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DIFFERENTIAL INEQUALITIES AND MEROMORPHIC STARLIKE FUNCTIONS

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Abstract. In the present paper, authors investigate a differential inequality involving certain differential operator. As consequences, sufficient conditions for starlikeness of meromorphic functions of certain order are obtained.

1. Introduction and preliminaries

Let $\Sigma_{p,n}$ denote the class of functions of the form

$$f(z)=\frac{a_{-1}}{z^p}+\sum_{k=n}^{\infty}a_{k-p}z^{k-p}\ (p,n\in\mathbb{N}=\{1,2,3,\ldots\}),$$

which are analytic and p-valent in the punctured unit disc $\mathbb{E}_0=\mathbb{E}\setminus\{0\}$, where $\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$. Define

$$\begin{aligned}D^0f(z)&=f(z),\\ D^1f(z)&=\frac{a_{-1}}{z^p}+2a_0+3a_1z+4a_2z^2+\ldots\\ &=\frac{(z^2f(z))'}{z},\\ D^2f(z)&=D^1(D^1f(z)),\end{aligned}$$

and for $n=1,2,3,\ldots$

$$D^nf(z)=D^1(D^{n-1}f(z))=\frac{(z^2D^{n-1}f(z))'}{z}.$$

A function $f\in\Sigma_{p,n}$ is said to be p-valent meromorphic starlike of order α if

$$-\Re\frac{1}{p}\left(\frac{D^{n+1}f(z)}{D^nf(z)}-(p+1)\right)>\alpha,\qquad(\alpha<1;z\in\mathbb{E}).$$

We denote by $\mathcal{MS}_n^*(p,\alpha)$, the class of all such functions. The class of meromorphic starlike functions of order α is denoted by $\mathcal{MS}^*(\alpha)$ and is defined as:

$$\mathcal{MS}^*(\alpha)=\left\{f\in\Sigma:-\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\qquad(0\le\alpha<1;z\in\mathbb{E})\right\},$$

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Note that $\Sigma_{1,n} = \Sigma_n$, and $\mathcal{MS}^*(\alpha) = \mathcal{MS}_0^*(1,\alpha)$.
A number of sufficient conditions for meromorphic starlike functions have been obtained in terms of differential inequalities in the literature of meromorphic functions. The study of such results is a source of motivation for us to produce the present paper. We refer to [1, 2, 3, 5] for the related work on the subject of meromorphic functions. In the present paper, we investigate the differential inequality

$$\Re\left(\frac{(1-\alpha)D^{n+1}[f](z)+\alpha D^{n+2}[f](z)}{(1-\beta)D^{n+1}[f](z)+\beta D^{n+1}[f](z)}\right) > L(\alpha,\beta,\mu),$$

where, α and β are real numbers and $L(\alpha,\beta,\mu)$ is a certain real number given in Section 2 for meomorphic starlikeness of $f \in \Sigma_n$. As particular cases , we derive certain sufficient conditions for meromorphic starlike functions.

We shall need the following lemma of Miller and Mocanu [4] to prove our main result.

Lemma 1.1. *Let Ω be a subset of $\mathbb{C}^2 \times \mathbb{E}$ ($\mathbb{C}$ is the complex plane) and let $\psi : \Omega \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ψ satisfy the following conditions:*
(i) $\psi(u,v)$ is continuous in Ω;
(ii) $(1,0) \in \Omega$ and $\Re\psi(1,0) > 0$; and
(iii) $\Re\,\{\psi(iu_2,v_1)\} \le 0$ for all $(iu_2,v_1) \in \Omega$ such that $v_1 \le -(1+u_2^2)/2$.
Let $p(z) = 1+p_1z+p_2z^2+\ldots$, be regular in the unit disc $\mathbb{E}$, such that $(p(z), zp'(z); z) \in \Omega$, for all $z \in \mathbb{E}$. If

$$\Re[\psi(p(z), zp'(z))] > 0, z \in \mathbb{E},$$

then $\Re\ p(z) > 0,\ z \in \mathbb{E}$.

2. Main Result

Theorem 2.1. *Let $\alpha \ge 0, \beta \le 1$ and $0 \le \mu < 1$, be real numbers such that $\beta\,(1-\mu) < -\dfrac{1}{2}$ and $\beta \le \alpha$. For $z \in \mathbb{E}$, let $f \in \Sigma_n$ satisfy the condition*

$$\Re\left(\frac{(1-\alpha)D^{n+1}[f](z)+\alpha D^{n+2}[f](z)}{(1-\beta)D^{n+1}[f](z)+\beta D^{n+1}[f](z)}\right) > L(\alpha,\beta,\mu) \tag{2.1}$$

then

$$\Re\left(2-\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \mu$$

in $\mathbb{E}$, i.e. $f \in \mathcal{MS}_n^*(\mu)$, where

$$L(\alpha,\beta,\mu) = \frac{(1-\alpha)\,(2-\mu)+\alpha\,(2-\mu)^2+\dfrac{\alpha}{2}\,(1-\mu)}{1+\beta\,(1-\mu)}$$

Proof. For $z \in \mathbb{E}$ and $0 \le \mu < 1$, define,

$$2-\frac{D^{n+1}f(z)}{D^nf(z)} = \mu+(1-\mu)w(z). \tag{2.2}$$

Then r is analytic in $\mathbb{E}$ and $r(0) = 1$. Differentiating (2.2) logarithmically, we get

$$\frac{z(D^{n+1}[f](z))'}{D^{n+1}[f](z)} - \frac{z(D^n[f](z))'}{D^n[f](z)} = -\frac{(1-\mu)zw'(z)}{(2-\mu)-(1-\mu)w(z)} \tag{2.3}$$

$$z(D^n[f](z))' = D^{n+1}[f](z) - 2D^n[f](z), \tag{2.4}$$

Using (2.4) in (2.3), we get

$$\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} = -\frac{(1-\mu)zw'(z)}{(2-\mu)-(1-\mu)w(z)} + (2-\mu) - (1-\mu)w(z)$$

Now, a simple calculation yields

$$\begin{aligned}
&\frac{(1-\alpha)D^{n+1}[f](z) + \alpha D^{n+2}[f](z)}{(1-\beta)D^{n+1}[f](z) + \beta D^{n+1}[f](z)} \\
&= \frac{(1-\alpha) + \alpha\left((2-\mu) - (1-\mu)w(z) - \frac{(1-\mu)zw'(z)}{(2-\mu)-(1-\mu)w(z)}\right)}{(1-\beta) + \beta[(2-\mu)-(1-\mu)w(z)]}[(2-\mu)-(1-\mu)w(z)] \\
&= \frac{(1-\alpha)[(2-\mu)-(1-\mu)w(z)] + \alpha\left([(2-\mu)-(1-\mu)w(z)]^2 - (1-\mu)zw'(z)\right)}{(1-\beta)+\beta[(2-\mu)-(1-\mu)w(z)]} \\
&= \psi(w(z), zw'(z)),
\end{aligned} \tag{2.5}$$

where

$$\psi(u,v) = \frac{(1-\alpha)[(2-\mu)-(1-\mu)u] + \alpha\left([(2-\mu)-(1-\mu)u]^2 - (1-\mu)v\right)}{(1-\beta)+\beta[(2-\mu)-(1-\mu)u]}$$

Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, where u_1, u_2, v_1, v_2 are all real with $v_1 \leq -(1+u_2^2)/2$. Then, we have

$$\begin{aligned}
&\Re\, \psi(iu_2, v_1; z) \\
&= \frac{[(1-\alpha)(2-\mu) + \alpha(2-\mu)^2][1+\beta(1-\mu)]}{[1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2{u_2}^2} \\
&\quad + \frac{(1-\mu)^2[(1-\alpha)\beta - \alpha(1+\beta(1-\mu)) + 2\alpha\beta(2-\mu)]{u_2}^2 - \alpha(1-\mu)(1+\beta(1-\mu))v_1}{[1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2{u_2}^2} \\
&\leq \frac{[(1-\alpha)(2-\mu) + \alpha(2-\mu)^2 + \frac{\alpha}{2}(1-\mu)][1+\beta(1-\mu)]}{[1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2{u_2}^2} \\
&\quad + \frac{(1-\mu)^2[(1-\alpha)\beta - \alpha(1+\beta(1-\mu)) + 2\alpha\beta(2-\mu) + \frac{\alpha}{2}(1-\mu)(1+\beta(1-\mu))]{u_2}^2}{[1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2{u_2}^2} \\
&= \frac{A + B{u_2}^2}{[1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2{u_2}^2} \\
&= \phi(u_2) \quad \text{(say)}, \\
&\leq \max \phi(u_2),
\end{aligned} \tag{2.6}$$

where

$$A = [(1-\alpha)(2-\mu) + \alpha(2-\mu)^2 + \frac{\alpha}{2}(1-\mu)][1+\beta(1-\mu)],$$

and

$$B = (1-\mu)^2[(1-\alpha)\beta - \alpha(1+\beta(1-\mu)) + 2\alpha\beta(2-\mu) + \frac{\alpha}{2}(1-\mu)(1+\beta(1-\mu))]$$

A little calculation yields

$$\phi'(u_2) = \frac{2u_2[B([1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 {u_2}^2) - (A + B{u_2}^2)\beta^2(1-\mu)^2]}{[1+\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 {u_2}^2}$$

It can be easily verified that $\phi'(u_2) = 0$ at $u_2 = 0$ and $\phi''(0) < 0$. Therefore,

$$max\ \phi(u_2) = \phi(0) = L(\alpha, \beta, \mu). \tag{2.7}$$

Let

$$\Omega = \{w : \ \Re\, w > L(\alpha, \beta, \mu)\}.$$

Then from (2.1) and (2.5), we have $\psi(w(z), zw'(z); z) \in \Omega$ for all $z \in \mathbb{E}$, but in view of (2.6) and (2.7), $\psi(iu_2, v_1; z) \notin \Omega$. In the light of Lemma 1.1, from (2.2), we conclude that then

$$\Re\left(2 - \frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \mu$$

in $\mathbb{E}$.
This completes the proof of the theorem. □

3. Criteria for Starlikeness

When we assign particular values to various parameters involved in Theorem 2.1, we obtain following results as:
Setting $n = 0$ in Theorem 2.1, we obtain the following result.

Corollary 3.1. *Let α, β, μ be real numbers such that*
(i) $\alpha \geq 0, \beta \leq 1$ and $0 \leq \mu < 1$
(ii) $\beta(1-\mu) < -\frac{1}{2}$ and $\beta \leq \alpha$.
For all $z \in \mathbb{E}$, if a function $f \in \Sigma$ satisfies the condition

$$\Re\left(\frac{\alpha z^2 f''(z) + (1+4\alpha)zf'(z) + 2(1+\alpha)f(z)}{(1+\beta)f(z) + \beta z f'(z)}\right)$$
$$> \frac{(1-\alpha)(2-\mu) + \alpha(2-\mu)^2 + \frac{\alpha}{2}(1-\mu)}{1+\beta(1-\mu)}$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \mu,$$

i.e. f is meromorphic starlike of order μ.

Writing $\beta = -1$ in above corollary, we get:

Corollary 3.2. *Let α and μ be real numbers such that $\alpha \geq 0$ and $0 < \mu < \frac{1}{2}$. If $f \in \Sigma$ satisfies the condition*

$$-\Re\left(\frac{\alpha z^2 f''(z) + (1+4\alpha)zf'(z) + 2(1+\alpha)f(z)}{zf'(z)}\right) > \frac{(1-\alpha)(2-\mu) + \alpha(2-\mu)^2 + \frac{\alpha}{2}(1-\mu)}{\mu}$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \mu, \;\; z \in \mathbb{E}$$

Taking $\alpha = 0$ in above corollary, we have:

Corollary 3.3. *Let μ be a real number such that $0 < \mu < \frac{1}{2}$. If $f \in \Sigma$ satisfies the condition*

$$\Re\left(\frac{zf'(z) + 2f(z)}{zf'(z)}\right) < \frac{\mu - 2}{\mu}$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \mu, \;\; z \in \mathbb{E}$$

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