Then, using (2.4) and (2.24), we deduce that

$$\begin{aligned}
\text{meas}(J) &= 1/\beta \int_J \beta dx, \\
&\leq \int_{\Omega} \omega [\Phi(\nabla u_n - \theta(u_n)) - \Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u_n)))] \\
&\quad \nabla T_{\mu}(T_{k+\mu}(u_n) - T_k(u)) dx, \\
&\leq \int_{\Omega} \omega \Phi(\nabla u_n - \theta(u_n)) \nabla T_{\mu}(T_{k+\mu}(u_n) - T_k(u)) dx \\
&\quad - \int_{\Omega} \omega \Phi(\nabla T_k(u) - \theta(T_{k+\mu}(u_n))) \nabla T_{\mu}(T_{k+\mu}(u_n) - T_k(u)) dx, \\
&\leq \mu \|f\|_1 + \frac{\delta}{2}, \\
&\leq \mu C_9 + \frac{\delta}{2}, \\
&\leq \delta \\
&\leq \frac{\varepsilon}{4}
\end{aligned} \tag{2.28}$$

where

$$\mu < \delta/2C_9.$$

Eventually, using (2.14) - (2.16) and (2.28), we get that

$$\text{meas}\{|\nabla u_n - \nabla u| > t\} \leq \varepsilon. \tag{2.29}$$

This implies that the sequence $(\nabla u_n)_{n \in \mathbb{N}}$ converges in measure to ∇u .

Now, we will prove that limit function u is an entropy solution of problem (1.1).

Let $\varphi \in W_0^{1,p(\cdot)}(\Omega, \omega) \cap L^\infty(\Omega)$ and we take $v = T_k(u_n - \varphi)$ in equality (2.4), we get

$$\begin{aligned}
\int_{\Omega} \omega \Phi(\nabla u_n - \theta(u_n)) \nabla T_k(u_n - \varphi) dx &+ \int_{\Omega} T_n(\omega |u_n|^{p(u_n)-2} u_n) T_k(u_n - \varphi) dx \\
&= \int_{\Omega} T_n(f) T_k(u_n - \varphi) dx.
\end{aligned} \tag{2.30}$$

Let $\bar{k} = k + \|\varphi\|_\infty$ and $\Omega_{n,\bar{k}} = \{|T_{\bar{k}}(u_n) - \varphi| < k\}$, we obtain that

$$\begin{aligned}
& \int_{\Omega} \omega \Phi(\nabla u_n - \theta(u_n)) \nabla T_k(u_n - \varphi) dx \\
&= \int_{\Omega} \omega \Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_k(T_{\bar{k}}(u_n) - \varphi) dx, \\
&= \int_{\Omega} \omega \Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_{\bar{k}}(u_n) \chi_{\Omega_{n,\bar{k}}} dx \\
&\quad - \int_{\Omega} \omega \Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla \varphi \chi_{\Omega_{n,\bar{k}}} dx, \\
&= \int_{\Omega} \omega (\Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_{\bar{k}}(u_n) + 1/p^+ |\theta(T_{\bar{k}}(u_n))|^{p(u_n)}) \chi_{\Omega_{n,\bar{k}}} dx \\
&\quad - \int_{\Omega} \omega \Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla \varphi \chi_{\Omega_{n,\bar{k}}} dx,
\end{aligned} \tag{2.31}$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$.

This implies that

$$\begin{aligned}
& \int_{\Omega} \omega (\Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_{\bar{k}}(u_n) + 1/p^+ \omega |\theta(T_{\bar{k}}(u_n))|^{p(u_n)}) \chi_{\Omega_{n,\bar{k}}} dx \\
&\quad - \int_{\Omega} \omega \Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla \varphi \chi_{\Omega_{n,\bar{k}}} dx + \int_{\Omega} T_n(\omega g(u_n)) T_k(u_n - \varphi) dx \\
&\quad = \int_{\Omega} T_n(f) T_k(u_n - \varphi) dx + 1/p^+ \int_{\Omega} \omega |\theta(T_{\bar{k}}(u_n))|^{p(u_n)} \chi_{\Omega_{n,\bar{k}}} dx,
\end{aligned} \tag{2.32}$$

where

$$g(u_n) = |u_n|^{p(u_n)-2} u_n.$$

As the function $T_{\bar{k}}(u_n)$ is bounded in $W_0^{1,p(\cdot)}(\Omega, \omega)$, then by hypothesis (H_3) , $\theta(T_{\bar{k}}(u_n))$ is also bounded in $(L^{p(\cdot)}(\Omega, \omega))^N$.

This implies that $\Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n)))$ is bounded and weakly converges in $(L^{p(\cdot)}(\Omega, \omega))'$, where $(L^{p(\cdot)}(\Omega, \omega))'$ is the dual space of $L^{p(\cdot)}(\Omega, \omega)$.

However, we have

$$\begin{aligned}
u_n &\rightarrow u \text{ a.e in } \Omega, \\
\nabla u_n &\rightarrow \nabla u \text{ a.e in } \Omega.
\end{aligned} \tag{2.33}$$

Hence,

$$\begin{aligned}
\theta(T_{\bar{k}}(u_n)) &\rightarrow \theta(T_{\bar{k}}(u)) \text{ a.e in } \Omega, \\
\nabla T_{\bar{k}}(u_n) &\rightarrow \nabla T_{\bar{k}}(u) \text{ a.e in } \Omega.
\end{aligned} \tag{2.34}$$

This implies that

$$\Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \rightarrow \Phi(\nabla T_{\bar{k}}(u) - \theta(T_{\bar{k}}(u))) \text{ in } (L^{p(\cdot)}(\Omega, \omega))' \tag{2.35}$$

As,

$$\nabla \varphi \chi_{\Omega_{n,\bar{k}}} \text{ converges in } (L^{p(\cdot)}(\Omega, \omega))'. \tag{2.36}$$

Where

$$\Omega_{n,\bar{k}} = \{|T_{\bar{k}}(u)| \leq k\}.$$

Then, using (2.32) - (2.35), we obtain that

$$\int_{\Omega} \omega \Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_{\bar{k}}(u_n) \chi_{\Omega_{n,\bar{k}}} dx \rightarrow \int_{\Omega} \omega h_k(u) \chi_{\bar{k}} dx, \quad (2.37)$$

where,

$$h_k(u) = \Phi(\nabla T_{\bar{k}}(u) - \theta(T_{\bar{k}}(u))) \nabla T_{\bar{k}}(u).$$

Now, by using (H_3) and properties of the truncated function, there exist a positive constant C_9 such that

$$|\theta(T_{\bar{k}}(u_n))|^{p(u_n)} \leq (\bar{k}C_9)^{p^+}. \quad (2.38)$$

This implies by using (2.34), (2.38) and Dominated Convergence Theorem that

$$1/p^+ \int_{\Omega} \omega |\theta(T_{\bar{k}}(u_n))|^{p(u_n)} \chi_{\Omega_{n,\bar{k}}} dx \rightarrow 1/p^+ \int_{\Omega} \omega |\theta(T_{\bar{k}}(u))|^{p(u)} \chi_{\Omega_{\bar{k}}} dx \quad (2.39)$$

Now, by using lemma (1.8), we obtain that

$$\left(\Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_{\bar{k}}(u_n) + 1/p^+ |\theta(T_{\bar{k}}(u_n))|^{p(u_n)} \chi_{\Omega_{n,\bar{k}}} \right) \geq 0 \quad (2.40)$$

a.e in Ω

Therefore, by using (2.33), (2.38), (2.39) and Fatou's lemma, we have

$$\begin{aligned} & \int_{\Omega} \omega \left(\Phi(\nabla T_{\bar{k}}(u) - \theta(T_{\bar{k}}(u))) \nabla T_{\bar{k}}(u) + 1/p^+ |\theta(T_{\bar{k}}(u))|^{p(u)} \right) \chi_{\Omega_{\bar{k}}} dx \quad (2.41) \\ & \leq \liminf_{n \rightarrow \infty} \left(\Phi(\nabla T_{\bar{k}}(u_n) - \theta(T_{\bar{k}}(u_n))) \nabla T_{\bar{k}}(u_n) + 1/p^+ |\theta(T_{\bar{k}}(u_n))|^{p(u_n)} \right) \chi_{\Omega_{n,\bar{k}}} dx. \end{aligned}$$

Finally, taking limits as n goes to infinity in (2.32) and using the above results to conclude that u satisfies the entropy inequality (2.1). $\square$

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