Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 22 (2024) 62-80 https://doi.org/10.56947/amcs.v22.279

FIXED POINT RESULTS FOR ALMOST $\mathcal{Z}$-CONTRACTIONS IN PARTIAL b-METRIC SPACES VIA SIMULATION FUNCTION

GURUCHARAN SINGH SALUJA

Abstract. In this paper, we investigate the existence and uniqueness of a fixed point of almost $\mathcal{Z}$-contractions via simulation function in complete partial b-metric spaces using (α, β)-admissibility. Also, some illustrative examples are given to validate the results. Furthermore, an application to the BVP is given. Our results extend and generalize several previous works from the existing literature.

1. Introduction

The study of fixed point is a well-established theory in mathematics which is applied to solve a wide area of problems. The well-known Banach contraction principle [8] established the existence and uniqueness of fixed point of a contraction on a complete metric space (see also, [16], [19], [24], [25], [30]). Since then, several authors generalized this principle by introducing the various contractions on usual metric spaces such as b-metric space, partial metric space, metric like space, partial b-metric space etc. As a generalization of metric spaces, *Bakhtin* [7] in 1989 initially introduced an extension version of metric, called b-metric space, and later formally defined by *Czerwik* [13] in 1993. *Czerwik* also generalized the well-known Banach Contraction Principle within this generalized space. The topological properties of such metric spaces and the fixed point theorems of KKM mappings in metric type spaces were first discussed by Khamsi and Hussain [22]. In 1994, *Matthews* (see, [26, 27]) introduced the notion of partial metric spaces as a part of the study of denotational semantics of data flow networks and proved Banach contraction principle in such spaces. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Further, *Matthews* showed that the Banach contraction principle [8] is valid in partial metric space. *Shukla* [39] generalized both the notions of b-metric and partial metric spaces by introducing partial b-metric spaces. He proved Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces. Also some examples are given which illustrate the results obtained in this new space.

Date: Received: Feb 18, 2024; Accepted: Mar 15, 2024.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Fixed point, partial b-metric space, simulation function, $\mathcal{Z}$-contraction, (α, β)-admissible mapping.

Later on, Mustafa *et al.* [28] proved some common fixed point theorems in the setting of partial b-metric space. Recently, some authors have obtained fixed point results in partial b-metric spaces, see for example, [15], [40], [2] and some others.

2. Preliminaries

In this section, we need the following concepts to prove our main results.

Definition 2.1. ([7]) Let Ξ be a nonempty set and the mapping $d\colon \Xi \times \Xi \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) satisfies:
(bM_1) $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$ for all $\alpha, \beta \in \Xi$;
(bM_2) $d(\alpha, \beta) = d(\beta, \alpha)$ for all $\alpha, \beta \in \Xi$;
(bM_3) there exists a real number $s \geq 1$ such that $d(\alpha, \beta) \leq s[d(\alpha, \gamma) + d(\gamma, \beta)]$ for all $\alpha, \beta, \gamma \in \Xi$.

Then d is called a b-metric on Ξ and the pair (Ξ, d) is called a b-metric space with coefficient s.

Definition 2.2. ([27]) A partial metric on a nonempty set Ξ is a function $\mathcal{P}\colon \Xi \times \Xi \to \mathbb{R}^+$ such that for all $\alpha, \beta, \gamma \in \Xi$:
$(\mathcal{P}M_1)$ $\alpha = \beta$ if and only if $\mathcal{P}(\alpha, \alpha) = \mathcal{P}(\alpha, \beta) = \mathcal{P}(\beta, \beta)$;
$(\mathcal{P}M_2)$ $\mathcal{P}(\alpha, \alpha) \leq \mathcal{P}(\alpha, \beta)$;
$(\mathcal{P}M_3)$ $\mathcal{P}(\alpha, \beta) = \mathcal{P}(\beta, \alpha)$;
$(\mathcal{P}M_4)$ $\mathcal{P}(\alpha, \beta) \leq \mathcal{P}(\alpha, \gamma) + \mathcal{P}(\gamma, \beta) - \mathcal{P}(\gamma, \gamma)$.

A partial metric space is a pair $(\Xi, \mathcal{P})$ such that Ξ is a nonempty set and $\mathcal{P}$ is a partial metric on Ξ.

Definition 2.3. ([39]) A partial b-metric on a nonempty set Ξ is a function $\mathcal{P}_b\colon \Xi \times \Xi \to \mathbb{R}^+$ such that for all $\alpha, \beta, \gamma \in \Xi$:
$(\mathcal{P}b_1)$ $\alpha = \beta$ if and only if $\mathcal{P}_b(\alpha, \alpha) = \mathcal{P}_b(\alpha, \beta) = \mathcal{P}_b(\beta, \beta)$;
$(\mathcal{P}b_2)$ $\mathcal{P}_b(\alpha, \alpha) \leq \mathcal{P}_b(\alpha, \beta)$;
$(\mathcal{P}b_3)$ $\mathcal{P}_b(\alpha, \beta) = \mathcal{P}_b(\beta, \alpha)$;
$(\mathcal{P}b_4)$ there exists a real number $s \geq 1$ such that $\mathcal{P}_b(\alpha, \beta) \leq s[\mathcal{P}_b(\alpha, \gamma) + \mathcal{P}_b(\gamma, \beta)] - \mathcal{P}_b(\gamma, \gamma)$.

A partial b-metric space is a pair $(\Xi, \mathcal{P}_b)$ such that Ξ is a nonempty set and $\mathcal{P}_b$ is a partial b-metric on Ξ. The number s is called the coefficient of $(\Xi, \mathcal{P}_b)$.

Remark 2.4. In a partial b-metric space $(\Xi, \mathcal{P}_b)$ if $\alpha, \beta \in \Xi$ and $\mathcal{P}_b(\alpha, \beta) = 0$, then $\alpha = \beta$, but the converse may not be true. A basic example of a partial b-metric space is a pair $(\mathbb{R}_0^+, \mathcal{P}_b)$, where $\mathcal{P}_b(\alpha, \beta) = (\max\{\alpha, \beta\})^2$ for all $\alpha, \beta \in \mathbb{R}_0^+$.

Remark 2.5. Every partial b-metric space is a generalization of the partial metric space and the b-metric space. However, converse is not true in general.

Example 2.6. ([39]) Let $\Xi = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, $p > 1$ a constant and $\mathcal{P}_b\colon \Xi \times \Xi \to \mathbb{R}^+$ be defined by

$$\mathcal{P}_b(\alpha, \beta) = [\max\{\alpha, \beta\}]^p + |\alpha - \beta| \quad \text{for all } \alpha, \beta \in \Xi.$$

Then $(\Xi, \mathcal{P}_b)$ is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space.

Example 2.7. ([2]) Let $\Xi = [0, +\infty)$ and $k > 1$ a constant. Define $\mathcal{P}_b\colon \Xi \times \Xi \to [0, +\infty)$ by

$$\mathcal{P}_b(\alpha, \beta) = [(\alpha \vee \beta)^k + |\alpha - \beta|^k] \quad \text{for all } \alpha, \beta \in \Xi,$$

is a partial b-metric space on Ξ with coefficient $s = 2^k > 1$. Also note that $\mathcal{P}_b(\alpha, \alpha) = \alpha^k \neq 0$. Thus $\mathcal{P}_b$ is not a b-metric on Ξ.

Now let α, β and $\delta \in \Xi$ such that $\alpha > \delta > \beta$. Then

$$(\alpha - \beta)^k > (\alpha - \delta)^k + (\delta - \beta)^k.$$

$$\begin{aligned} \mathcal{P}_b(\alpha, \beta) &= \alpha^k + |\alpha - \beta|^k \\ \mathcal{P}_b(\alpha, \delta) + \mathcal{P}_b(\delta, \beta) - \mathcal{P}_b(\delta, \delta) &= \alpha^k + |\alpha - \delta|^k + |\delta - \beta|^k. \end{aligned}$$

Thus

$$\mathcal{P}_b(\alpha, \beta) > \mathcal{P}_b(\alpha, \delta) + \mathcal{P}_b(\delta, \beta) - \mathcal{P}_b(\delta, \delta).$$

Hence, $\mathcal{P}_b$ is not a partial metric on Ξ.

Every partial b-metric "$\mathcal{P}_b$" on a nonempty set Ξ generates a topology $\tau_{\mathcal{P}_b}$ on Ξ whose base is the family of open $\mathcal{P}_b$-balls where $\tau_{\mathcal{P}_b} = \{B_{\mathcal{P}_b}(\alpha, \varepsilon) : \alpha \in \Xi, \varepsilon > 0\}$ and

$$B_{\mathcal{P}_b}(\alpha, \varepsilon) = \{\beta \in \Xi : \mathcal{P}_b(\alpha, \beta) < \mathcal{P}_b(\alpha, \alpha) + \varepsilon\},$$

for all $\alpha \in \Xi$ and $\varepsilon > 0$. Obviously, the topological space $(\Xi, \tau_{\mathcal{P}_b})$ is T_0, but need not be T_1.

Now, we recall the following concept in partial b-metric spaces.

Definition 2.8. ([39]) Let $(\Xi, \mathcal{P}_b)$ be a partial b-metric space with coefficient s. Then:

- a sequence $\{r_n\}$ in $(\Xi, \mathcal{P}_b)$ is said to be convergent with respect to $\tau_{\mathcal{P}_b}$ and converges to a point $r \in \Xi$, if $\lim_{n\to\infty} \mathcal{P}_b(r_n, r) = p_b(r, r)$;

- a sequence $\{r_n\}$ is said to be Cauchy sequence in $(\Xi, \mathcal{P}_b)$ if $\lim_{n,m\to\infty} \mathcal{P}_b(r_n, r_m)$ exists and is finite.

- $(\Xi, \mathcal{P}_b)$ is said to be a complete partial b-metric space if for every Cauchy sequence $\{r_n\}$ in Ξ there exists $r \in \Xi$ such that

$$\lim_{n,m\to\infty} \mathcal{P}_b(r_n, r_m) = \lim_{n\to\infty} \mathcal{P}_b(r_n, r) = \mathcal{P}_b(r, r).$$

- A mapping $\mathcal{G}\colon \Xi \to \Xi$ is said to be continuous at $e_0 \in \Xi$ if for every $\varepsilon > 0$, there exists $\mu > 0$ such that $\mathcal{G}\Big(B_{\mathcal{P}_b}(e_0, \mu)\Big) \subset B_{\mathcal{P}_b}\Big(\mathcal{G}(e_0), \varepsilon\Big)$.

Note that in a partial b-metric space, the limit of a convergent sequence may not be unique.

Example 2.9. ([39]) Let $\Xi = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, $c > 0$ be any constant and define $\mathcal{P}_b\colon \Xi \times \Xi \to \mathbb{R}^+$ by

$$\mathcal{P}_b(\alpha, \beta) = \max\{\alpha, \beta\} + c \quad \text{for all } \alpha, \beta \in \Xi.$$

Then $(\Xi, \mathcal{P}_b)$ is a partial b-metric space with arbitrary coefficient $s \geq 1$. If $\alpha_n = 1$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \alpha_n = \beta$ for all $\beta \geq 1$. Thus, the limit of a convergent sequence in partial b-metric space need not be unique.

Let $(\Xi, \mathcal{P}_b)$ be a partial b-metric space. The function $b\colon \Xi \times \Xi \to [0, +\infty)$ defined by

$$b(\acute{u}, \acute{v}) = 2\mathcal{P}_b(\acute{u}, \acute{v}) - \mathcal{P}_b(\acute{u}, \acute{u}) - \mathcal{P}_b(\acute{v}, \acute{v}), \qquad (*)$$

for all $\acute{u}, \acute{v} \in \Xi$, satisfies all axioms of the b-metric. The pair (Ξ, b) is a b-metric space. It is called an associated b-metric space.

Another associated b-metric is defined as follows.

Let $(\Xi, \mathcal{P}_b)$ be a partial b-metric space and the mapping $b\colon \Xi \times \Xi \to [0, +\infty)$ defined by

$$b(u, v) = \begin{cases} \mathcal{P}_b(u, v), & \text{if } u \neq v, \\ 0, & \text{if } u = v, \end{cases}$$

for $u, v \in \Xi$. Then b is a b-metric associated with $\mathcal{P}_b$.

In 2012, *Samet et al.* [36] introduced the concept of α-admissible mappings.

Definition 2.10. ([36]) Let Ξ be a non-empty set. Let $\mathcal{T}\colon \Xi \to \Xi$ and $\alpha\colon \Xi\times\Xi \to [0, +\infty)$ be given mappings. We say that $\mathcal{T}$ is α-admissible if for all $u, v \in \Xi$, we have

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(\mathcal{T}u, \mathcal{T}v) \geq 1. \tag{2.1}$$

The α-admissible mappings notion has been used in many works, see for example [4, 5, 17, 21, 31, 35]. Later, Karapinar *et al.* [20] introduced the concept of triangular α-admissible mappings in 2013.

Definition 2.11. ([20]) Let Ξ be a non-empty set. Let $\mathcal{T}\colon \Xi \to \Xi$ and $\alpha\colon \Xi\times\Xi \to [0, +\infty)$ be given mappings. A mapping $\mathcal{T}$ is called a triangular α-admissible if: for all $u, v, w \in \Xi$, we have

$(\mathcal{T}_1)$ $\mathcal{T}$ is α-admissible;

$(\mathcal{T}_2)$ $\alpha(u, v) \geq 1$ and $\alpha(v, w) \geq 1$ implies $\alpha(u, w) \geq 1$.

Recently, Chandok [11] introduced the notion of (α, β)-admissible Geraghty type contractive mapping with sufficient condition for the existence of a fixed point of such class of generalized nonlinear contractive mapping in metric space and proved some fixed point results (see, also [1], [12], [32], [33]).

Definition 2.12. ([11]) Let Ξ be a non-empty set. Let $\mathcal{T}\colon \Xi \to \Xi$ and $\alpha, \beta\colon \Xi \times \Xi \to [0, +\infty)$ be given mappings, we say that $\mathcal{T}$ is an (α, β)-admissible if $\alpha(u, v) \geq 1$ and $\beta(u, v) \geq 1$ implies $\alpha(\mathcal{T}u, \mathcal{T}v) \geq 1$ and $\beta(\mathcal{T}u, \mathcal{T}v) \geq 1$ for all $u, v \in \Xi$.

Berinde [9, 10] extended the class of contractive mappings, introducing the notion of almost contractions defined as follows:

Definition 2.13. Let (Ξ, d) be a metric space. A self mapping $\mathcal{T}\colon \Xi \to \Xi$ is called an almost contraction if there are constants $\lambda \in (0, 1)$ and $\theta \geq 0$ such that

$$d(\mathcal{T}u, \mathcal{T}v) \leq \lambda\, d(u, v) + \theta\, d(v, \mathcal{T}u), \text{ for all } u, v \in \Xi. \tag{2.2}$$

Berinde [9] then proved every almost contraction mapping defined in complete metric space has at least one fixed point. Subsequently, Babu *et al.* [6] demonstrated that almost contraction type mappings have a unique fixed point under conditions that present the notion of B-almost contraction (see, [37, 38]).

Definition 2.14. Let (Ξ, d) be a metric space. A self mapping $\mathcal{T} \colon \Xi \to \Xi$ is called an B-almost contraction if there are constants $\lambda \in (0,1)$ and $\theta \geq 0$ such that

$$d(\mathcal{T}u, \mathcal{T}v) \leq \lambda\, d(u,v) + \theta\, N(u,v), \text{ for all } u, v \in \Xi, \tag{2.3}$$

where

$$N(u,v) = \min\big\{d(u,\mathcal{T}), d(v,\mathcal{T}v), d(u,\mathcal{T}v), d(v,\mathcal{T}u)\big\}.$$

Recently, *Khojasteh et al.* [23] introduced the notion of $\mathcal{Z}$-contraction involving a new class of mappings, namely simulation functions to prove the following result.

Theorem 2.15. *([23]) Let (Ξ, d) be a metric space and let $\mathcal{T} \colon \Xi \to \Xi$ be a $\mathcal{Z}$-contraction with respect to a function ζ satisfying certain conditions, that is,*

$$\zeta\Big(d(\mathcal{T}u, \mathcal{T}v), d(u,v)\Big) \geq 0, \tag{2.4}$$

for all $u, v \in \Xi$. Then, $\mathcal{T}$ has a unique fixed point, and for every initial point $e_0 \in \Xi$, the Picard sequence $\{\mathcal{T}^n e_0\}$ converges to this fixed point.

Remark 2.16. ([23]) Every $\mathcal{Z}$-contraction mapping is contractive and therefore it is continuous.

A simple example of $\mathcal{Z}$-contraction is the Banach contraction, which can be obtained by defining $\zeta \colon [0,\infty)\times[0,\infty) \to \mathbb{R}$ via $\zeta(t,s) = \lambda\, s - t$ for all $t, s \in [0,\infty)$, where $\lambda \in [0,1)$.

Definition 2.17. ([23]) Let $\zeta \colon [0,\infty)\times[0,\infty) \to \mathbb{R}$ be a function. Consider the following conditions:

(ζ_1) $\zeta(0,0) = 0$.

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$.

(ζ_3) If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

(ζ_4) If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

If the function ζ satisfies the conditions $(\zeta_1)-(\zeta_3)$, we say that ζ is a simulation function according to the sense of Khojasteh *et al.* [23]. If it satisfies (ζ_2) and (ζ_3), it is a simulation function in the sense of Argoubi *et al.* [3] and if it satisfies (ζ_1), (ζ_2) and (ζ_4), then it is a simulation function according to sense of Roldan-Lopez-de-Hierro *et al.* [34].

Remark 2.18. ([23]) It is clear from the definition of simulation function that $\zeta(t,s) < 0$ for all $t \geq s > 0$. Therefore, if $\mathcal{T}$ is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then for all $u, v \in \Xi$ such that $d(\mathcal{T}u, \mathcal{T}v) < d(u,v)$. This shows that every $\mathcal{Z}$-contraction mapping is contraction and therefore it is continuous.

The following are examples of simulation functions.

Example 2.19. Let $\zeta\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ be defined by:
(*i*) $\zeta(s,t)=\frac{s}{1+s}-t$ for all $s,t\in[0,\infty)$.
(*ii*) $\zeta(t,s)=\lambda\, s-t$ for all $s,t\in[0,\infty)$, where $\lambda\in[0,1)$.
(*iii*) $\zeta(t,s)=s-\lambda\, t$ for all $s,t\in[0,\infty)$, where $\lambda>1$.
(*iv*) $\zeta(t,s)=\frac{1}{s+1}-(t+1)$ for all $s,t\in[0,\infty)$.
(*v*) $\zeta(t,s)=\frac{s}{s+1}-te^t$ for all $s,t\in[0,\infty)$.
(*vi*) $\zeta(t,s)=\frac{k}{r}s-t$ for all $s,t\in[0,\infty)$ where $k\in[0,1)$ and $r\in(1,\infty)$.
(*vii*) $\zeta(t,s)=\phi(s)-t$ for all $s,t\in[0,\infty)$, where $\phi\colon[0,\infty)\to[0,\infty)$ is an upper semi-continuous function with $\phi(t)<t$ for all $t>0$ and $\phi(0)=0$.

Definition 2.20. ([18]) Let (Ξ,d) be a metric space and $\zeta\in\mathcal{Z}$. We say that $\mathcal{T}\colon\Xi\to\Xi$ is an almost $\mathcal{Z}$-contraction if there is a constant $\theta\ge 0$ such that

$$\zeta\Big(\alpha(d(\mathcal{T}u,\mathcal{T}v)),d(u,v)+\theta\,N(u,v)\Big)\ge 0, \tag{2.5}$$

for all $u,v\in\Xi$, where $N(u,v)=\min\{d(u,\mathcal{T}u),d(v,\mathcal{T}v),d(u,\mathcal{T}v),d(v,\mathcal{T}u)\}$.

Remark 2.21. If $\mathcal{T}$ is an almost $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$, then

$$d(\mathcal{T}u,\mathcal{T}v)<d(u,v)+\theta\,N(u,v), \tag{2.6}$$

for all $u,v\in\Xi$.

Lemma 2.22. *([28], Lemma 1)*
(1) *A sequence $\{r_n\}$ is a $\mathcal{P}_b$-Cauchy sequence in a partial b-metric space $(\Xi,\mathcal{P}_b)$ if only if it is a b-Cauchy sequence in the b-metric space $(\Xi,d_{\mathcal{P}_b})$.*
(2) *A partial b-metric space $(\Xi,\mathcal{P}_b)$ is complete if and only if the b-metric space $(\Xi,d_{\mathcal{P}_b})$ is b-complete. Moreover, $\lim_{n\to\infty}d_{\mathcal{P}_b}(r,r_n)=0$ if and only if*

$$\lim_{n\to\infty}\mathcal{P}_b(r,r_n)=\lim_{n,m\to\infty}\mathcal{P}_b(r_n,r_m)=\mathcal{P}_b(r,r).$$

Proposition 2.23. *([28], Proposition 3) Every partial b-metric space $\mathcal{P}_b$ defines a b-metric $d_{\mathcal{P}_b}$, where*

$$d_{\mathcal{P}_b}(u,v)=2\mathcal{P}_b(u,v)-\mathcal{P}_b(u,u)-\mathcal{P}_b(v,v),$$

for all $u,v\in\Xi$.

In this paper, we investigate the existence and uniqueness of a fixed point of almost $\mathcal{Z}$-contractions via simulation function in complete partial b-metric spaces using (α,β)-admissibility. Also, some illustrative examples are given to validate the results. Moreover, an application to the BVP is given. Our results extend, generalize and enrich several results from the existing literature.

3. Main results

Firstly, we give the following definition.

Definition 3.1. Let $(\Xi,\mathcal{P}_b)$ be a complete partial b-metric space with arbitrary coefficient $s\ge 1$, let $\mathcal{U}\colon\Xi\to\Xi$ be a mapping and $\alpha,\beta\colon\Xi\to[0,+\infty)$ be two functions. Then $\mathcal{U}$ is said to be a (α,β)-admissible almost $\mathcal{Z}$-contraction, if it satisfies the following conditions:
(1) $\mathcal{U}$ is (α,β)-admissible,

(2) there exists a simulation function $\zeta \in \mathcal{Z}$ such that

$$\begin{aligned} 0 &\leq \zeta\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)),\mathcal{P}_b(x,y) \\ &\qquad +\theta\Omega^b_{\mathcal{P}}(x,y)\Big), \end{aligned} \tag{3.1}$$

for all $x, y \in \Xi$, where

$$\Omega^b_{\mathcal{P}}(x,y) = \min\Big\{\mathcal{P}_b(x,\mathcal{U}x),\mathcal{P}_b(y,\mathcal{U}y),\mathcal{P}_b(x,\mathcal{U}y),\mathcal{P}_b(y,\mathcal{U}x)\Big\}.$$

Now, we are in a position to prove our main result.

Theorem 3.2. *Let $(\Xi, \mathcal{P}_b)$ be a complete partial b-metric space with coefficient $s \geq 1$ and a continuous mapping $\mathcal{U}\colon \Xi \to \Xi$ be a (α, β)-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ simulation function satisfying the inequality (3.1) and there exists $e_0 \in \Xi$ such that $\alpha(e_0,\mathcal{U}e_0) \geq 1$, $\beta(e_0,\mathcal{U}e_0) \geq 1$. Then $\mathcal{U}$ has a unique fixed point $u \in \Xi$ such that $\mathcal{P}_b(u,u) = 0$.*

Proof. Let $\{e_n\}$ be a sequence in Ξ such that $e_{n+1} = \mathcal{U}e_n$ for all $n = 0, 1, 2, 3, \dots$. If $e_n = e_{n+1}$, then $\mathcal{U}e_n = e_{n+1} = e_n$, that is, e_n is a fixed point of $\mathcal{U}$. Thus, the proof is finished. So, we consider $e_n \neq e_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Since $\alpha(e_0,\mathcal{U}e_0) \geq 1$ implies $\alpha(e_0, e_1) \geq 1$ and $\mathcal{U}$ is an (α, β)-admissible, so

$$\alpha(\mathcal{U}e_0,\mathcal{U}e_1) \geq 1 \quad \Rightarrow \quad \alpha(e_1, e_2) \geq 1.$$

Now, continuing the above manner, we get for all $n \geq 0$

$$\alpha(e_n, e_{n+1}) \geq 1. \tag{3.2}$$

Likewise, for all $n \geq 0$, we obtain

$$\beta(e_n, e_{n+1}) \geq 1. \tag{3.3}$$

From equation (3.1), we have

$$\begin{aligned} 0 &\leq \zeta\Big(\alpha(\mathcal{U}(e_{n-1}),\mathcal{U}(e_n))\beta(\mathcal{U}(e_{n-1}),\mathcal{U}(e_n)\mathcal{P}_b(\mathcal{U}(e_{n-1}),\mathcal{U}(e_n)),\mathcal{P}_b(x_{n-1},x_n) \\ &\qquad +\theta\Omega^b_{\mathcal{P}}(e_{n-1},e_n)\Big), \end{aligned}$$

that is,

$$\begin{aligned} 0 &\leq \zeta\Big(\alpha(e_n,e_{n+1})\beta(e_n,e_{n+1})\mathcal{P}_b(e_n,e_{n+1}),\mathcal{P}_b(e_{n-1},e_n) \\ &\qquad +\theta\Omega^b_{\mathcal{P}}(e_{n-1},e_n)\Big), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Omega^b_{\mathcal{P}}(e_{n-1},e_n) &= \min\Big\{\mathcal{P}_b(e_{n-1},\mathcal{U}e_n),\mathcal{P}_b(e_n,\mathcal{U}e_n),\mathcal{P}_b(e_{n-1},\mathcal{U}e_n), \\ &\qquad \mathcal{P}_b(e_n,\mathcal{U}e_{n-1})\Big\} \\ &= \min\Big\{\mathcal{P}_b(e_{n-1},e_{n+1}),\mathcal{P}_b(e_n,e_{n+1}),\mathcal{P}_b(e_{n-1},e_{n+1}), \\ &\qquad \mathcal{P}_b(e_n,e_n)\Big\} \\ &= 0. \end{aligned}$$

Therefore, from equation (3.4) and (ζ_2), we have

$$\begin{aligned} 0 &\leq \zeta\Big(\alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1}), \mathcal{P}_b(e_{n-1}, e_n)\Big) \\ &< \mathcal{P}_b(e_{n-1}, e_n) - \alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1}), \end{aligned}$$

that is,

$$\alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1}) < \mathcal{P}_b(e_{n-1}, e_n). \tag{3.5}$$

Now, we know that

$$\mathcal{P}_b(e_n, e_{n+1}) \leq \alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1}), \tag{3.6}$$

since $\alpha(e_n, e_{n+1}) \geq 1$ and $\beta(e_n, e_{n+1}) \geq 1$.

From equations (3.5) and (3.6) for all $n \geq 0$, we have

$$\begin{aligned} \mathcal{P}_b(e_n, e_{n+1}) &\leq \alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1}) \\ &< \mathcal{P}_b(e_{n-1}, e_n), \end{aligned}$$

that is,

$$\mathcal{P}_b(e_n, e_{n+1}) < \mathcal{P}_b(e_{n-1}, e_n). \tag{3.7}$$

The sequence $\{\mathcal{P}_b(e_n, e_{n+1})\}$ is non-decreasing. So, there exists $c \geq 0$ such that $\lim_{n\to\infty} \mathcal{P}_b(e_n, e_{n+1}) = c$. We claim that

$$\mathcal{P}_b(e_n, e_{n+1}) = 0. \tag{3.8}$$

Now, we assume on the contrary that $c > 0$. By (3.6), we have

$$\lim_{n\to\infty} \big\{\alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1})\big\} = c. \tag{3.9}$$

Since $c > 0$, letting $s_n = \alpha(e_n, e_{n+1})\beta(e_n, e_{n+1})\mathcal{P}_b(e_n, e_{n+1})$ and $t_n = \mathcal{P}_b(e_n, e_{n+1})$ such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = c$, then by (ζ_3) $\limsup_{n\to\infty} \zeta(s_n, t_n) < 0$. Since $\zeta(s_n, t_n) \geq 0$, so $0 \leq \limsup_{n\to\infty} \zeta(s_n, t_n) < 0$, which is a contradiction. Thus, our assumption that $c > 0$ is false. Hence $c = 0$. Now, we show that $\{e_n\}$ is a Cauchy sequence in $(\Xi, \mathcal{P}_b)$, that is,

$$\lim_{n,m\to\infty} \mathcal{P}_b(e_n, e_m) = 0. \tag{3.10}$$

Suppose the contrary, that is, $\{e_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{e_{n_{(k)}}\}$ and $\{e_{m_{(k)}}\}$ of $\{e_n\}$ such that $m_{(k)}$ is the smallest index for which

$$m_{(k)} > n_{(k)} > k, \quad d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}}) \geq \varepsilon. \tag{3.11}$$

This means that

$$d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) < \varepsilon. \tag{3.12}$$

From equation (3.11) and using the triangular inequality, we have

$$\begin{aligned} \varepsilon &\leq d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}}) \\ &\leq s[d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) + d_{\mathcal{P}_b}(e_{m_{(k)}-1}, e_{m_{(k)}})] \\ &\quad -d_{\mathcal{P}_b}(e_{m_{(k)}-1}, e_{m_{(k)}-1}) \\ &\leq s[d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) + d_{\mathcal{P}_b}(e_{m_{(k)}-1}, e_{m_{(k)}})]. \end{aligned} \tag{3.13}$$

Taking the limit as $k \to \infty$ in the above inequality and using equations (3.10) and (3.12), we get

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) \leq \limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) \leq \varepsilon. \tag{3.14}$$

Also from (3.11) and (3.13), we have

$$\varepsilon \leq \limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}}) \leq s\varepsilon. \tag{3.15}$$

Again, we have

$$d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}+1}) \leq s d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}}) + s d_{\mathcal{P}_b}(e_{m_{(k)}}, e_{m_{(k)}+1}),$$

and hence

$$\limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}+1}) \leq s^2\varepsilon. \tag{3.16}$$

Further,

$$d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}+1}) \leq s d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}}) + s d_{\mathcal{P}_b}(e_{m_{(k)}}, e_{m_{(k)}+1}),$$

and hence

$$\limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}+1}) \leq s\varepsilon. \tag{3.17}$$

Again, we have

$$d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}-1}) \leq s d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{n_{(k)}}) + s d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}),$$

and hence

$$\limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}-1}) \leq s\varepsilon. \tag{3.18}$$

Finally, we have

$$d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) \leq s d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{n_{(k)}-1}) + s d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}-1}),$$

and hence

$$\limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}}, e_{m_{(k)}-1}) \leq s^2\varepsilon. \tag{3.19}$$

On the other hand, by the definition of $d_{\mathcal{P}_b}$ and (3.10),

$$\limsup_{k\to\infty} d_{\mathcal{P}_b}(e_{n_{(k)}-1}, e_{m_{(k)}}) = 2 \limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}}).$$

Hence by equation (3.13), we have

$$\frac{\varepsilon}{2s} \leq \liminf_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}}) \leq \limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}}) \leq \frac{\varepsilon}{2}. \tag{3.20}$$

Likewise,

$$\limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}}, e_{m_{(k)}}) \le \frac{s\varepsilon}{2}, \tag{3.21}$$

$$\frac{\varepsilon}{2s} \le \limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}}, e_{m_{(k)}+1}), \tag{3.22}$$

$$\limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}+1}) \le \frac{s\varepsilon}{2}, \tag{3.23}$$

$$\limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}-1}) \le \frac{s\varepsilon}{2}, \tag{3.24}$$

$$\limsup_{k\to\infty} \mathcal{P}_b(e_{n_{(k)}}, e_{m_{(k)}-1}) \le \frac{s^2\varepsilon}{2}. \tag{3.25}$$

Since $\alpha(e_n, e_{n+1}) \ge 1$ and $\beta(e_n, e_{n+1}) \ge 1$ for all $n = 1, 2, 3, \ldots$, we conclude that

$$\alpha(e_{n_{(k)}-1}, e_{n_{(k)}})\beta(e_{n_{(k)}-1}, e_{n_{(k)}}) \ge 1. \tag{3.26}$$

Since $\mathcal{U}$ is (α, β)-admissible $\mathcal{Z}$-contraction. Hence from equation (3.1), we have

$$\begin{aligned} 0 &\le \zeta\Big(\alpha(\mathcal{U}(e_{n_{(k)}-1}), \mathcal{U}(e_{m_{(k)}-1}))\beta(\mathcal{U}(e_{n_{(k)}-1}), \mathcal{U}(e_{m_{(k)}-1}))\mathcal{P}_b(\mathcal{U}(e_{n_{(k)}-1}), \\ &\qquad \mathcal{U}(e_{m_{(k)}-1})), \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}-1}) + \theta\Omega^b_{\mathcal{P}}(e_{n_{(k)}-1}, e_{m_{(k)}-1})\Big) \\ &= \zeta\Big(\alpha(e_{n_{(k)}}, e_{m_{(k)}})\beta(e_{n_{(k)}}, e_{m_{(k)}})\mathcal{P}_b(e_{n_{(k)}}, e_{m_{(k)}}), \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}-1}) \\ &\qquad +\theta\Omega^b_{\mathcal{P}}(e_{n_{(k)}-1}, e_{m_{(k)}-1})\Big), \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} \Omega^b_{\mathcal{P}}(e_{n_{(k)}-1}, e_{m_{(k)}-1}) &= \min\Big\{\mathcal{P}_b(e_{n_{(k)}-1}, \mathcal{U}e_{n_{(k)}-1}), \mathcal{P}_b(e_{m_{(k)}-1}, \mathcal{U}e_{m_{(k)}-1}), \\ &\qquad \mathcal{P}_b(e_{n_{(k)}-1}, \mathcal{U}e_{m_{(k)}-1}), \mathcal{P}_b(e_{m_{(k)}-1}, \mathcal{U}e_{n_{(k)}-1})\Big\} \\ &= \min\Big\{\mathcal{P}_b(e_{n_{(k)}-1}, e_{n_{(k)}}), \mathcal{P}_b(e_{m_{(k)}-1}, e_{m_{(k)}}), \\ &\qquad \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}}), \mathcal{P}_b(e_{m_{(k)}-1}, e_{n_{(k)}})\Big\}, \end{aligned}$$

taking limit as $k \to \infty$ in the above and using equation (3.10), we obtain

$$\Omega^b_{\mathcal{P}}(e_{n_{(k)}-1}, e_{m_{(k)}-1}) \to 0 \text{ as } k \to \infty. \tag{3.28}$$

From equations (3.27) and (3.28), we have

$$\begin{aligned} 0 &\le \limsup_{k\to\infty} \zeta\Big(\alpha(e_{n_{(k)}}, e_{m_{(k)}})\beta(e_{n_{(k)}}, e_{m_{(k)}})\mathcal{P}_b(e_{n_{(k)}}, e_{m_{(k)}}), \\ &\qquad \mathcal{P}_b(e_{n_{(k)}-1}, e_{m_{(k)}-1})\Big) < 0, \end{aligned} \tag{3.29}$$

which is a contraction of our assumption. Consequently, $\{e_n\}$ is a b-Cauchy sequence in the b-metric space $(\Xi, d_{\mathcal{P}_b})$.

Finally, we prove that $\mathcal{U}$ has a fixed point. Since $(\Xi, d_{\mathcal{P}_b})$ is a complete partial b-metric space, then from Lemma 2.22, $(\Xi, d_{\mathcal{P}_b})$ is a complete b-metric space.

Therefore, the sequence converges to some $u \in \Xi$, that is, $\lim_{n\to\infty} d_{\mathcal{P}_b}(e_n, u) = 0$. Again, from Lemma 2.22,

$$\lim_{n\to\infty} \mathcal{P}_b(u, e_n) = \lim_{n\to\infty} \mathcal{P}_b(e_n, e_n) = \mathcal{P}_b(u, u). \tag{3.30}$$

On the other hand, from equation (3.10) and $(\mathcal{P}b_2)$, $\lim_{n\to\infty} \mathcal{P}_b(e_n, e_n) = 0$, which yields that

$$\lim_{n\to\infty} \mathcal{P}_b(u, e_n) = \lim_{n\to\infty} \mathcal{P}_b(e_n, e_n) = \mathcal{P}_b(u, u) = 0. \tag{3.31}$$

Now, since $\lim_{n\to\infty} \mathcal{P}_b(u, e_n) = 0$ or $e_n \to u$ as $n \to \infty$, the continuity of $\mathcal{U}$ implies that $\mathcal{U}e_{2n} \to \mathcal{U}(u)$. Since $e_{2n+1} = \mathcal{U}e_{2n}$ and $e_{2n+1} \to u$ as $n \to \infty$, by uniqueness of limit, we get $\mathcal{U}(u) = u$. So, u is a fixed point of $\mathcal{U}$.

Now, we shall show that the uniqueness of the fixed point u. Suppose that u' is another fixed point of $\mathcal{U}$ such that $\mathcal{U}(u') = u'$ with $u \neq u'$. From the given contraction (3.1), taking $x = u$ and $y = u'$, we have

$$\begin{aligned} 0 &\leq \zeta\Big(\alpha(\mathcal{U}(u),\mathcal{U}(u'))\beta(\mathcal{U}(u),\mathcal{U}(u'))\mathcal{P}_b(\mathcal{U}(u),\mathcal{U}(u')), \mathcal{P}_b(u, u') \\ &\qquad +\theta\Omega^b_{\mathcal{P}}(u, u')\Big) \\ &= \zeta\Big(\alpha(\mathcal{U}(u),\mathcal{U}(u'))\beta(\mathcal{U}(u),\mathcal{U}(u'))\mathcal{P}_b(u, u'), \mathcal{P}_b(u, u') \\ &\qquad +\theta\Omega^b_{\mathcal{P}}(u, u')\Big), \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} \Omega^b_{\mathcal{P}}(u, u') &= \min\Big\{\mathcal{P}_b(u,\mathcal{U}u), \mathcal{P}_b(u',\mathcal{U}u'), \mathcal{P}_b(u,\mathcal{U}u'), \\ &\qquad \mathcal{P}_b(u',\mathcal{U}u)\Big\} \\ &= \min\Big\{\mathcal{P}_b(u, u), \mathcal{P}_b(u', u'), \mathcal{P}_b(u, u'), \mathcal{P}_b(u', u)\Big\} \\ &= 0. \end{aligned}$$

From equations (3.32) and (3.33), we obtain

$$\begin{aligned} 0 &\leq \zeta\Big(\alpha(\mathcal{U}(u),\mathcal{U}(u'))\beta(\mathcal{U}(u),\mathcal{U}(u'))\mathcal{P}_b(u, u'), \mathcal{P}_b(u, u')\Big) \\ &\leq \mathcal{P}_b(u, u') - \alpha(\mathcal{U}(u),\mathcal{U}(u'))\beta(\mathcal{U}(u),\mathcal{U}(u'))\mathcal{P}_b(u, u') \\ &= \mathcal{P}_b(u, u') - \alpha(u, u')\beta(u, u')\mathcal{P}_b(u, u') \\ &= \mathcal{P}_b(u, u')[1 - \alpha(u, u')\beta(u, u')] < 0, \end{aligned}$$

which is a contradiction, since $\alpha(u, u') \geq 1$ and $\beta(u, u') \geq 1$. Hence, we conclude that $\mathcal{P}_b(u, u') = 0$, that is, $u = u'$. This shows that $\mathcal{U}$ has a unique fixed point in Ξ. This completes the proof. □

From Theorem 3.2, we obtain the following result as corollary.

Corollary 3.3. *In Theorem 3.2, if we replace contractive condition (3.1) by one of the following contractive conditions given below, then we have the same* *conclusion as in Theorem 3.2 and the proofs of these corollaries are similar to that of Theorem 3.2.*

$$0 \leq \zeta\Big(\alpha(x,\mathcal{U}(x))\beta(y,\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x,y) + \theta\Omega^b_{\mathcal{P}}(x,y)\Big), \tag{3.33}$$

$$0 \leq \zeta\Big(\alpha(x,y)\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x,y) + \theta\Omega^b_{\mathcal{P}}(x,y)\Big), \tag{3.34}$$

$$0 \leq \zeta\Big(\alpha(x,y)\beta(x,y)\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x,y) + \theta\Omega^b_{\mathcal{P}}(x,y)\Big), \tag{3.35}$$

$$0 \leq \zeta\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(x,y)\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x,y) + \theta\Omega^b_{\mathcal{P}}(x,y)\Big), \tag{3.36}$$

for all $x, y \in \Xi$, where $\Omega^b_{\mathcal{P}}(x,y)$ is as in Theorem 3.2.

The following corollary is a result of *Shukla* type [39].

Corollary 3.4. *Let $(\Xi, \mathcal{P}_b)$ be a complete partial b-metric space with coefficient $s \geq 1$ and let $\mathcal{U}\colon \Xi \to \Xi$ be a mapping. Suppose that there exists a constant $\lambda \in [0,1)$ such that*

$$\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)) \leq \lambda\,\mathcal{P}_b(x,y) + \theta\Omega^b_{\mathcal{P}}(x,y),$$

for all $x, y \in \Xi$, where $\Omega^b_{\mathcal{P}}(x,y)$ is as in Theorem 3.2. Then $\mathcal{U}$ has a unique fixed point $u \in \Xi$ such that $\mathcal{P}_b(u,u) = 0$.

Proof. The result follows from Theorem 3.2 by taking as simulation function

$$\zeta(t,s) = \lambda\, s - t,$$

for all $t, s \geq 0$ and $\lambda \in [0,1)$. □

From Corollary 3.4, we deduce the following result.

Corollary 3.5. *([39], Theorem 1) Let $(\Xi, \mathcal{P}_b)$ be a complete partial b-metric space with coefficient $s \geq 1$ and let $\mathcal{U}\colon \Xi \to \Xi$ be a mapping. Suppose that there exists a constant $\lambda \in [0,1)$ such that*

$$\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)) \leq \lambda\,\mathcal{P}_b(x,y),$$

for all $x, y \in \Xi$. Then $\mathcal{U}$ has a unique fixed point $u \in \Xi$ such that $\mathcal{P}_b(u,u) = 0$.

Proof. By taking $\alpha(\mathcal{U}(x),\mathcal{U}(y)) = \beta(\mathcal{U}(x),\mathcal{U}(y)) = 1$ for all $x, y \in \Xi$ and $\theta = 0$ in Corollary 3.4. □

Remark 3.6. Corollary 3.5 extends Banach contraction mapping principle [8] from complete metric space to the setting of complete partial b-metric space.

If we take $\zeta(t,s) = \phi(s) - t$, $\alpha(\mathcal{U}(x),\mathcal{U}(y)) = \beta(\mathcal{U}(x),\mathcal{U}(y)) = 1$ for all $x, y \in \Xi$ and $\theta = 0$ in Theorem 3.2, then we have the following result.

Corollary 3.7. *Let $(\Xi, \mathcal{P}_b)$ be a complete partial b-metric space with coefficient $s \geq 1$ and let $\mathcal{U}\colon \Xi \to \Xi$ be a given mapping satisfying*

$$\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)) \leq \phi\big(\mathcal{P}_b(x,y)\big),$$

for all $x, y \in \Xi$, where $\phi\colon [0,\infty) \to [0,\infty)$ is an upper semi-continuous function with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Then, $\mathcal{U}$ has a unique fixed point $u \in \Xi$ such that $\mathcal{P}_b(u,u) = 0$.

From Theorem 3.2, we easily deduce the following fixed point theorem in partial b-metric spaces.

Theorem 3.8. *Let $(\Xi, \mathcal{P}_b)$ be a complete partial b-metric space with coefficient $s \geq 1$ and let $\mathcal{U}\colon \Xi \to \Xi$ be a mapping. Suppose that there exists a simulation function ζ such that*

$$\zeta\big(\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x,y)\big) \geq 0,$$

for all $x, y \in \Xi$. Then $\mathcal{U}$ has a unique fixed point $u \in \Xi$ such that $\mathcal{P}_b(u,u) = 0$.

Remark 3.9. (1) Theorem 3.8 generalizes Theorem 5.1 of [29] from complete partial metric space to the setting of complete partial b-metric space.

(2) Corollary 3.5 is obtained from Theorem 3.8 by taking as simulation function

$$\zeta(t,s) = \lambda\, s - t,$$

for all $t, s \geq 0$ and $\lambda \in [0,1)$.

Now, we give examples in support of Theorem 3.2 and Corollary 3.5.

Example 3.10. Let $\Xi = [0,+\infty)$. Define $\mathcal{P}_b\colon \Xi \times \Xi \to \mathbb{R}^+$ by

$$\mathcal{P}_b(x,y) = [\max\{x,y\}]^2,$$

for all $x, y \in \Xi$. Then $(\Xi, \mathcal{P}_b)$ is a complete partial b-metric space with coefficient $s = 4 > 1$.

Now, consider the mapping $\mathcal{U}\colon \Xi \to \Xi$ given by

$$\mathcal{U}(x) = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,1], \\ x+1, & \text{if } x > 1, \end{cases}$$

for all $x \in \Xi$.

Define two functions $\alpha, \beta\colon \Xi \times \Xi \to [0,+\infty)$ by:

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

for all $x, y \in \Xi$.

Let $\zeta(t,s) = \lambda\, s - t$ for all $t, s \in [0,\infty)$ and $\lambda \in [0,1)$. Note that $\mathcal{U}$ is (α,β)-admissible. In fact, let $x, y \in \Xi$ such that $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$. By definition of α and β, this implies that $x, y \in [0,1]$. Thus,

$$\alpha\big(\mathcal{U}x, \mathcal{U}y\big) = \alpha\big(\frac{x}{4}, \frac{y}{4}\big) = 1.$$

Likewise
$$\beta\big(\mathcal{U}x,\mathcal{U}y\big) = 1.$$
Now, we show that the contraction condition (3.1) is verified. Let $x, y \in \Xi$ and suppose $x \geq y$ such that $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case, we have

$$\begin{aligned}
&\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x, y) \\
&\qquad +\theta\Omega^b_{\mathcal{P}}(x, y)\Big) \\
&= \Big(\frac{x^2}{4}, x^2 + \theta\Omega^b_{\mathcal{P}}(x, y)\Big). \qquad (3.37)
\end{aligned}$$

Here $\theta \geq 0$ and

$$\begin{aligned}
\Omega^b_{\mathcal{P}}(x, y) &= \min\Big\{\mathcal{P}_b(x,\mathcal{U}x), \mathcal{P}_b(y,\mathcal{U}y), \mathcal{P}_b(x,\mathcal{U}y), \\
&\qquad \mathcal{P}_b(y,\mathcal{U}x)\Big\} \\
&= \min\Big\{x^2, y^2, x^2, y^2\Big\}.
\end{aligned}$$

Since $x, y \in [0, 1]$, we have
$$\Omega^b_{\mathcal{P}}(x, y) = 0.$$
From equations (3.37) and (3.38), we have

$$\begin{aligned}
&\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x, y) \\
&\qquad +\theta\Omega^b_{\mathcal{P}}(x, y)\Big) \\
&= \Big(\frac{x^2}{4}, x^2\Big). \qquad (3.38)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\zeta\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x, y) \\
&\qquad +\theta\Omega^b_{\mathcal{P}}(x, y)\Big) \\
&= \zeta\Big(\frac{x^2}{4}, x^2\Big) \\
&= \lambda\, x^2 - \frac{x^2}{4}.
\end{aligned}$$

If we take $\lambda = \frac{1}{2}$, we get

$$\begin{aligned}
&\zeta\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)), \mathcal{P}_b(x, y) \\
&\qquad +\theta\Omega^b_{\mathcal{P}}(x, y)\Big)
\end{aligned}$$

$$= \frac{1}{2}x^2 - \frac{x^2}{4} \geq 0,$$

that is,

$$\zeta\Big(\alpha(\mathcal{U}(x),\mathcal{U}(y))\beta(\mathcal{U}(x),\mathcal{U}(y))\mathcal{P}_b(\mathcal{U}(x),\mathcal{U}(y)),\mathcal{P}_b(x,y) \\ +\theta\Omega^b_{\mathcal{P}}(x,y)\Big) \geq 0.$$

Also, let $\{e_n\}$ be a sequence in Ξ such that $\alpha(e_n, e_{n+1}) \geq 1$, $\beta(e_n, e_{n+1}) \geq 1$ for all n and $e_n \to u \in \Xi$. Then, $\{e_n\} \subset [0,1]$ and $[\max\{e_n, u\}]^2 \to u^2$ as $n \to \infty$. Thus, $e_n \to u$ as $n \to \infty$ in $(\Xi, |\cdot|)$. This implies that $x \in [0,1]$ and so $\alpha(e_n, u) = 1$ and $\beta(e_n, u) = 1$ for all n. Moreover, there exists $e_0 \in \Xi$ such that $\alpha(e_0, \mathcal{U}e_0) \geq 1$ and $\beta(e_0, \mathcal{U}e_0) \geq 1$. In fact, for $e_0 = 1$, we have $\alpha(1, \mathcal{U}1) = \alpha(1, 1/4) = 1 = \beta(1, \mathcal{U}1)$. Thus, all the assumptions of Theorem 3.2 are verified. Hence, $u = 0$ is the unique fixed point of $\mathcal{U}$.

Example 3.11. Let $\Xi = \{1,2,3,4\}$ and $\mathcal{P}_b \colon \Xi \times \Xi \to \mathbb{R}$ be defined by

$$\mathcal{P}_b(x,y) = \begin{cases} |x-y|^2 + \max\{x,y\}, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1, \end{cases}$$

for all $x, y \in \Xi$. Then $(\Xi, \mathcal{P}_b)$ is a complete partial b-metric space with the coefficient $s = 4 > 1$.

Define the mapping $\mathcal{U} \colon \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{U}(1) = 1, \mathcal{U}(2) = 1, \mathcal{U}(3) = 2, \mathcal{U}(4) = 2.$$

Now, we have

$$\mathcal{P}_b(\mathcal{U}(1),\mathcal{U}(2)) = \mathcal{P}_b(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}\mathcal{P}_b(1,2),$$

$$\mathcal{P}_b(\mathcal{U}(1),\mathcal{U}(3)) = \mathcal{P}_b(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}\mathcal{P}_b(1,3),$$

$$\mathcal{P}_b(\mathcal{U}(1),\mathcal{U}(4)) = \mathcal{P}_b(1,2) = 3 \leq \frac{3}{4}.13 = \frac{3}{4}\mathcal{P}_b(1,4),$$

$$\mathcal{P}_b(\mathcal{U}(2),\mathcal{U}(3)) = \mathcal{P}_b(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}\mathcal{P}_b(2,3),$$

$$\mathcal{P}_b(\mathcal{U}(2),\mathcal{U}(4)) = \mathcal{P}_b(1,2) = 3 \leq \frac{3}{4}.8 = \frac{3}{4}\mathcal{P}_b(2,4),$$

$$\mathcal{P}_b(\mathcal{U}(3),\mathcal{U}(4)) = \mathcal{P}_b(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}\mathcal{P}_b(3,4).$$

Thus, $\mathcal{U}$ satisfies all the conditions of Corollary 3.5 with $\lambda = \frac{3}{4} < 1$. Now by applying Corollary 3.5, $\mathcal{U}$ has a unique fixed point namely $u = 1 \in \Xi$ is the unique fixed point of $\mathcal{U}$.

4. Application to the BVP

This section contains an existence theorem for the solution to the following boundary value problem (BVP):

$$-\frac{d^2\bar{y}}{dt^2} = g(t, \bar{y}(t));\ \forall\, t \in [0,1] = J,\ \bar{y}(0) = \bar{y}(1) = 0. \tag{4.1}$$

The associated Green's function $\mathcal{G}\colon J \times J \to J$ to (4.1) can be defined as follows:

$$\mathcal{G}(t,s) = \begin{cases} t(1-w), & \text{if } 0 \le t \le w \le 1, \\ w(1-t), & \text{if } 0 \le w \le t \le 1. \end{cases}$$

Let $C(J)$ represents the set of all continuous functions defined on J. Let the mapping $b\colon C(J) \times C(J) \to [0, +\infty)$ be defined by

$$b(f,h) = \|(f-h)^2\| = \sup |f(t) - h(t)|^2,\ \forall\, f, h \in C(J),\ \text{and}\, t \in J.$$

The pair $(C(J), b)$ is a complete b-metric space with $s = 2$. The associated integral operator $\mathcal{R}\colon C(J) \to C(J)$ to (4.1) is defined by:

$$\mathcal{R}(f)(t) = \int_0^1 \mathcal{G}(t,b) g(b, f(b)) db.$$

It is remarked that the fixed point of the operator $\mathcal{R}$ is a solution of (4.1). The following theorem states the condition under which the BVP has a solution.

Theorem 4.1. *Let the function $g\colon J \times C(J) \to \mathbb{R}$ is continuous and satisfies the following conditions:*

(1)

$$|g(t,f) - g(t,h)|^2 \le \phi\Big(|f(t) - h(t)|^2\Big),$$

for all $t \in J$ and $f, h \in C(J)$, where $\phi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing upper semi-continuous function with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

(2) $\Big(\sup \int_0^1 \mathcal{G}(t,b) db\Big)^2 \le 1$ *for all $t \in J$.*

Then the BVP (4.1) has a solution.

Proof. The proof of Theorem 4.1 will be done by the application of Corollary 3.7. Since, the function g is continuous, so, the operator $\mathcal{R}\colon C(J) \to C(J)$ defined above is continuous. To show that the function $\mathcal{R}$ form a contraction, we now proceed as follows:

$$\begin{aligned} |\mathcal{R}(f)(t) - \mathcal{R}(h)(t)|^2 &= \Big| \int_0^1 \mathcal{G}(t,b)[g(b,f) - g(b,h)] db \Big|^2 \\ &\le \Big(\int_0^1 \mathcal{G}(t,b) \sqrt{\Big|\phi\Big(|f(t) - h(t)|^2\Big)\Big|} db \Big)^2. \end{aligned}$$

Using condition (2) and taking supremum on both sides of above inequality, we have

$$\mathcal{P}_b(\mathcal{R}(f), \mathcal{R}(h)) \le \phi(b(f,h))\, \forall\, f, h \in C(J).$$

Now for any partial b-metric $\mathcal{P}_b$ on $C(J)$, we can have a b-metric b on $C(J)$ by

$$b(f,h) = \begin{cases} \mathcal{P}_b(f,h), & \text{if } f \neq h, \\ 0, & \text{if } f = h. \end{cases}$$

The last inequality can be written as:

$$\mathcal{P}_b(\mathcal{R}(f), \mathcal{R}(h)) \leq \phi\big(\mathcal{P}_b(f,h)\big) \, \forall \, f, h \in C(J).$$

Hence, by Corollary 3.7, the BVP (4.1) has a solution in $C(J)$. □

5. Conclusion

In this paper, we investigate the existence and uniqueness of a fixed point of almost $\mathcal{Z}$-contractions via simulation function in complete partial b-metric spaces using (α, β)-admissibility. Also, some illustrative examples are given to validate the results. Moreover, an application to the BVP is given. Our results extend and generalize several previously published results from the existing literature.

Acknowledgement. I would like to thank the anonymous learned referee and the Editor-in-Chief for their careful reading and valuable suggestions that helped us to improve the present work.

References

1. S. Alizadeh, F. Moradlou and P. Salimi, *Some fixed point results for $(\alpha, \beta) - (\psi, \phi)$-contractive mappings*, Filomat **28(3)** (2014), 635-647. https://doi.org/10.2298/FIL1403635A
2. S. Anwar, M. Nazam, Hamed H Al Sulami, A. Hussain, K. Javed and M. Arshad, *Existence fixed-point theorems in the partial b-metric spaces and an application to the boundary value problem*, AIMS Math. **7(5)** (2022), 8188-8205. doi:10.3934/math.2022456
3. H. Argoubi, B. Samet and C. Valero, *Nonlinear contraction involving simulation function in a metric space with a partial order*, J. Nonlinear Sci. Appl. **8** (2015), 1082-1094.
4. H. Aydi and E. Karapinar, *Fixed point results for generalized $\alpha - \psi$-contractions in metric-like spaces and applications*, Electro. J. Differential Equations, **133(2015)** (2015), 1-15. http://ejde.math.txstate.edu
5. H. Aydi and A. Felhi, *Fixed points in modular spaces via α-admissible mappings and simulation functions*, J. Nonlinear Sci. Appl. **9** (2016), 3686-3701. http://dx.doi.org/10.22436/jnsa.009.06.20
6. G. V. R. Babu, M. L. Sandhya and M. V. R. Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. **24(1)** (2008), 8-12.
7. I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. Gos. Ped. Inst. Unianowsk **30** (1989), 26-37.
8. S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3** (1922), 133-181.
9. V. Berinde, *Approximating fixed point of weak contractions using the Picard iteration*, Nonlinear Anal. Forum **9(1)** (2004), 43-53.
10. V. Berinde, *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math. **24(2)** (2008), 10-19.
11. S. Chandok, *Some fixed point theorems for (α, β)-admissible Geraghty type contractive mappings and related results*, Math. Sci. **9(3)** (2015), 127-135. DOI:10.1007/s40096-015-0159-4
12. S. H. Cho, *Fixed point theorem for (α, β)-z contraction in metric spaces*, Int. J. Math. Anal. **13(4)** (2019), 161-174. https://doi.org/10.12988/ijma.2019.9318

13. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Mathematica Et Informatica Universitatis Ostraviensis **1** (1993), 5-11. http://dml.cz/dmlcz/120469
14. S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti. Semin. Mat. Fis. Univ. Modena **46** (1998), 263-276.
15. N. V. Dung and Vo Thi Le Hang, *Remarks on partial b-metric spaces and fixed point theorems*, Mat. Vesnik **69(4)** (2017), 231-240.
16. K. Goebel and W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, 1990.
17. N. Hussain, E. Karapinar, P. Salimi and F. Akbar, *α-admissible mappings and related fixed point theorems*, J. Inequal. Appl. **2013** (2013), 114.
18. H. Isik, N. B. Gungor, C. Park and S. Y. Jang, *Fixed point theorems for almost $\mathcal{Z}$-contraction with an application*, Mathematics **6(3), 37** (2018), 1-8. http://doi.org/10.3390/math6030037
19. F. Kamalov and H. H. Leung, *Fixed point theory: A review*, https://doi.org/10.48550/arXiv.2309.03226
20. E. Karapinar, P. Kumam and P. Salimi, *On $\alpha-\psi$-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **94** (2013), 12 pages.
21. E. Karapinar, *$\alpha-\psi$-Geraghty contraction type mappings and some related fixed point results*, Filomat **28(1)** (2014), 37-48. DOI:10.2298/FIL1401037K
22. M. A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal. **73(9)** (2010), 3123-3129. https://doi.org/10.1016/j.na.2010.06.084
23. F. Khojasteh, S. Shukla and S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat **29** (2015), 1189-1194. DOI:10.2298/FIL1506189K
24. W. A. Kirk and M. A. Khamsi, An introduction to metric spaces and fixed point theory, New York: John Wiley, 2001.
25. W. A. Kirk and B. Sims, Handbook of metric fixed point theory, Springer, 2001.
26. S. G. Matthews, *Partial metric topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.
27. S. G. Matthews, Partial metric topology, *Proceedings of the 8th summer conference on topology and its applications*, Annals of the New York Academy of Sciences, **728** (1994), 183-197.
28. Z. Mustafa, J. R. Roshan, V. Parvaneh and Z. Kadelburg, *Some common fixed point results in ordered partial b-metric spaces*, J. Ineualities Appl. 2013, **2013:562**.
29. A. Nastasi and P. Vetro, *Fixed point results on metric and partial metric spaces via simulation functions*, J. Nonlinear Sci. Appl. **8** (2015), 1059-1069.
30. P. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal. (TMA), **64(3)** (2006), 546-557. https://doi.org/10.1016/j.na.2005.04.044
31. H. Qawaqneh, M. S. M. Noorani, W. Shatanawi and H. Alsamir, *Common fixed points for pairs of triangular α-admissible mappings*, J. Nonlinear Sci. Appl. **10(12)** (2017), 6192-6204. https://dx.doi.org/10.22436/jnsa.010.12.06
32. H. Qawaqneh, M. S. Noorani, W. Shatanawi, K. Abodayeh and H. Alsamir, *Fixed point for mappings under contractive condition based on simulation functions and (α, β)-admissibility*, J. Math. Anal. **9(1)** (2018), 38-51.
33. H. Qawaqneh, *New contraction embedded with simulation function and cyclic (α, β)-admissible in metric-like spaces*, Int. J. Math. Comput. Sci. **15(1)** (2020), 1029-1044.
34. A. Roldán, E. Karapinar, C. Roldán and J. Martinez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345-355. DOI:10.1016/j.cam.2014.07.011
35. V. L. A. Rosa and P. Vetro, *Common fixed points for α, ψ, ϕ-contractions in generalized metric spaces*, Nonlinear Anal. Model. Control **19(1)** (2014), 43-54. https://doi.org/10.15388/NA.2014.1.3
36. B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for $(\alpha-\psi)$-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154-2165.

37. A. Shoaib, M. Arshad and M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers type on a closed ball in ordered partial metric spaces*, J. Comput. Anal. Appl. **17** (2014), 255-264.
38. A. Shoaib, $\alpha - \eta$ *dominated mappings and related common fixed point results in closed ball*, J. Concrete Appl. Math. **13** (2015), 152-170.
39. S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math. **11** (2014), 703-711. DOI:10.1007/s00009-013-0327-4.
40. J. Zhou, D. Zheng and G. Zhang, *Fixed point theorems in partial b-metric spaces*, Appl. Math. Sci. **12** (2018), No. 13, 617-624. https://doi.org/10.12988/ams.2018.8460

H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India.
Email address: saluja1963@gmail.com