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EQUAL-NORM PARSEVAL CONTINUOUS K-FRAMES IN HILBERT SPACES

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This paper is dedicated to Professor Samir Kabbaj

Abstract. In this paper, we present some fundamental results for the Parseval continuous $K-$frames in Hilbert space and we study some properties of an equal-norm of vectors. Furthermore, we show that a set of $K-$norm vectors can be extended to become a $K-$norm of $K-$frame.

1. Introduction and preliminaries

The concept of frame in Hilbert spaces has been introduced by Duffin and Schaffer [4] in 1952 to study some deep problems in nonharmonic Fourier series, after the fundamental paper [3] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. The majority of these applications requires frames in finite-dimensional spaces. For example, Jamali et al. [9] and Javanshiri et al. [10], were obtained results that are interesting in applications of frames. The continuous frames has been defined by Ali, Antoine and Gazeau [1], called these kinds of frames, frames associated with measurable space. The concept of continuous $K-$frame in Hilbert space have been introduced by Găvruta, in [6] for Hilbert spaces to study atomic systems with respect to a bounded linear operator. Due to the structure of $K-$frames, there are many differences between $K-$frames and standard frames. $K-$frames, which are a generalization of frames, allow us in a stable way, to reconstruct elements from the range of a bounded linear operator in a Hilbert space. For more, see [7, 8].

In this paper we try to give a generalization of the results given in [15] moving from the discrete case to the continuous case, we get some new results on the Parseval continuous $K-$frames.

Throughout this paper, assume that $(\mathfrak{A}, \mu)$ be a measure space with positive measure μ, $\mathcal{H}$, $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. $(\mathfrak{A}, \mu)$ is a σ-finite measures pace, $L(\mathcal{H}_0; \mathcal{H})$ is the set of all linear mappings of $\mathcal{H}_0$ to $\mathcal{H}$ and $B(\mathcal{H}_0; \mathcal{H})$ is the set of all bounded linear mappings. Instead of $B(\mathcal{H}; \mathcal{H})$, we simply write $B(\mathcal{H})$.

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Lemma 1.1. [2] *Let $\Gamma \in B(H,K)$ has a closed range, then there exists a unique operator $\tilde{\Gamma} \in B(K,H)$, called the pseudo-inverse of Γ, satisfying :*

$$\Gamma\tilde{\Gamma}\Gamma = \Gamma, \ (\Gamma\tilde{\Gamma})^* = \Gamma\tilde{\Gamma}$$

$$\tilde{\Gamma}\Gamma\tilde{\Gamma} = \tilde{\Gamma}, \quad \tilde{\Gamma}\Gamma = (\tilde{\Gamma}\Gamma)^*$$

$$(\tilde{\Gamma})^* = (\tilde{\Gamma^*}).$$

Lemma 1.2. [2] *Let $\Gamma \in B(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded operator with closed range $R(\Gamma)$. Then, there exists a bounded operator $\tilde{\Gamma} \in B(\mathcal{H}_1, \mathcal{H}_2)$, for which $\Gamma\tilde{\Gamma}h = h$, $h \in R(\Gamma)$.*

Lemma 1.3. [2] *Let $V \in B(\mathcal{H}_1, \mathcal{H}_2)$, then, the following assertions hold :*

1) $(\tilde{V^*}) = (\tilde{V})^*$.
2) *The orthogonal projection of $\mathcal{H}_2$ onto $R(V)$ is given by $V\tilde{V}$*
3) *The orthogonal projection of $\mathcal{H}_1$ onto $R(\tilde{V})$ is given by $\tilde{V}V$.*
4) $ker\tilde{V} = R^{\perp}(V)$ *and* $R(\tilde{V}) = (ker(V))^{\perp}$.
5) *On $R(V)$ we have $\tilde{V} = V^*(VV^*)^{-1}$.*

Definition 1.4. [13] Let $\mathcal{H}$ be a Hilbert space, and $(\mathfrak{A}, \mu)$ be a measure space. Then a map $F : \mathfrak{A} \to \mathcal{H}$ is called continuous frame in $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_{\mathfrak{A}} |\langle F_\varsigma, f\rangle|^2 d\mu(\varsigma) \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The elements A and B are called the continuous frame bounds.
If $A = B$, we call this a tight frame.
If $A = B = 1$, it is called a Parseval frame. If only the right hand inequality of 1.1 is satisfied, we call F a continuous Bessel map with bound B.

If F is a Bessel map, then $T_F : L^2(\mathfrak{A}, \mu) \to \mathcal{H}$, defined by $T_F(f) = \int_{\mathfrak{A}} \langle f, F(\varsigma)\rangle F(\varsigma) d\mu(\varsigma)$, is a bounded linear operator. T_F is surjective and bounded if and only if F is an integral frame. This operator is called the synthesis operator.

The adjoint of T_F, which is called the analysis operator, is defined by

$$T_F^* : \mathcal{H} \to L^2(\mathfrak{A}, \mu), \quad T_F^*(f)(\varsigma) = \langle f, F(\varsigma)\rangle, \ \varsigma \in \mathfrak{A}.$$

The continuous frame operator is defined to be $S_F = T_F T_F^*$, it is invertible and positive.

Recall that a Bessel map F is a frame if and only if there exists a continuous Bessel mapping G is a dual of F if for any $f, g \in \mathcal{H}$

$$\langle f, g\rangle = \int_{\mathfrak{A}} \langle f, G(\varsigma)\rangle \langle g, F(\varsigma)\rangle d\mu(\varsigma),$$

G is called a dual frame for F and $S_F^{-1}F$ is a dual of F.

Lemma 1.5. [15] *A set $\{F_\varsigma\}_\varsigma$ in $\mathcal{H}$ is a Parseval continuous frame if there exists $W \supseteq \mathcal{H}$ and an orthonormal basis $\{a_\varsigma\}$ for W so that the orthonormal projection P of W onto $\mathcal{H}$ satisfies $F_\varsigma = Pa_\varsigma$, we have for $f \in W$*

$$\int_{\mathfrak{A}} |\langle F_\varsigma, f\rangle|^2 d\mu = \int_{\mathfrak{A}} |\langle F_\varsigma, Pf\rangle|^2 d\mu = \int_{\mathfrak{A}} |\langle P(F_\varsigma), f\rangle|^2 d\mu.$$

Definition 1.6. [12] Let $K \in B(\mathcal{H}_0;\mathcal{H})$ and $F : \mathfrak{A} \to \mathcal{H}$ be weakly measurable. Then the map F is called a continuous K-frame with respect to $\mathcal{H}_0$, if there exist constants $A; B > 0$ such that for each $f \in \mathcal{H}$

$$A\|K^*f\|^2 \leq \int_{\mathfrak{A}} |\langle F_\varsigma, f\rangle|^2 d\mu(\varsigma) \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{1.2}$$

The constants A and B are called the lower and upper continuous $K-$frame bounds.
If $A = B$, F is called a tight continuous $K-$frame.
If $A = B = 1$, F is called the parseval continuous $K-$frame.
whenever for every $f \in \mathcal{H}$

$$\int_{\mathfrak{A}} |\langle F_\varsigma, f\rangle|^2 d\mu(\varsigma) = \|K^*f\|^2.$$

We say that $\{F_\varsigma\}_\varsigma$ is an equal norm $K-$frame for $\mathcal{H}$, if $\{F_\varsigma\}_\varsigma$ is a $K-$frame and $\|F_\varsigma\| = r$.
If $\|F_\varsigma\| = \|K\|$, we say that $\{F_\varsigma\}_\varsigma$ is a $K-$ norm-frame.

Theorem 1.7. [12] *Suppose that $F : \mathfrak{A} \to \mathcal{H}$ is a continuous $K-$frame for $\mathcal{H}$ and $Q \in B(\mathcal{H})$ with $R(Q) \subset R(K)$. Then $F$is a continuous $Q-$frame for $\mathcal{H}$.*

Lemma 1.8. [12] *Let $\mathcal{H}_0 \subset \mathcal{H}$. Let $F_\varsigma : \mathfrak{A} \to \mathcal{H}$ be weakly measurable, and $K \in B(\mathcal{H}_0, \mathcal{H})$. Then the following assertions are equivalent:*

1) *F is a continuous $K-$frame for $\mathcal{H}$ with respect to $\mathcal{H}_0$.*
2) *F is a continuous-Bessel mapping for $\mathcal{H}$ and there exists $Q \in B(\mathcal{H}_0, L^2(\mathfrak{A}))$ such that*

$$Kf = \int_{\mathfrak{A}} Q(f)F_\varsigma d\mu.$$

Theorem 1.9. [5] *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous $K-$frame of $\mathcal{H}$ with analysis operator T_F. Then F has a unique dual continuous K-Bessel sequence if and only if $R(T_F) = L^2(\mathfrak{A})$.*

2. Main results

Lemma 2.1. *Let $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a continuous $K-$frame for $\mathcal{H}$ and K has a closed range. If $S(R(K)) = R(K)$, then*

(i) *$\{S^{\frac{1}{2}} P_{(R(K))} F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a Parseval continuous frame for $R(K)$.*
(ii) *$\{S^{\frac{1}{2}} P_{(R(K))} F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a Parseval continuous $P_{(R(K))}$frame for $\mathcal{H}$.*

Proof. (i) The operator S is invertible and positive on $R(K)$, then, $S^{-1} |_{S(R(K))}$ has a unique positive root as $S^{\frac{1}{2}}$ and this operator commutes with S^{-1} and so

with S.
We have for $f \in R(K)$

$$\begin{aligned} f &= S^{-1}S \mid_{S(R(K))} f \\ &= S^{\frac{-1}{2}} S \mid_{S(R(K))} S^{\frac{-1}{2}} f \\ &= \int_{\mathfrak{A}} \langle S^{\frac{-1}{2}} f, P_{R(K)}F_{\varsigma}\rangle S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma} d\mu \\ &= \int_{\mathfrak{A}} \langle f, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma} d\mu. \end{aligned}$$

Therefore, $\|f\|^2 = \langle f, f\rangle = \int_{\mathfrak{A}} |\langle f, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle|^2 d\mu$.
(ii) Let $f \in \mathcal{H}$, such that $f = f_1 + f_2$, where $f_1 \in R(K)$ and $f_2 \in R(K)^{\perp}$.
Then, $\langle S^{\frac{-1}{2}} P_{R(K)}f_1, f_2\rangle = 0$ and

$$\begin{aligned} \int_{\mathfrak{A}} |\langle f, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle|^2 d\mu &= \int_{\mathfrak{A}} |\langle f_1, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle + \langle f_2, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle|^2 d\mu \\ &= \int_{\mathfrak{A}} |\langle f_1, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle|^2 d\mu + \int_{\mathfrak{A}} |\langle f_2, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle|^2 d\mu \\ &+ 2\int_{\mathfrak{A}} Re\langle f_1, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle\langle S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}, f_2\rangle d\mu \\ &= \int_{\mathfrak{A}} |\langle f_1, S^{\frac{-1}{2}} P_{R(K)}F_{\varsigma}\rangle|^2 d\mu. \end{aligned}$$

Then, we obtain $\int_{\mathfrak{A}} |\langle f, S^{\frac{-1}{2}} P_{R(K)}F\varsigma\rangle|^2 d\mu = \|P_{R(K)}f\|^2$. $\square$

Proposition 2.2. *Let K be a bounded operator on $\mathcal{H}$ with closed range, and let $F = \{F_{\varsigma}\}$ be a continuous K-frame with A and B bounds, respectively. Then, $\{K^*S_F^{-1}P_{S_F(R(K))}F_{\varsigma}\}$ is a K-dual of $P_{R(K)}F$ with B^{-1} and $A^{-1}\|K\|^2\|\tilde{K}\|^2$ bounds, respectively.*

Proof. Note that S_F is a bounded operator. The mapping $S_F : R(K) \to S_F(R(K))$ is invertible, and $\{K^*S_F^{-1}P_{S_F(R(K))}F_{\varsigma}\}$ is a Bessel sequence.
We have $S_F = T_F T_F^*$ is self-adjoint on $\mathcal{H}$ and $S_F^{-1}S_F \mid R(K) = I_{R(K)}$.

$$\begin{aligned} Kf &= (S_F^{-1}S_F)^*Kf = S_F^*(S_F^{-1})^*Kf \\ &= S_F^* P_{S_F(R(K))}(S_F^{-1})^*Kf \\ &= \int_{\mathfrak{A}} \langle P_{S_F(R(K))}(S_F^{-1})^*Kf, F_{\varsigma}\rangle P_{R(K)}F_{\varsigma} d\mu \\ &= \int_{\mathfrak{A}} \langle f, K^*S_F^{-1}P_{S_F(R(K))}F_{\varsigma}\rangle P_{R(K)}F_{\varsigma} d\mu. \end{aligned}$$

Therefore, $\{K^*S_F^{-1}PS_F(R(K))F_{\varsigma}\}_{\varsigma}$ is a $K-$dual of $P(R(K))F_{\varsigma}$ with the lower bound B^{-1}.
We have $\|(S_F^{-1})^*f\|^2 = \langle S_F^{-1}(S_F^{-1})^*f, f\rangle \leq A^{-1}\|\tilde{K}\|^2\|(S_F^{-1})^*f\|\|f\|$.

Then, for $\forall f \in \mathcal{H}$

$$\begin{aligned}\int_{\mathfrak{A}} |\langle f, K^* S_F^{-1} P_{S_F(R(K))} F_\varsigma\rangle|^2 d\mu &= \int_{\mathfrak{A}} |\langle (S_F^{-1})^* K f, F_\varsigma\rangle|^2 d\mu \\ &= \langle S_F (S_F^{-1})^* K f, (S_F^{-1})^* K f\rangle \\ &= \langle K f, (S_F^{-1})^* K f\rangle \\ &\leq A^{-1} \|K\|^2 \|\tilde{K}\|^2 \|f\|^2.\end{aligned}$$

□

Theorem 2.3. *Let $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a continuous $K-$frame. Then $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a continuous $K-$dual of $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ if and only if $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}} = \{V\sigma_\varsigma\}_\varsigma$ where $\{\sigma_\varsigma\}$ is the standard orthonormal basis of $L^2(\mathfrak{A})$ and $V : L^2(\mathfrak{A}) \to \mathcal{H}$ is a bounded operator such that $T_F V^* = K$. In this case, $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a Parseval continuous $V-$frame.*

Proof. Suppose that $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a continuous $K-$dual of $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ with the synthesis operator T_G. If $V = T_G$, then $T_F V^* = K$. On the other hand, if $V \in B(L^2(\mathfrak{A}), \mathcal{H})$ with $T_F V^* = K$ and $G_\varsigma = V\sigma_\varsigma$. Then, $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a Bessel sequence and we have for $\forall f \in \mathcal{H}$

$$\int_{\mathfrak{A}} \langle f, G_\varsigma\rangle F_\varsigma d\mu = \int_{\mathfrak{A}} \langle V^* f, \sigma_\varsigma\rangle F_\varsigma d\mu = T_F V^* f = K f.$$

Then, $\{G_\varsigma\}$ is a continuous $K-$dual of $\{F_\varsigma\}$ and $\{G_\varsigma\}$ is a continuous K^*-frame.

$$\int_{\mathfrak{A}} |\langle f, G_\varsigma\rangle|^2 d\mu = \int_{\mathfrak{A}} |\langle V^* f, \sigma_\varsigma\rangle|^2 d\mu = \|V^* f\|^2 \quad f \in \mathcal{H}.$$

Then, $\{G_\varsigma\}$ is a Parseval continuous $V-$frame. □

Corollary 2.4. *Let $K \in B(\mathcal{H})$ be a closed range operator and $\{F_\varsigma\}$ a continuous $K-$frame for $\mathcal{H}$. Then $\{D_\varsigma\}$ satisfies for $f \in R(K)$, $f = \int_{\mathfrak{A}} \langle f, D_\varsigma\rangle F_\varsigma d\mu$ if and only if $\{D_\varsigma\}_\varsigma = \{V\sigma_\varsigma\}_\varsigma$, where $\{\sigma_\varsigma\}$ is the standard orthonormal basis of $L^2(\mathfrak{A})$ and $V : L^2(\mathfrak{A}) \to R(K)$ is a left inverse of $T_F^* \mid R(K)$. In this case, $\{D_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a Parseval continuous $V-$frame.*

Proposition 2.5. *Let $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a Parseval continuous $K-$frame for $\mathcal{H}$ and $\{\alpha_i\}_i, 1 \leq i \leq n$ be the eigenvalues for the frame operator S, where $n = dim\mathcal{H}$. Then, $\alpha_i > 0$ for each $i = 1, 2, 3, .., n$ and $n\|K^* e_i\|^2 = \int_{\mathfrak{A}} \|F_\varsigma\|^2 d\mu$, where $\{e_i\}_{1<i<n}$ is an orthonormal basis for $\mathcal{H}$.*

Proof. We note that $\sum_{i=1}^n |\langle F_\varsigma, \alpha_i\rangle|^2 = \|F_\varsigma\|^2$. Suppose that $\{\alpha_i\}_i$ are the eigenvalues of S and $Sf = \alpha_i f$, for $f \in \mathcal{H}$. Since $S = KK^*$, therefore $\|K^* f\|^2 = \alpha_i \|f\|^2$ and we conclude that $\alpha_i > 0$ for each $i = 1, 2, ..., n$ and we obtain

$$\frac{n\|K^* f\|^2}{\|f\|^2} = \int_{\mathfrak{A}} \|F_\varsigma\|^2 d\mu.$$

□

Theorem 2.6. *If $F = \{f_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a set of $K-$norm vectors in $\mathcal{H}$, then there is a $K-$norm frame for $\mathcal{H}$ that contains the set F.*

Proof. Let $1 \leq i \leq n$ and $\{a_{\varsigma i}\}_{\varsigma\in\mathfrak{A}}$ be an orthonormal basis for $\mathcal{H}$ that contain the vector v_i where $v_i := \dfrac{f_i}{\|K\|}$. For $\forall f \in \mathcal{H}$ we have

$$n\|K^* f\|^2 \leq n\|K^*\|^2\|f\|^2 \leq \sum_{i=1}^{n}\int_{\mathfrak{A}} |\langle f, \|K\|a_{\varsigma i}\rangle|^2 d\mu.$$

Therefore, $\{\|K\|a_{\varsigma i}\}_{i,\varsigma}$, is a continuous $K-$frame for $\mathcal{H}$ with bounds n and $n\|K\|^2$. □

Corollary 2.7. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a set of continuous $K-$norm vectors in $\mathcal{H}$ and $\|K\|\|\tilde{K}\| = 1$ such that K has closed range, then there is a $K-$norm-tight frame for $R(K)$ that contains the set F.*

Proof. Suppose that $f \in R(K)$, such that $\|K\|\|f\| = \|K\|\|\tilde{K}^*K^*f\| \leq \|K^*F\|$ then, we obtain $\|K^*f\| = \|K\|\|f\|$.
By Theorem 2.6, if $\{a_{\varsigma i}\}_{\varsigma\in\mathfrak{A}}$ be an orthonormal basis for $R(\mathcal{H})$, then $\{\|K\|a_{\varsigma i}\}_{\varsigma\in\mathfrak{A}}$ is a continuous $K-$norm tight frame for $R(K)$.

□

Theorem 2.8. *Let $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a Parseval $K-$frame for $\mathcal{H}$ and K be closed range. Then, there is a larger Hilbert space $W \supseteq R(\tilde{K})$ and an orthonormal basis $\{u_\varsigma\}_{\varsigma\in\mathfrak{A}}$ for W so the orthonormal projection P of W onto $R(\tilde{K})$ satisfies $f_\varsigma = KPu_\varsigma$ for each $\varsigma \in \mathfrak{A}$.*

Proof. Suppose that $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a Parseval $K-$frame for $\mathcal{H}$. By Lemma 1.8, there exists an orthonormal basis $\{v_\varsigma\}_\varsigma$ such that $F_\varsigma = Tv_\varsigma$, $\forall\varsigma \in \mathfrak{A}$.
Since $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a Parseval continuous $K-$frame, we have $KK^* = TT^*$.
By lemma 1.8, we obtain $R(T) = R(K)$. Then, $F_\varsigma \in R(\tilde{K})$.
If $f \in R(\tilde{K})$, by Lemma 1.3, we have

$$\int_{\mathfrak{A}} |\langle f, \tilde{K}F_\varsigma\rangle|^2 d\mu = \int_{\mathfrak{A}} |\langle \tilde{K}^* f, F_\varsigma\rangle|^2 d\mu = \|K^*\tilde{K}^*f\|^2 = \|f\|^2.$$

Hence, $\{\tilde{K}F_\varsigma\}_\varsigma$ is a Parseval continuous frame for $R(\tilde{K})$. By Lemma 1.5, there exists space $W \supseteq R(\tilde{K})$ and an orthonormal basis $\{u_\varsigma\}_\varsigma$ for W, then $\tilde{K}F_\varsigma = Pu_\varsigma$, $\forall\varsigma \in \mathfrak{A}$ or $F_\varsigma = KPu_\varsigma$. This completes the proof. □

We give a general case for continuous $K-$frames in finite dimensional Hilbert spaces.

Theorem 2.9. *Let K be closed range and P be an orthonormal projection on $\mathcal{H}$ onto $R(\tilde{K})$ such that KP be a rank-s.*
Define

$$\mathcal{C} = \{\textit{all Parseval K-frames for } KP\mathcal{H}\}$$

$$\mathcal{B} = \{V \prec \mathcal{H}, dimV = s\}.$$

Then, there exists a natural one to one correspondence between $\mathcal{C}$ and $\mathcal{B}$.

Proof. Let $\mathcal{H}$ be an $n-$dimensional Hilbert space and $F = \{F_i\} \in \mathcal{C}$, there exists an orthonormal basis $\{\alpha_i\}$ such that $F_i = KP\alpha_i$ for $1 \leq i \leq n$. Define a unitary operator U on $\mathcal{H}$ by $Ue_i = \alpha_i$. Hence, we have $F_i = KPU_F e_i$, which is unitarily equivalent to $\tilde{F}_i = U^* KPUe_i$.
Define

$$\phi : \mathcal{C} \to \mathcal{B}$$
$$\phi(F_i) = U_F^* KPU_F \mathcal{H}.$$

Assume that $V \in \mathcal{B}$, there exists a unitary operator U_F on $\mathcal{H}$ such that $U_F V = KP\mathcal{H}$. Then , we obtain $U_F^* KPU_F \mathcal{H} = V$ and $\{KPe_i\}_i \in \mathcal{C}$, this implies that ϕ is surjective.
Suppose that $G = \{G_i\}_{1 \leq i \leq n} \in \mathcal{C}$ and U_G is a unitary operator on $\mathcal{H}$ such that $U_F^* KPU_F \mathcal{H} = U_G^* KPU_G \mathcal{H}$ which gives $Ue_i = \beta_i$, and $G_i = KP\beta_i = KPU_G e_i$. We have $U_F^* KPU_F e_i = U_G^* KPU_G e_i$ for each $1 \leq i \leq n$, then, $U_F^* F_i = U_G^* G_i$. Hence, F and G are two identical continuous $K-$ Parseval frames, then ϕ is injective.

□

Theorem 2.10. *Let $\{F_\varsigma\}_{\varsigma \in \mathfrak{A}}$ be a Parseval continuous $K-$frame for $\mathcal{H}$ with the synthesis operator T_F and $\{G_\varsigma\}_{\varsigma \in \mathfrak{A}}$ be a Parseval continuous K^*-frame for $\mathcal{H}$ with the synthesis operator T_G. If $\{G_\varsigma\}_{\varsigma \in \mathfrak{A}}$ is a $K-$dual for $\{F_\varsigma\}_{\varsigma \in \mathfrak{A}}$, then we have*
$\|(T_F^* - T_G^*)f\|^2 = \|KK^* f\|^2 - \|K^* f\|^2, \forall f \in \mathcal{H}.$
Moreover, if K has closed range, then we can obtain the following inequality:

$$\|(KK^*)^{-1}\|^2 (1 - \|\tilde{K}\|^2) \leq \|T_F - KT_G\|^2 \leq \|K\|^2 (\|K\|^2 - 1).$$

Proof. For $f, g \in \mathcal{H}$, we have $T_F T_G^* = K$ and $T_F T_F^* = KK^*$, then $T_F(T_F^* - T_G^* K^*) = 0$ and

$$\langle T_F^* g, (T_F^* - T_G^* K^*) f \rangle = \langle g, T_F(T_F^* - T_G^* K^*) f \rangle = 0$$

We have

$$\begin{aligned}
\|T_G^* K^* f\|^2 &= \|T_F^* f - T_F^* f + T_G^* K^* f\|^2 \\
&= \|T_F^* f - (T_F^* - T_G^* K^*) f\|^2 \\
&= \|T_F^* f\|^2 + \|(T_F^* - T_G^* K^*) f\|^2 - \langle T_F^* f, (T_F^* - T_G^* K^*) f \rangle - \overline{\langle T_F^* f, (T_F^* - T_G^* K^*) f \rangle} \\
&= \|T_F^* f\|^2 + \|(T_F^* - T_G^* K^*) f\|^2.
\end{aligned}$$

We have $\|T_F^* f\|^2 = \|K^* f\|^2$ and $\|T_G^* f\|^2 = \|K^* f\|^2$.
Then,

$$\|f\|^2 \leq \|(KK^*)^{-1}\|^2 \|KK^* f\|^2.$$

Moreover,

$$\|K^* f\|^2 = \|\tilde{K} KK^* f\|^2 \leq \|\tilde{K}\|^2 \|KK^* f\|^2.$$

Hence, we deduce that

$$\|T_F - KT_G\|^2 = \|T_F^* - T_G^* K^*\|^2 \geq \frac{\|(T_F^* - T_G^* K^*) f\|^2}{\|f\|^2} \geq \|(KK^*)^{-1}\|^2 (1 - \|\tilde{K}\|^2).$$

□

In the next result, we define $\tilde{K}$ as a left inverse of K and $\{\beta_\varsigma\}_\varsigma$ is the orthonormal basis for $L^2(\mathfrak{A})$.

Theorem 2.11. *Let $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be an equal -norm-Parseval continuous $K-$frame for the finite Hilbert space $\mathcal{H}$ with the synthesis operator T such that $T^*\mathcal{H} \perp V^*\mathcal{H}$ and $\tilde{K}^*F_\varsigma \perp V\beta_\varsigma$, where $V \in B(L^2(\mathfrak{A}), \mathcal{H})$ and $\tilde{K}$ are two isometries. Then, $\{F_\varsigma\}$ has infinitely many dual equal-norm continuous $K-$frames.*

Proof. Choose $Z = \lambda V^*$, where $\lambda \neq 0$. we have $(\tilde{K}T + V)T^* = K^*$, if $G_\varsigma = (\tilde{K}T + Z^*)\beta_\varsigma$ then, $\{G_\varsigma\}$ is a continuous dual $K-$frame of $\{F_\varsigma\}_\varsigma$.
Let $P : L^2(\mathfrak{A}) \to T^*\mathcal{H}$ is an orthonormal projection so that $P\beta_\varsigma = T^*F_\varsigma$ and $\{e_1, ..., e_n\}$ be an orthonormal basis for $\mathcal{H}$. Since, $T\beta_\varsigma = F_\varsigma$, then $G_\varsigma = \tilde{K}F_\varsigma + Z^*(I-P)\beta_\varsigma$. We can easily to check that $\langle \tilde{K}F_\varsigma, Z(I-P)\beta_\varsigma\rangle = 0$.
From Proposition 2.7, we obtain

$$\|F_\varsigma\|^2 = n\|K^*e_j\|^2$$

for $j = 1, 2, ..., n$, and

$$\|P\beta_\varsigma\|^2 = \|T^*F_\varsigma\|^2 = \|K^*F_\varsigma\|^2 = n\|K^*e_j\|^2\delta_j,$$

where δ_j is the eigenvalue of the frame operator S_F.
We have

$$\langle \beta_\varsigma, P\beta_\varsigma\rangle = \langle \beta_\varsigma, P^2\beta_\varsigma\rangle = \langle P\beta_\varsigma, P\beta_\varsigma\rangle = \|P\beta_\varsigma\|^2$$

and

$$\|(I-P)\beta_\varsigma\|^2 = \langle \beta_\varsigma - P\beta_\varsigma, \beta_\varsigma - P\beta_\varsigma\rangle = 1 - \|P\beta_\varsigma\|^2.$$

We conclude that

$$\begin{aligned}
\|G_\varsigma\| &= \langle \tilde{K}F_\varsigma + Z(I-P)\beta_\varsigma\rangle \\
&= \|\tilde{K}F_\varsigma\|^2 + \|Z(I-P)\beta_\varsigma\|^2 \\
&= \|F_\varsigma\|^2 + \lambda^2\|(I-P)\beta_\varsigma\|^2 \\
&= n\|K^*e_j\|^2 + \lambda(1 - n\|K^*e_j\|^2)\delta_j \\
&= \lambda^2 + n(1-\lambda^2\delta_j)\|K^*e_j\|^2.
\end{aligned}$$

□

Declarations

Availablity of data and materials
Not applicable.

Competing interest
The authors declare that they have no competing interests.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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