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STABILITY ANALYSIS OF LINEAR QUATERNION-VALUED DIFFERENTIAL EQUATION USING INTEGRAL TRANSFORM

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ABSTRACT. In this article, we examine the stability of first-order linear quaternion-valued differential equations using the Mittag-Leffler-Hyers-Ulam approach. We achieve this by transforming a linear quaternion-valued differential equation into a real differential system. The stability outcomes for these linear quaternion-valued differential equations are determined through the use of quaternion module and Fourier transform techniques.

1. INTRODUCTION

A quaternion is a mathematical concept that extends the idea of complex numbers into a higher dimensional space. A quaternions are a four-dimensional extension of complex numbers, and they have proven to be useful in various fields, especially in representing 3D rotations and orientations.

Quaternion-valued differential equations involve the use of quaternions in the context of differential equations. Differential equations describe how quantities change in relation to each other, often involving variables like time or space. When quaternions are used in these equations, it is often in the context of modeling systems that have both rotational and translational components, such as rigid body motion.

Quaternion-valued differential equations can describe the dynamics of objects that undergo rotations and translations simultaneously, which can be quite common in fields like robotics [15], aerospace engineering [20], and physics [10, 3]. They provide a concise and elegant way to represent and solve problems involving rotational motion and orientation changes. These equations can be quite complex due to the algebraic properties of quaternions and the interactions between their components. Solving quaternion differential equations often requires specialized mathematical techniques and software tools. However, their benefits in accurately modeling real-world scenarios involving 3D rotations make them a valuable tool in various applications.

Advances in the fundamental theory of quaternion-valued differential equations (QDEs) have been made in recent years by a number of academics. For example,

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Kou and Xia [12] examined the solutions of linear QDEs and proposed two new techniques for calculating the basic matrix. They also introduced the idea of the Liouville formula and Wronskian in the context of quaternions. A method for figuring out the basic matrix of linear systems with many eigenvalues was presented by Kou et al. [13]. In the meantime, Xia et al. [22] established the variation of constants formula in the quaternion sense, gave stability results for quaternion periodic systems, and presented an algorithm for solving linear nonhomogeneous QDEs. Suo et al. [19] simultaneously presented solutions in both the complex numbers and quaternion settings for linear quaternion-valued impulsive differential equations. Additionally, they investigated periodic solutions for quaternion-valued linear impulsive differential equations that are homogeneous and nonhomogeneous.

Chen et al. [2] developed a new method for analyzing the controllability and observability of linear quaternion-valued systems and used Laplace transforms to prove the Hyers-Ulam stability of linear QDEs. Using delayed quaternion matrix exponentials and constant variation, Fu et al. [4] found solutions for homogeneous and nonhomogeneous linear QDEs under the permutation matrix hypothesis. While Feckan et al. [5] investigated the Hyers-Ulam stability of linear recurrence equations with constant coefficients in the quaternion sense, Lv et al. [14] examined the Hyers-Ulam stability of linear QDEs using Fourier transforms. Furthermore, Zahid et al. [21] computed the exponential matrix of QDEs, and Huang et al. [6] examined the stability of QDEs in the context of quaternions using the second Lyapunov technique.

The MLHU stability involves studying the stability properties of solutions to functional equations that include Mittag-Leffler functions. The MLHU stability is a mathematical framework that merges the Mittag-Leffler function and the Hyers-Ulam stability concept. This stability analysis is based on the Hyers-Ulam stability principle, which examines how small changes in the equations lead to small changes in the solutions. It is used to study the behavior and stability of solutions to differential equations or functional equations that have Mittag-Leffler functions or similar behaviors in their solutions. This stability concept has applications in various fields where complex relaxation dynamics, fractional calculus, and stability analysis are relevant.

Jung's significant contribution, highlighted in [7], lies in establishing the Hyers-Ulam stability for first-order differential systems with constant coefficients using the matrix method. Additionally, Jung, as noted in [8, 9], showcased the generalized Hyers-Ulam stability, extending its applicability to both differential equations and first-order matrix differential equations. Mohanapriya et al. [16] conducted a comprehensive study on the MLHU stability of linear differential equations of first, second, and nth order, employing the Fourier transform method. Following this, Rassias et al. [18] adopted the Fourier transform method to investigate the MLHU stability of linear differential equations, offering an alternative approach to address such problems.Inspired by the works of Jung [7, 8, 9] and Rassias [18], our research investigates the MLHU stability of first-order linear QDEs through a distinct approach, as outlined in [16, 18].

In this present study, we study the MLHU stability of homogeneous QDE given by

$$\aleph'(\psi) = \delta\aleph(\psi), \quad \psi \in R, \tag{1.1}$$

and non-homogeneous QDEs given by

$$\aleph'(\psi) = \delta\aleph(\psi) + u(\psi), \quad \psi \in R, \tag{1.2}$$

where $\delta = a + bi + cj + dk \in H$ is a quaternion constant, $u : R \to H$ and $\aleph : R \to H$ are continuous and continuously differentiable functions respectively.

2. Preliminaries

Throughout this paper, we present H be a quaternion set and F be the complex or real field.

A quaternion function is a mathematical function that operates on quaternions, which are a four-dimensional extension of complex numbers. Quaternions are represented as a combination of a scalar part (real number) and a vector part (three-dimensional vector). Quaternion functions take quaternion values as inputs and produce quaternion values as outputs. Mathematically, a quaternion can be written as:

$$q = a + bi + cj + dk,$$

where a is the scalar part (real number), b, c, d are the vector components (real numbers), i, j, k are the quaternion units, each of which satisfies $i^2+j^2+k^2 = -1$, and they obey the rules of multiplication of quaternion algebra. A quaternion function $\aleph : R \to H$ might look like:

$$\aleph(\psi) = \aleph_1(\psi) + \aleph_2(\psi)i + \aleph_3(\psi)j + \aleph_4(\psi)k.$$

The derivative of $\aleph(\psi)$ is

$$\aleph'(\psi) = \aleph_1^{'}(\psi) + \aleph_2^{'}(\psi)i + \aleph_3^{'}(\psi)j + \aleph_4^{'}(\psi)k. \tag{2.1}$$

Denote $H \otimes R$ be the set of quaternion-valued function and a real matrix A (4×4) by

$$A = aE + B, \tag{2.2}$$

where

$$B = \begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -d & c \\ c & d & 0 & -b \\ d & c & b & 0 \end{pmatrix}, \tag{2.3}$$

E is a unit matrix and $b, c, d \in R$ are not all zero.

Lemma 2.1. [17] *If A is a real matrix (2.3), then the characteristic polynomial $P(\nu) = \nu^4 + a_2\nu^2 + a_0$, where $a_2 = 2(b^2 + c^2 + d^2)$, $a_0 = (b^2 + d^2 - c^2)$.*

Corollary 2.2. [17] *The eigen value of* (2.3) *are*

$$\nu_1 = \pm i\sqrt{\left|\frac{a_2 - \sqrt{a_2^2 - 4a_0}}{2}\right|}$$

$$\nu_2 = -i\sqrt{\left|\frac{a_2 - \sqrt{a_2^2 - 4a_0}}{2}\right|}$$

$$\nu_3 = +i\sqrt{\left|\frac{a_2 + \sqrt{a_2^2 - 4a_0}}{2}\right|}$$

$$\nu_4 = -i\sqrt{\left|\frac{a_2 + \sqrt{a_2^2 - 4a_0}}{2}\right|}$$

Theorem 2.3. [17] *The eigen value of a matrix A in* (2.2) *are*

$$\delta_{A1} = a + \delta_1, \quad \delta_{A2} = a + \delta_2,$$
$$\delta_{A3} = a + \delta_3, \quad \delta_{A4} = a + \delta_4.$$

Lemma 2.4. *For any matrix $A \in R^{n\times n}$, $|exp(A\psi)| = exp(\delta_{\max}\psi)$ for $\psi \in R$, where $\delta_{\max}$ is largest eigen value of A.*

Proof.

$$\begin{aligned} \left|e^{At}\right| &= \left|\sum_{n=0}^{\infty}\frac{A^n\psi^n}{n!}\right| \\ &\le \sum_{n=0}^{\infty}\frac{\psi^n}{n!}|A^n| \\ &\le \sum_{n=0}^{\infty}\psi^n\frac{\delta_{\max}}{n!} \\ &= e^{\delta_{\max}\psi}, \quad \psi \in R. \end{aligned}$$

The proof is complete. □

Definition 2.5. If a function $\aleph : R \to H$ is absolutely integrable in R and piecewise continuous in each finite interval, then the Fourier transform associated to $\aleph \in L^1(R)$ is a mapping $\widehat{F}(\varphi) : R \to H$ is given by the integral

$$\widehat{F}(\varphi) = \int_{-\infty}^{\infty}\mathcal{F}(\psi)e^{i\varphi\psi}d\psi, \quad \forall\psi \in R. \tag{2.4}$$

The inverse Fourier transform associated to $\widehat{\mathcal{F}}(\varphi)$ is given by

$$F(\psi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\mathcal{F}}(\varphi)e^{-i\varphi\psi}d\varphi, \tag{2.5}$$

for any $\psi \in R$, the relation $F^{-1}F(\aleph) = \aleph$ holds true almost every where on R.

Lemma 2.6. [11] *Let* $\aleph \in L^1(R)$, $F(\aleph)(\psi) = \widehat{\aleph}(\varphi)$, *and* $\theta(\psi)$ *be the Heaviside step function defined by* $\theta(\psi) = 1$ *for* $\psi \geq 0$ *and* $\theta(\psi) = 0$ *for* $\psi < 0$, *then*

(1) $F(\aleph(\psi \pm a)) = e^{\pm ia}\widehat{\aleph}(\varphi)$
(2) $F\left(e^{-z\psi}\theta(\psi)\right)(\varphi) = \frac{1}{i\varphi+z}$ *provided that* $Re(z) > 0$
(3) $F((-i\psi)^n(\aleph(\psi)))(\varphi) = \widehat{\aleph}^n(\varphi)$
(4) $F((\aleph)^n(\psi))(\varphi) = (-i\varphi)^n\widehat{\aleph}(\varphi)$

The convolution of two functions $\aleph_1(\psi)$ and $\aleph_2(\psi)$ is defined as

$$\aleph_1(\psi) * \aleph_2(\psi) = \int_{-\infty}^{\infty} \aleph_1(\tau)\aleph_2(\psi - \tau)d\tau.$$

Theorem 2.7. [11] *Let* $\aleph_1, \aleph_2 \in L^1(R)$ *then*

(1) $F(\aleph_1 * \aleph_2) = F(\aleph_1)F(\aleph_2)$
(2) $F^{-1}(\aleph_1\aleph_2) = F^{-1}(\aleph_1) * F^{-1}(\aleph_2)$.

Notice that if $\theta(\psi)$ is the Heaviside step function then

$$(\aleph * \theta)(\psi) = (\theta * \aleph)(\psi) = \int_{-\infty}^{\infty} \theta(\psi - \tau)\aleph(\tau)d\tau = \int_{0}^{\infty} \aleph(\tau)d\tau.$$

Definition 2.8. The Mittag-Leffler function of one and two parameters are defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\alpha k + 1} z^k; \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\alpha k + \beta} z^k,$$

where $Re(\alpha) > 0$ and $z, \alpha \in C$.

3. Stability of QDE

In this section, we consider the MLHU stability of QDE with real co-efficient and quaternion co-efficient.

3.1. **Stability of linear QDE with real co-efficient.** Consider the QDE

$$\aleph'(\psi) = \delta\aleph(\psi), \quad \psi, \delta \in R, \tag{3.1}$$

For $\epsilon > 0$, we consider the inequality

$$|\aleph'(\psi) - \delta\aleph(\psi)| \leq \epsilon E_\alpha(\psi^\alpha), \quad \psi \in R. \tag{3.2}$$

Definition 3.1. The QDE (3.1) is said to be MLHU stable if there exist $K \in R^+$ and for every solution $\aleph : R \to H$ of the inequality (3.2) there exist a solution $\aleph_\alpha : R \to H$ of QDE with

$$|\aleph(\psi) - \aleph_\alpha(\psi)| \leq K\epsilon E_\alpha(\psi^\alpha), \quad \psi \in R.$$

Further, the QDE is generalized MLHU stable if we can find a function $\Psi : R \to H$ with $\Psi(0) = 0$ such that $|\aleph(\psi) - \aleph_\alpha(\psi)| \leq \Psi(\epsilon)E_\alpha(\psi^\alpha)$, $\psi \in R$.

Theorem 3.2. *If* $\delta \in R$, *then the continuously differentiable function* $\aleph : R \to H$ *satisfying* $|\aleph'(\psi) - \delta\aleph(\psi)| \leq \epsilon E_\alpha(\psi^\alpha)$, $\psi \in R$, *for any* $\epsilon > 0$, *there exists a solution* $\aleph_0 : R \to H$ *of* (3.1) *such that* $|\aleph(\psi) - \aleph_0(\psi)| \leq K\epsilon E_{\alpha,2}(\psi^\alpha)$, $\psi \in R$. *That is, QDE* (3.1) *is MLHU stable.*

Proof. Let

$$v(\psi) = \aleph'(\psi) - \delta\aleph(\psi). \tag{3.3}$$

Then $|v(\psi)| \leq \epsilon E_\alpha(\psi^\alpha)$ and $\aleph'(\psi) = v(\psi) + \delta\aleph(\psi)$. By (2.1), we have

$$\begin{aligned}
\aleph'(\psi) &= \aleph_1'(\psi) + \aleph_2'(\psi)i + \aleph_3'(\psi)j + \aleph_4'(\psi)k \\
&= \delta\left(\aleph_1(\psi) + \aleph_2(\psi)i + \aleph_3(\psi)j + \aleph_4(\psi)k\right) + v_1(\psi) + v_2(\psi)i + v_3(\psi)j + v_4(\psi)k \\
&= \left(\delta\aleph_1(\psi) + v_1(\psi)\right) + \left(\delta\aleph_2(\psi) + v_1(\psi)\right)i + \left(\delta\aleph_3(\psi) + v_3(\psi)\right)j + \left(\delta\aleph_4(\psi) + v_4(\psi)\right)k.
\end{aligned}$$

The real and imaginary parts of two quaternoin-valued functions are equivalent if they are equal. Therefore,

$$\begin{aligned}
&\aleph_l'(\psi) = \delta\aleph_l(\psi) + v_l(\psi), \\
&v_l(\psi) = \aleph_l'(\psi) - \delta\aleph_l(\psi) \quad l = 1, 2, 3, 4.
\end{aligned} \tag{3.4}$$

By taking the Fourier transform of (3.4), we have

$$\hat{V}_l(\varphi) = (-i\varphi)\hat{F}_l(\varphi) - \delta\hat{F}_l(\varphi). \tag{3.5}$$

The method of variation of constant gives the unique solution of (3.4), which is

$$\aleph_l(\psi) = C_l e^{\delta\psi} + F^{-1}\left(\hat{V}_l(\varphi)\frac{1}{i\varphi + \delta}\right).$$

Applying the properties of Fourier transform and the formula for convolution, we obtain

$$\begin{aligned}
\aleph_l(\psi) &= C_l e^{\delta\psi} + v_l(\psi)e^{\delta\psi}\theta(\psi) \\
&= C_l e^{\delta\psi} + \int_{-\infty}^{\infty} v_l(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau, \quad C_l = \aleph_l(0) \in R, \ l = 1, 2, 3, 4.
\end{aligned}$$

We derive the solution of (3.3), we obtain

$$\begin{aligned}
\aleph(\psi) &= \aleph_1(\psi) + \aleph_2(\psi)i + \aleph_3(\psi)j + \aleph_4(\psi)k \\
&= C_1 e^{\delta\psi} + \int_{-\infty}^{\infty} v_1(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau \\
&\quad + \left(C_2 e^{\delta\psi} + \int_{-\infty}^{\infty} v_2(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau\right)i \\
&\quad + \left(C_3 e^{\delta\psi} + \int_{-\infty}^{\infty} v_3(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau\right)j \\
&\quad + \left(C_4 e^{\delta\psi} + \int_{-\infty}^{\infty} v_4(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau\right)k \\
&= e^{\delta\psi}q + \int_{-\infty}^{\infty} v(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau,
\end{aligned} \tag{3.6}$$

where $q = C_1 + C_2 i + C_3 j + C_4 k = \aleph(0) \in H$. Let

$$\aleph_0(\psi) = e^{\delta\psi}q. \tag{3.7}$$

Obviously, $\aleph_0(\psi)$ is a solution of (3.1). It follows from (3.6) and (3.7) that,

$$\begin{aligned}
|\aleph(\psi)-\aleph_0(\psi)| &\leq \left|\int_{-\infty}^{\infty} v(\tau)e^{\delta(\psi-\tau)}\theta(\psi-\tau)d\tau\right| \\
&\leq \int_0^{\psi} |v(\tau)|\left|e^{\delta(\psi-\tau)}\right| d\tau \\
&\leq K\epsilon\sum_{k=0}^{\infty}\frac{1}{\sqrt{\alpha k+1}}\int_0^{\psi}\tau^{\alpha k}d\tau \\
&\leq K\epsilon\sum_{k=0}^{\infty}\frac{\psi^{\alpha k+1}}{\sqrt{\alpha k+1}(\alpha k+1)} \\
&\leq K\epsilon E_{\alpha,2}(\psi^{\alpha}).
\end{aligned}$$

□

Remark 3.3. The QDE (3.1) is not MLHU stable when $\delta \neq 0$

3.2. Stability of linear QDE with Quaternion co-efficient.

Theorem 3.4. *If $\delta \in H$, then the QDE*

$$\aleph'(\psi) = \delta\aleph(\psi),\ \ \psi \in R \tag{3.8}$$

is MLHU stable.

Proof. Let $v(\psi) = \aleph'(\psi) - \delta\aleph(\psi),\ \delta \in H$. Then $|v(\psi)| \leq \epsilon E_\alpha(\psi^\alpha)$ and $\aleph'(\psi) = v(\psi) + \delta\aleph(\psi)$.

By (2.1), we have

$$\begin{aligned}
\aleph'(\psi) = \aleph_1'(\psi)+\aleph_2'(\psi)i+\aleph_3'(\psi)j+\aleph_4'(\psi)k &= (a+bi+cj+dk)\left(\aleph_1(\psi)+\aleph_2(\psi)i+\aleph_3(\psi)j+\aleph_4(\psi)k\right) \\
&\quad + \left(v_1(\psi)+v_2(\psi)i+v_3(\psi)j+v_4(\psi)k\right) \\
&= a\aleph_1(\psi)-b\aleph_2(\psi)-c\aleph_3(\psi)-d\aleph_4(\psi) \\
&\quad + \left(b\aleph_1(\psi)+a\aleph_2(\psi)-d\aleph_3(\psi)+c\aleph_4(\psi)\right)i \\
&\quad + \left(c\aleph_1(\psi)+d\aleph_2(\psi)+a\aleph_3(\psi)-b\aleph_4(\psi)\right)j \\
&\quad + \left(d\aleph_1(\psi)-c\aleph_2(\psi)+b\aleph_3(\psi)+a\aleph_4(\psi)\right)k \\
&\quad + v_1(\psi)+v_2(\psi)i+v_3(\psi)j+v_4(\psi)k.
\end{aligned} \tag{3.9}$$

Obviously, (3.9) is equivalent to the linear differential system

$$\begin{pmatrix}\aleph_1'\\ \aleph_2'\\ \aleph_3'\\ \aleph_4'\end{pmatrix} = \begin{pmatrix} a & -b & -c & -d\\ b & a & -d & c\\ c & d & a & -b\\ d & -c & b & a\end{pmatrix}\begin{pmatrix}\aleph_1\\ \aleph_2\\ \aleph_3\\ \aleph_4\end{pmatrix} + \begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix} \tag{3.10}$$

Let $y = (\aleph_1\ \aleph_2\ \aleph_3\ \aleph_4)^T$, $v = (v_1\ v_2\ v_3\ v_4)^T$ and $A = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$. Then the system (3.10) can be written as

$$\begin{aligned} y'(\psi) &= Ay(\psi) + v(\psi) \\ v(\psi) &= y'(\psi) - Ay(\psi). \end{aligned} \tag{3.11}$$

Using Fourier transform of (3.11), we arrive

$$\hat{V}(\varphi) = (-i\varphi)\hat{Y}_l(\varphi) - A\hat{Y}_l(\varphi). \tag{3.12}$$

By the method of variation, equation (3.12) gives the unique solution

$$y(\psi) = Ce^{A\psi} + F^{-1}\left(\hat{V}(\varphi)\frac{1}{i\varphi + A}\right).$$

By utilizing the convolution formula and the Fourier transform properties, we obtain

$$\begin{aligned} y(\psi) &= Ce^{A\psi} + v(\psi)e^{-At}\theta(\psi) \\ &= Ce^{At} + \int_{-\infty}^{\infty} v(\tau)e^{A(\psi-\tau)}\theta(\psi-\tau)d\tau, \end{aligned} \tag{3.13}$$

where $C = y(0) = (\aleph_1(0)\ \aleph_2(0)\ \aleph_3(0)\ \aleph_4(0))^T \in R^4$. In addition, QDE ((3.8)) can be represented as $y'(\psi) = Ay(\psi),\ \psi \in R$. Let

$$y_0(\psi) = Ce^{At},\ \psi \in R. \tag{3.14}$$

Obviously, $y_0 = (\aleph_1\ \aleph_2\ \aleph_3\ \aleph_4)^T$ is a solution of QDE ((3.8)). It follows from (3.13) and (3.14) and definition (3.1), we arrive

$$\begin{aligned} |y(\psi) - y_0(\psi)| &\leq \int_{-\infty}^{\infty} |v(\tau)| \left|e^{-A(\psi-\tau)}\theta(\psi-\tau)\right| d\tau \\ &\leq \epsilon \int_0^{\psi} e^{-\delta_{\max}(\psi-\tau)}E_\alpha(\tau)d\tau \\ &\leq \epsilon \sum_{k=0}^{\infty} \frac{1}{\sqrt{\alpha k + 1}} \int_0^{\psi} e^{-\delta_{\max}(\psi-\tau)}\tau^{\alpha k}d\tau \\ &\leq \frac{\epsilon}{\delta_{\max}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{\alpha k + 1}} \\ &\leq K\epsilon E_\alpha(\psi^\alpha),\ \psi \in R. \end{aligned}$$

□

4. Example

Example 4.1. Consider the initial value problem $\aleph'(\psi) = (1 + i - j + k)\aleph(\psi)$, $\aleph(0) = 1+j$, $\psi \in R$. Let $v(\psi) = \aleph'(\psi) - (1+i-j+k)\aleph(\psi)$. Then $|v(\psi) \leq \epsilon E_\alpha(\psi^\alpha)$ and

$$\aleph'(\psi) = (1 + i - j + k)\aleph(\psi) + v(\psi). \tag{4.1}$$

Equation (4.1) can be rewritten as

$$\begin{pmatrix} \aleph_1' \\ \aleph_2' \\ \aleph_3' \\ \aleph_4' \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \aleph_1 \\ \aleph_2 \\ \aleph_3 \\ \aleph_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{4.2}$$

The system (4.2) can be represented as

$$y'((\psi) = Ay(\psi) + v(\psi), \tag{4.3}$$

where $y' = (\aleph_1' \ \aleph_2' \ \aleph_3' \ \aleph_4')^T$, $y = (\aleph_1, \aleph_2, \aleph_3, \aleph_4)$, $v = (v_1 \ v_2 \ v_3 \ v_4)^T$ and $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ We can arrive that the eigen value of A are $\delta_1 = 1 + i\sqrt{3}$, $\delta_2 = 1 - i\sqrt{3}$, $\delta_3 = 1 + i\sqrt{3}$ and $\delta_4 = 1 - i\sqrt{3}$. We note that $e^{|\delta_{\max}|t} = e^{\sqrt{4}\psi}$.

By theorem 3.2, the solution of (4.3) is

$$y(\psi) = e^{A\psi}(1 \ 0 \ 1 \ 0)^T + \int_{-\infty}^{\infty} v(\tau)e^{-A(\psi-\tau)}\theta(\psi-\tau)d\tau. \tag{4.4}$$

Equation (4.1) can be represented as $y'(\psi) = Ay(\psi)$. Let

$$y_0(\psi) = e^{A\psi}(1 \ 0 \ 1 \ 0)^T \tag{4.5}$$

Obviously, $y_0 = (\aleph_1 \ \aleph_2 \ \aleph_3 \ \aleph_4)^T$ is a solution of (4.1). By theorem 3.2, QDE (4.1) has a solution and is MLHU stable with

$$\begin{aligned} |y(\psi) - y_0(\psi)| &\leq \left| \int_{-\infty}^{\infty} v(\tau)e^{-A(\psi-\tau)}\theta(\psi-\tau)d\tau \right| \\ &\leq \epsilon \int_0^{\psi} e^{-\delta_{\max}(\psi-\tau)} E_\alpha(\tau)d\tau \\ &\leq \epsilon \int_0^{\psi} e^{-\sqrt{4}(\psi-\tau)} E_\alpha(\tau)d\tau \\ &\leq \epsilon \sum_{k=0}^{\infty} \frac{1}{\sqrt{\alpha k+1}} \int_0^{\psi} e^{-\sqrt{4}(\psi-\tau)}\tau^{\alpha k}d\tau \\ &\leq \frac{\epsilon}{\sqrt{4}} \sum_{k=0}^{\infty} \frac{\psi^{\alpha k}}{\sqrt{\alpha k+1}} \\ &\leq K\epsilon E_\alpha(\psi^\alpha), \ \psi \in R, \end{aligned}$$

where $K = \dfrac{1}{\sqrt{4}}$.

Example 4.2. Consider the initial value problem $\aleph'(\psi) = (-i-j-k)\aleph(\psi) - ie^{i\psi}$, $\aleph(0) = 1$, $\psi \in R$. Let $v(\psi) = \aleph'(\psi) - (-i-j-k)\aleph(\psi) + ie^{i\psi}$, then $|v(\psi) \leq \epsilon E_\alpha(\psi^\alpha)$ and

$$\aleph'(\psi) = (-i-j-k)\aleph(\psi) + ie^{i\psi} + v(\psi). \tag{4.6}$$

The equation (4.6) can be rewritten as

$$\begin{pmatrix} \aleph_1' \\ \aleph_2' \\ \aleph_3' \\ \aleph_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \aleph_1 \\ \aleph_2 \\ \aleph_3 \\ \aleph_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 + e^{i\psi} \\ v_3 \\ v_4 \end{pmatrix} \tag{4.7}$$

The system (4.7) can be represented as

$$y'((\psi) = Ay(\psi) + w(\psi), \tag{4.8}$$

where $y' = (\aleph_1' \ \aleph_2' \ \aleph_3' \ \aleph_4')^T$, $y = (\aleph_1, \aleph_2, \aleph_3, \aleph_4)$, $w = (v_1 \ v_2 + e^{i\psi} \ v_3 \ v_4)^T$ and $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$ We can show the eigen value of A are $\delta_1 = i\sqrt{3}$, $\delta_2 = -i\sqrt{3}$, $\delta_3 = i\sqrt{3}$ and $\delta_4 = -i\sqrt{3}$. We note that $e^{|\delta_{\max}|\psi} = e^{\sqrt{4}\psi}$.

By theorem 3.3, the solution of (4.8) is

$$y(\psi) = e^{A\psi}(1\ 0\ 0\ 0)^T + \int_{-\infty}^{\infty} w(\tau)e^{-A(\psi-\tau)}\theta(\psi-\tau)d\tau. \tag{4.9}$$

Equation (4.6) can be represented as $y'(\psi) = Ay(\psi) + u(\psi)$. Let

$$y_0(\psi) = e^{A\psi}(1\ 0\ 0\ 0)^T + \int_{-\infty}^{\infty} u(\tau)e^{-A(\psi-\tau)}\theta(\psi-\tau)d\tau. \tag{4.10}$$

Obviously, $y_0 = (\aleph_1\ \aleph_2\ \aleph_3\ \aleph_4)^T$ is a solution of (4.1). By theorem 3.2, the solution of equation (4.6) is MLHU stable with

$$\begin{aligned} |y(\psi) - y_0(\psi)| &\leq \left| \int_{-\infty}^{\infty} v(\tau)e^{-A(\psi-\tau)}\theta(\psi-\tau)d\tau \right| \\ &\leq \epsilon \int_0^{\psi} e^{-\delta_{\max}(\psi-\tau)}E_\alpha(\tau)d\tau \\ &\leq \epsilon \int_0^{\psi} e^{-\sqrt{3}(\psi-\tau)}E_\alpha(\tau)d\tau \\ &\leq \epsilon \sum_{k=0}^{\infty} \frac{1}{\sqrt{\alpha k+1}} \int_0^{\psi} e^{-\sqrt{3}(\psi-\tau)}\tau^{\alpha k}d\tau \\ &\leq \frac{\epsilon}{\sqrt{3}} \sum_{k=0}^{\infty} \frac{\psi^{\alpha k}}{\sqrt{\alpha k+1}} \\ &\leq K\epsilon E_\alpha(\psi^\alpha), \end{aligned}$$

where $\psi \in R$ and $K = \dfrac{1}{\sqrt{3}}$. Notice that (4.6) is MLHU stable.

5. Conclusion

In conclusion, this article has explored the stability analysis of first-order linear QDEs through the application of the MLHU approach. Our approach involved transforming these equations into a real differential system. The assessment of stability for these linear QDEs was successfully carried out using quaternion module and Fourier transform techniques. This research contributes to a deeper understanding of the stability properties of QDEs, which has important implications in various scientific and engineering applications.

References

1. Bachmann, J.L., Marins, M.J., Zyda, R.B., Mcghee & Yun, X. (2011). An extended Kalman filter for quaternion-based orientation estimation using MARG sensors. IEEE/RSJ Intelligent Robotics and Systems, 4, 2001-2003. https://doi.org/10.1109/IROS.2001.976367
2. Chen, D., Feckan, M., & Wang, J. (2022). Investigation of controllability and observability for linear quaternion-valued systems from its complex-valued systems. Qualitative Theory of Dynamical Systems, 21, 66. https://doi.org/10.1007/s12346-022-00599-6
3. Chou, J.C.K. (1992). Quaternion kinematic and dynamic differential equation. IEEE Transaction on Robotics and Animation, 8(1), 53-64. https://doi.org/10.1109/70.127239
4. Fu, T., Kou,K., & Wang, T. (2022). Representation of solutions to linear quaternion differential equations with delay. Qualitative Theory of Dynamical Systems, 21, 118. https://doi.org/10.1007/s12346-022-00648-0
5. Feckan, T., and Wang, J. (2023). Hyers-Ulam-Rassias stability of linear recurrence over the quaternion skew field. Rocky Mountain Journal of Mathematics, 53(2), 661-670. http://dx.doi.org/10.1216/rmj.2023.53.661
6. Huang, S., & He, F. (2021). On the second Lyapunov method for quaternionic differential equations. Qualitative Theory of Dynamical Systems, 20, 41. https://doi.org/10.1007/s12346-021-00476-8
7. Jung, S.M. (2006). Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. Journal of Mathematical Analysis and Applications, 320, 549-561. https://doi.org/10.1016/j.jmaa.2005.07.032
8. Jung, S.M. (2015). Hyers-Ulam stability of the first-order matrix differential equations. Journal of Function Spaces, 2015, 614745. https://doi.org/10.1186/s13660-021-02670-3
9. Jung, S.M. (2004). Hyers-Ulam stability of linear differential equations of first order. Applied Mathematics Letters, 17, 1135-1140. https://doi.org/10.1016/j.aml.2003.11.004
10. Kartiwa, A., Supriatna, A,K., Rusyyaman, E., & Sulaiman. (2023). Review of quaternion differential equation: Historical development. Applications, and Direction, Axioms, 12(5). http://dx.doi.org/10.3390/axioms12050483
11. Katzenelson, Y. (2012) An Introduction to Harmonic Analysis, Cambridge University Press.https://www.mat.uniroma2.it/~picard/SMC/didattica/materiali_did/Anal.Armon./Katznelson/Katznelson.pdf
12. Kou, K., & Xia, Y. (2018). Linear quaternion differential equations: Basic theory and fundamental results. Studies in Applied Mathematics, 141, 3-45. https://doi.org/10.1111/sapm.12211
13. Kou, K., Liu, W., & Xia, Y. (2019). Solve the linear quaternion-valued differential equations having multiple eigenvalues. Journal of Mathematical Physics, 60, 023510. https://doi.org/10.1063/1.5040237
14. Lv, J., Kou, K., & Wang, J. (2022). Hyers-Ulam stability of linear quaternion-valued differential equations with constant cofficients via Fourier transform. Qualitative Theory of Dynamical Systems, 21, 116. https://doi.org/10.1007/s12346-022-00649-z
15. Montgomery-Smith, S. (2012). Robot Kinematics, using Dual Quaternions. IAES International Journal of Robotics and Automation, 1(1). http://dx.doi.org/10.11591/ijra.v1i1.275

16. Mohanapriya, A., Park, C., Ganesh, A., & Govindan, V. (2020). Mittag-Leffler-Hyers-Ulam stability of differential equation using Fourier transform, Advances in Difference Equation, 2020:389, 1-16. https://doi.org/10.1186/s13662-020-02854-z
17. Poditi, T. (2001). A formula for the exponential of a real skew-symmetric matrix of order 4, BIT Numerical Mathematics, 41, 842-845. https://doi.org/10.1023/A:1021960405660
18. Mohanapriya, A., Sivakumar, V., & Prakash, P. (2021). A generalized approach of fractional Fourier transform to stability of fractional differential equation. Korean Journal of Mathematics, 29(4), 749-763. https://doi.org/10.11568/kjm.2021.29.4.749
19. Suo, L., Feckan, M., & Wang, J. (2021). Quaternion-valued linear impulsive differential equations. Qualiitative Theory of Dynamical Systems, 20, 33. https://doi.org/10.1007/s12346-021-00467-9
20. Terze, Z., Zlatar, D., & Pandza, V. (2020). Aircraft attitude reconstuction via novel quaternion-integration procedure, Aerospace Science and Technology, 97, 150617. https://doi.org/10.1016/j.ast.2019.105617
21. Zahid, M., Younus, A., Ghoneim, M.E., Monsour, F.Y. (2022). Quaternion-valued exponential matrices and its fundamental properties. International Journal of Modern Physics B, 37, 2350027. https://doi.org/10.1142/S0217979223500273
22. Xia, Y., Huang, H., & Kou, K. (2022). An algorithm for solving linear nonhomogeneous quaternion-valued differential equations and some open problems, Discrete and Continuous Dynamical Systems-Series S, 15, 1685-1697. https://doi.org/10.3934/dcdss.2021162

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