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MULTIPLICATIVE JORDAN TRIPLE HIGHER SEMI-DERIVATIONS ON RINGS AND STANDARD OPERATOR ALGEBRAS

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Abstract. In this paper we study the additivity of multiplicative Jordan triple higher semi-derivations on rings and standard operator algebras.

1. Introduction

The study of the additivity of maps on rings has been an active area of research. The first result about the additivity of multiplicative isomorphisms on rings was given by Martindale III [8]. Daif [5] introduced the definition of a multiplicative derivation of a ring and studied the additivity of such map. In particular, Jing and Lu [7] studied the additivity of multiplicative Jordan triple derivations on rings and Ashraf and Parveen [2] studied the additivity of multiplicative Jordan triple higher derivations on rings. Considering that Ferreira and Marietto [6] introduced the notion of a multiplicative Jordan triple semi-derivation and studied its additivity and inspired by the facts mentioned above and the notions of [1], in this paper we introduce the notions of a multiplicative higher semi-derivation of a ring and multiplicative Jordan triple higher semi-derivation of a ring and present a study on the additivity of the last class. We will conclude this paper by applying the obtained results on the standard operator algebras.

2. Multiplicative Jordan triple higher semi-derivations on rings

Let $\mathscr{R}$ be a ring and $g : \mathscr{R} \to \mathscr{R}$ an arbitrary map. We denote by $g^0 = id_{\mathscr{R}}$ (the identity map on $\mathscr{R}$) and $g^{n+1} = g^n \circ g$, for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\circ$ is the map composition operation.

A family $D = \{d_n\}_{n\in\mathbb{N}}$ of maps $d_n : \mathscr{R} \to \mathscr{R}$ is said to be a *multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g)* (resp., *multiplicative Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g)*) if $d_0 = id_{\mathscr{R}}$,

$$\begin{aligned} d_n(ab) &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(a))d_q(b) \\ &= \textstyle\sum_{p+q=n} d_p(a)d_q(g^{n-q}(b)), \end{aligned}$$

for all elements $a, b \in \mathscr{R}$ and for each $n \in \mathbb{N}$ (resp.,

$$d_n(aba) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(a))d_q(g^r(b))d_r(a)$$

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$$= \sum_{p+q+r=n} d_p(a)d_q(g^p(b))d_r(g^{n-r}(a)),$$

for all elements $a, b \in \mathscr{R}$ and for each $n \in \mathbb{N}$), and $d_n(g(a)) = g(d_n(a))$, for all element $a \in \mathscr{R}$ and for each integer $n \in \mathbb{N}$.

A multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g) (resp., multiplicative Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g)) $D = \{d_n\}_{n\in\mathbb{N}}$ is said to be *additive* if d_n is an additive map, for each integer $n \in \mathbb{N}$. An additive multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g) (resp., additive multiplicative Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g)) is called a *higher semi-derivation of $\mathscr{R}$ (associated with g)* (resp., *Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g)*).

Our main result reads as follows.

Theorem 2.1. *Let $\mathscr{R}$ be a ring containing a non-trivial idempotent e_1, $\mathscr{R} = \oplus_{i,j=1,2}\mathscr{R}_{ij}$ the Peirce decomposition of $\mathscr{R}$, relative to e_1, $g : \mathscr{R} \to \mathscr{R}$ an arbitrary endomorphism of $\mathscr{R}$ and t an element of $\mathscr{R}$. For each integer $n \in \mathbb{N}$, assume that the following properties hold:*

(♧) $g^n(x_{ij})tx_{ij} = 0$ *and* $x_{ij}tg^n(x_{ij}) = 0$, *for all elements* $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$, *implies* $t = 0$,

(♤) $g^n(x_{ij})tx_{ij} = 0$ *and* $x_{ij}tg^n(x_{ij}) = 0$, *for all elements* $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$, *implies* $t = 0$, *provided that the conditions are satisfied:*

$$g^n(x_{ji})tx_{ii} = 0 \text{ and } x_{ii}tg^n(x_{ij}) = 0,$$

for all elements $x_{ii} \in \mathscr{R}_{ii}$, $x_{ij} \in \mathscr{R}_{ij}$ *and* $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i, j = 1, 2)$.

Then every multiplicative Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g) is additive.

In addition, if $\mathscr{R}$ is a 2-torsion free prime ring and $g : \mathscr{R} \to \mathscr{R}$ is a ring isomorphism of $\mathscr{R}$, then every multiplicative Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g) is a higher semi-derivation of $\mathscr{R}$ (associated with g).

We will prove the Theorem 2.1 using the Second Principle of Mathematical Induction.

Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a multiplicative Jordan triple higher semi-derivation of $\mathscr{R}$ (associated with g). First of all, we note that $\{g^n\}_{n\in\mathbb{N}}$ is a family of endomorphisms of $\mathscr{R}$ (this basic fact will be used throughout this paper without explicit mention) and that d_0 and d_1 are additive maps, by definition and [6, Theorem 2.1], respectively. Hence, suppose that for each positive integer n, each map d_k $(0 \leq k < n)$ belonging to D is additive. Based on the techniques used by Jing and Lu [7], we shall organize the proof of Theorem 2.1 in a series of Steps.

Step 2.2. $d_n(0) = 0$.

Proof. By induction hypothesis on n, we have $d_k(0) = 0$, for each integer $0 \leq k < n$. Since $d_p(g^{n-p}(0))d_q(g^r(0))d_r(0) = 0$ (resp., $d_p(0)d_q(g^p(0))d_r(g^{n-r}(0)) = 0$), for each integers $p, q, r \in \mathbb{N}$ satisfying $p + q + r = n$, then

$$d_n(0) = d_n(000) = \sum_{p+q+r=n} d_p(g^{n-p}(0))d_q(g^r(0))d_r(0)$$
$$(\text{resp.}, = \sum_{p+q+r=n} d_p(0)d_q(g^p(0))d_r(g^{n-r}(0))) = 0.$$

□

Step 2.3. For arbitrary elements $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following holds: $d_n(a_{11} + a_{12} + a_{21} + a_{22}) = d_n(a_{11}) + d_n(a_{12}) + d_n(a_{21}) + d_n(a_{22})$.

Proof. For all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ we have

$$\begin{aligned} &d_n(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) \\ &= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{11} + a_{12} + a_{21} + a_{22}))d_r(x_{ij}), \end{aligned} \tag{2.1}$$

$$d_n(x_{ij}a_{11}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{11}))d_r(x_{ij}), \tag{2.2}$$

$$d_n(x_{ij}a_{12}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{12}))d_r(x_{ij}), \tag{2.3}$$

$$d_n(x_{ij}a_{21}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{21}))d_r(x_{ij}) \tag{2.4}$$

and

$$d_n(x_{ij}a_{22}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{22}))d_r(x_{ij}). \tag{2.5}$$

Adding (2.2), (2.3), (2.4) and (2.5) and subtracting this sum from (2.1), we arrive at

$$\begin{aligned} &d_n(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) - d_n(x_{ij}a_{11}x_{ij}) - d_n(x_{ij}a_{12}x_{ij}) \\ &\qquad - d_n(x_{ij}a_{21}x_{ij}) - d_n(x_{ij}a_{22}x_{ij}) \\ &= g^n(x_{ij})(d_n(a_{11} + a_{12} + a_{21} + a_{22}) - d_n(a_{11}) \\ &\qquad - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}))x_{ij}, \end{aligned} \tag{2.6}$$

by using the induction hypothesis on n. Writing $t = d_n(a_{11} + a_{12} + a_{21} + a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22})$ in (2.6), we get

$$g^n(x_{ij})tx_{ij} = 0. \tag{2.7}$$

Next, for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ we have

$$\begin{aligned} &d_n(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) \\ &= \textstyle\sum_{p+q+r=n} d_p(x_{ij})d_q(g^p(a_{11} + a_{12} + a_{21} + a_{22}))d_r(g^{n-r}(x_{ij})), \end{aligned} \tag{2.8}$$

$$d_n(x_{ij}a_{11}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(x_{ij})d_q(g^p(a_{11}))d_r(g^{n-r}(x_{ij})), \tag{2.9}$$

$$d_n(x_{ij}a_{12}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(x_{ij})d_q(g^p(a_{12}))d_r(g^{n-r}(x_{ij})), \tag{2.10}$$

$$d_n(x_{ij}a_{21}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(x_{ij})d_q(g^p(a_{21}))d_r(g^{n-r}(x_{ij})) \tag{2.11}$$

and

$$d_n(x_{ij}a_{22}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(x_{ij})d_q(g^p(a_{22}))d_r(g^{n-r}(x_{ij})). \tag{2.12}$$

Adding (2.9), (2.10), (2.11) and (2.12) and subtracting this sum from (2.8), we arrive at

$$\begin{aligned} &d_n(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) - d_n(x_{ij}a_{11}x_{ij}) - d_n(x_{ij}a_{12}x_{ij}) \\ &\qquad - d_n(x_{ij}a_{21}x_{ij}) - d_n(x_{ij}a_{22}x_{ij}) \end{aligned}$$

$$= x_{ij}(d_n(a_{11} + a_{12} + a_{21} + a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}))g^n(x_{ij}), \tag{2.13}$$

by using again the induction hypothesis on n. Replacing t in (2.13) we get

$$x_{ij}tg^n(x_{ij}) = 0. \tag{2.14}$$

The identities (2.7) and (2.14) and the property (♧) imply that $t = 0$. As a consequence of this we obtain $d_n(a_{11} + a_{12} + a_{21} + a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}) = 0$. □

Step 2.4. For arbitrary elements $a_{12}, b_{12} \in \mathscr{R}_{12}$, $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{12} + b_{12}t_{22}) = d_n(a_{12}) + d_n(b_{12}t_{22})$ and (ii) $d_n(a_{21} + t_{22}b_{21}) = d_n(a_{21}) + d_n(t_{22}b_{21})$.

Proof. In view of the identity

$$e_1 + a_{12} + b_{12}t_{22} = (e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22}),$$

for all elements $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $t_{22} \in \mathscr{R}_{22}$, of induction hypothesis on n and Step 2.3, we have

$$\begin{aligned}
&d_n(e_1) + d_n(a_{12} + a_{12}t_{22})\\
&= d_n(e_1 + a_{12} + a_{12}t_{22})\\
&= d_n((e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22}))\\
&= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(e_1 + a_{12} + t_{22}))d_q(g^r(e_1 + b_{12}))d_r(e_1 + a_{12} + t_{22})\\
&= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(e_1 + a_{12} + t_{22}))(d_q(g^r(e_1)) + d_q(g^r(b_{12})))d_r(e_1 + a_{12} + t_{22})\\
&= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(e_1 + a_{12} + t_{22}))d_q(g^r(e_1))d_r(e_1 + a_{12} + t_{22})\\
&+ \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(e_1 + a_{12} + t_{22}))d_q(g^r(b_{12}))d_r(e_1 + a_{12} + t_{22})\\
&= d_n((e_1 + a_{12} + t_{22})e_1(e_1 + a_{12} + t_{22}))\\
&+ d_n((e_1 + a_{12} + t_{22})b_{12}(e_1 + a_{12} + t_{22}))\\
&= d_n(e_1 + a_{12}) + d_n(b_{12}t_{22})\\
&= d_n(e_1) + d_n(a_{12}) + d_n(b_{12}t_{22}).
\end{aligned}$$

This implies that $d_n(a_{12} + a_{12}t_{22}) = d_n(a_{12}) + d_n(a_{12}t_{22})$. Similarly we show that $d_n(a_{21} + t_{22}a_{21}) = d_n(a_{21}) + d_n(t_{22}a_{21})$, from the identity

$$e_1 + a_{21} + t_{22}b_{21} = (e_1 + a_{21} + t_{22})(e_1 + b_{21})(e_1 + a_{21} + t_{22}),$$

for all elements $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$. □

Step 2.5. For arbitrary elements $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $a_{21}, b_{21} \in \mathscr{R}_{21}$ the following hold: (i) $d_n(a_{12} + b_{12}) = d_n(a_{12}) + d_n(b_{12})$ and (ii) $d_n(a_{21} + b_{21}) = d_n(a_{21}) + d_n(b_{21})$.

Proof. For all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ we have

$$d_n(x_{ij}(a_{12} + b_{12})x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{12} + b_{12}))d_r(x_{ij}), \tag{2.15}$$

$$d_n(x_{ij}a_{12}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{12}))d_r(x_{ij}) \tag{2.16}$$

and

$$d_s(x_{ij}b_{12}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(b_{12}))d_r(x_{ij}). \tag{2.17}$$

Adding (2.16) and (2.17) and subtracting this sum from (2.15), we obtain that

$$d_n(x_{ij}(a_{12}+b_{12})x_{ij}) - d_n(x_{ij}a_{12}x_{ij}) - d_n(x_{ij}b_{12}x_{ij})$$
$$= g^n(x_{ij})(d_n(a_{12}+b_{12}) - d_n(a_{12}) - d_n(b_{12}))x_{ij}. \tag{2.18}$$

Writing $t = d_n(a_{12}+b_{12}) - d_n(a_{12}) - d_n(b_{12})$ in (2.18) and using the Step 2.4(ii) we get

$$g^n(x_{ij})tx_{ij} = 0. \tag{2.19}$$

Similarly we show that

$$x_{ij}tg^n(x_{ij}) = 0, \tag{2.20}$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$. The identities (2.19) and (2.20) and the property (♧) imply that $t = 0$. Thus it follows that $d_n(a_{12}+b_{12}) - d_n(a_{12}) - d_n(b_{12}) = 0$.

Similarly, we show the other case. □

Step 2.6. For arbitrary elements $a_{11}, b_{11} \in \mathscr{R}_{11}$ and $a_{22}, b_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11}+b_{11}) = d_n(a_{11}) + d_n(b_{11})$ and (ii) $d_n(a_{22}+b_{22}) = d_n(a_{22}) + d_n(b_{22})$.

Proof. For all $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$ we have

$$d_n(x_{ij}(a_{11}+b_{11})x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{11}+b_{11}))d_r(x_{ij}), \tag{2.21}$$

$$d_n(x_{ij}a_{11}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(a_{11}))d_r(x_{ij}) \tag{2.22}$$

and

$$d_n(x_{ij}b_{11}x_{ij}) = \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ij}))d_q(g^r(b_{11}))d_r(x_{ij}). \tag{2.23}$$

Adding (2.22) and (2.23) and subtracting this sum from (2.21), we obtain that

$$d_n(x_{ij}(a_{11}+b_{11})x_{ij}) - d_n(x_{ij}a_{11}x_{ij}) - d_n(x_{ij}b_{11}x_{ij})$$
$$= g^n(x_{ij})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))x_{ij}. \tag{2.24}$$

Writing $t = d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11})$ in (2.24) we have

$$g^n(x_{ij})tx_{ij} = 0. \tag{2.25}$$

Similarly we show that

$$x_{ij}tg^n(x_{ij}) = 0, \tag{2.26}$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$. Next, for all $x_{ii} \in \mathscr{R}_{ii}$ and $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i, j = 1, 2)$ we have

$$d_n(x_{ii}(a_{11}+b_{11})x_{ii}) + d_n(x_{ji}(a_{11}+b_{11})x_{ii}) = d_n((x_{ii}+x_{ji})(a_{11}+b_{11})(x_{ii}+x_{ji}))$$
$$= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ii}+x_{ji}))d_q(g^r(a_{11}+b_{11}))d_r(x_{ii}+x_{ji}), \tag{2.27}$$

$$d_n(x_{ii}a_{11}x_{ii}) + d_n(x_{ji}a_{11}x_{ii}) = d_n((x_{ii}+x_{ji})a_{11}(x_{ii}+x_{ji}))$$
$$= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ii}+x_{ji}))d_q(g^r(a_{11}))d_r(x_{ii}+x_{ji}) \tag{2.28}$$

and

$$d_n(x_{ii}b_{11}x_{ii}) + d_n(x_{ji}b_{11}x_{ii}) = d_n((x_{ii}+x_{ji})b_{11}(x_{ii}+x_{ji}))$$
$$= \textstyle\sum_{p+q+r=n} d_p(g^{n-p}(x_{ii}+x_{ji}))d_q(g^r(b_{11}))d_r(x_{ii}+x_{ji}). \tag{2.29}$$

Adding (2.28) and (2.29) and subtracting this sum from (2.27), we obtain that

$$\begin{aligned} &d_n(x_{ii}(a_{11}+b_{11})x_{ii}) - d_n(x_{ii}a_{11}x_{ii}) - d_n(x_{ii}b_{11}x_{ii}) \\ &= g^n(x_{ii}+x_{ji})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))(x_{ii}+x_{ji}), \end{aligned} \tag{2.30}$$

by Steps 2.2 and 2.5(ii). Now, note that

$$\begin{aligned} &d_n(x_{ii}(a_{11}+b_{11})x_{ii}) - d_n(x_{ii}a_{11}x_{ii}) - d_n(x_{ii}b_{11}x_{ii}) \\ &= g^n(x_{ii})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))x_{ii}. \end{aligned} \tag{2.31}$$

Comparing the identities (2.30) and (2.31), we see that

$$\begin{aligned} &g^n(x_{ii}+x_{ji})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))(x_{ii}+x_{ji}) \\ &= g^n(x_{ii})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))x_{ii} \end{aligned}$$

and by (2.25) we get

$$\begin{aligned} &g^n(x_{ii})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))x_{ji} \\ &+ g^n(x_{ji})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))x_{ii} = 0. \end{aligned} \tag{2.32}$$

Since g is an endomorphism of $\mathscr{R}$, then multiplying the identity (2.32) on the left by $g^n(e_1)$, we conclude that

$$g^n(x_{ji})(d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}))x_{ii} = 0. \tag{2.33}$$

Thus, replacing t in (2.33) we get

$$g^n(x_{ji})tx_{ii} = 0. \tag{2.34}$$

Similarly we show that, for all $x_{ii} \in \mathscr{R}_{ii}$ and $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$ we have

$$x_{ii}tg^n(x_{ij}) = 0. \tag{2.35}$$

The identities (2.25), (2.26), (2.34) and (2.35) and the property ($\spadesuit$) imply that $t = 0$. Therefore, $d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11}) = 0$.

Similarly, we show the other case. □

Step 2.7. d_n is additive.

Proof. The result follows directly from Steps 2.3, 2.5 and 2.6. □

As a consequence of Step 2.7, applying the Second Principle of Mathematical Induction on D, we see that D is additive.

To complete the proof of the Theorem 2.1 we will assume in addition that $\mathscr{R}$ is a 2-torsion free prime ring and $g : \mathscr{R} \to \mathscr{R}$ is a ring isomorphism of $\mathscr{R}$.

Step 2.8. D is a higher semi-derivation of $\mathscr{R}$ (associated with g).

Proof. First, note that D is simultaneously a Jordan triple $(g, id_{\mathscr{R}})$-higher derivation of $\mathscr{R}$ and a Jordan triple $(id_{\mathscr{R}}, g)$-higher derivation of $\mathscr{R}$, according to the definition presented in [1, pp. 24]. Thus, by [1, Theorem 2.1], D is simultaneously a $(g, id_{\mathscr{R}})$-higher derivation of $\mathscr{R}$ and a $(id_{\mathscr{R}}, g)$-higher derivation of $\mathscr{R}$, respectively. As a consequence of these last two results we see that D is a higher semi-derivation of $\mathscr{R}$ (associated with g). □

3. Multiplicative Jordan triple higher semi-derivations on standard operator algebras

In order to present the main result of this section, we will take into account the results of Chang [4, Theorem 1], that proved that all maps which are associated with arbitrary semi-derivations of a prime ring are always homomorphisms, the obtained result of Ferreira and Marietto in [6] and of Brešar and Šemrl [3, Corollary 7.3.], that characterized the class of isomorphisms between standard operator algebras. Thus, in view of these facts and the main result of the previous section, we can state the following.

Theorem 3.1. *Let $\mathscr{X}$ be a Hilbert space with* $\dim \mathscr{X} \geq 2$, *$\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$, $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ a standard operator algebra and $g : \mathscr{A} \to \mathscr{A}$ an isomorphism of $\mathscr{A}$. Then the following statement holds: every multiplicative Jordan triple higher semi-derivation of $\mathscr{A}$ (associated with g) is a higher semi-derivation of $\mathscr{A}$ (associated with g).*

Proof. First of all, it is well known that $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$ are prime rings and $\mathscr{A}$ contains a non-trivial idempotent e_1. Also, $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$ under the strong operator topology and, by [3, Corollary 7.3.], there exist a linear invertible operator of $\mathscr{B}(\mathscr{X})$, $y : \mathscr{X} \to \mathscr{X}$, such that $g(a) = yay^{-1}$, for all element $a \in \mathscr{A}$. Write $e_2 = 1_{\mathscr{B}(\mathscr{X})} - e_1$, where $1_{\mathscr{B}(\mathscr{X})}$ is a multiplicative identity of $\mathscr{B}(\mathscr{X})$.

Let $\mathscr{B}(\mathscr{X}) = \oplus_{i,j=1,2}\mathscr{B}(\mathscr{X})_{ij}$ and $\mathscr{A} = \oplus_{i,j=1,2}\mathscr{A}_{ij}$ be the Peirce decompositions of $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$, relative to e_1, respectively, and an arbitrary element $t \in \mathscr{A}$. For each integer $n \in \mathbb{N}$, we consider two cases, according as whether $n = 0$ or $n \geq 1$. Case 1: $n = 0$. In this case, the properties (♧) and (♤) follow immediately from [7, Lemma 1.6(P1), (P2) and (P4)]. Case 2: $n \geq 1$. Consider $(y^{-1})^n t = ((y^{-1})^n t)_{11} + ((y^{-1})^n t)_{12} + ((y^{-1})^n t)_{21} + ((y^{-1})^n t)_{22}$ and $ty^n = (ty^n)_{11} + (ty^n)_{12} + (ty^n)_{21} + (ty^n)_{22}$ be the Peirce decompositions of $(y^{-1})^n t$ and ty^n, relative to e_1, and assume that the following conditions hold: $g^n(x_{ij})tx_{ij} = 0$ and $x_{ij}tg^n(x_{ij}) = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$. Then $y^n x_{ij}(y^{-1})^n tx_{ij} = 0$ and $x_{ij}ty^n x_{ij}(y^{-1})^n = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$, which implies that $x_{ij}(y^{-1})^n tx_{ij} = 0$ and $x_{ij}ty^n x_{ij} = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$. As $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, we get that $x_{ij}(y^{-1})^n tx_{ij} = 0$ and $x_{ij}ty^n x_{ij} = 0$, for all elements $x_{ij} \in \mathscr{B}(\mathscr{X})_{ij}$ $(i, j = 1, 2)$. Thus it follows from [7, Lemma 1.6(P4)] that $((y^{-1})^n t)_{ji} = 0$ and $(ty^n)_{ji} = 0$ $(i, j = 1, 2)$, respectively. This leads to $(y^{-1})^n t = 0$ and $ty^n = 0$. From either of these two identities, we obtain $t = 0$. Now, assume that the conditions $g^n(x_{ij})tx_{ij} = 0$ and $x_{ij}tg^n(x_{ij}) = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i \neq j; i, j = 1, 2)$, hold provided that the following conditions are satisfied: $g^n(x_{ji})tx_{ii} = 0$ and $x_{ii}tg^n(x_{ij}) = 0$, for all elements $x_{ii} \in \mathscr{A}_{ii}$, $x_{ij} \in \mathscr{A}_{ij}$ and $x_{ji} \in \mathscr{A}_{ji}$ $(i \neq j; i, j = 1, 2)$. First, note that by a similar reasoning as in the previous case we get $((y^{-1})^n t)_{ji} = 0$ and $(ty^n)_{ji} = 0$ $(i \neq j; i, j = 1, 2)$. Next, $y^n x_{ji}(y^{-1})^n tx_{ii} = 0$ and $x_{ii}ty^n x_{ij}(y^{-1})^n = 0$, for all elements $x_{ii} \in \mathscr{A}_{ii}$, $x_{ij} \in \mathscr{A}_{ij}$ and $x_{ji} \in \mathscr{A}_{ji}$ $(i \neq j; i, j = 1, 2)$ implies the identities $x_{ji}(y^{-1})^n tx_{ii} = 0$ and $x_{ii}ty^n x_{ij} = 0$, for all elements $x_{ii} \in \mathscr{A}_{ii}$, $x_{ij} \in \mathscr{A}_{ij}$ and $x_{ji} \in \mathscr{A}_{ji}$. As $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, we get that $x_{ji}(y^{-1})^n tx_{ii} = 0$ and $x_{ii}ty^n x_{ij} = 0$, for all elements $x_{ii} \in \mathscr{B}(\mathscr{X})_{ii}$, $x_{ij} \in \mathscr{B}(\mathscr{X})_{ij}$ and $x_{ji} \in \mathscr{B}(\mathscr{X})_{ji}$ $(i \neq j; i, j = 1, 2)$. Thus it follows that $((y^{-1})^n t)_{ii} = 0$ and $(ty^n)_{ii} = 0$, by [7, Lemma 1.6(P1) and (P2)]. This leads to $(y^{-1})^n t = 0$ and $ty^n = 0$, respectively. From either of these two identities, we obtain $t = 0$. The last two results on t show that the properties (♧) and (♤) also hold, for this second case. Therefore, by Theorem 2.1, D is additive and we get the required result. □

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