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ADDITIVE MULTIPLIERS AND BI-SEMIDERIVATIONS ON RINGS

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Abstract. In this paper, our attempt is to investigate commutativity of rings satisfying certain identities involving multipliers and bi-semiderivations on prime(semiprime) ring.

1. Introduction and preliminaries

A mapping $D : R \times R \to R$ is said to be symmetric if $D(a, b) = D(b, a)$ for all $a, b \in R$. A mapping $D : R \times R \to R$ is called bi-additive if it is additive in both slot. Now we introduce the concept of symmetric bi-derivations as follows: A bi-additive mapping $D : R \times R \longrightarrow R$ is called a bi-derivation if for every $a \in R$, the map $b \mapsto D(a, b)$ as well as for every $b \in R$, the map $a \mapsto D(a, b)$ is a derivation of R. For ideational reading in the related matter one can turned to [9]. For a symmetric mapping D, a map $d : R \to R$ defined as $d(c) = D(c, c)$ is called the trace of D.

Bergen [2] set out the concept of semiderivation on a ring R. An additive mapping $f : R \longrightarrow R$ is called a semiderivation if there exists a function $g : R \longrightarrow R$ such that $f(ab) = f(a)g(b) + af(b) = f(a)b + g(a)f(b)$ and $f(g(a)) = g(f(a))$ for all $a, b \in R$. In case g is an identity map of R; then all semiderivations associated with g are merely ordinary derivations. On the other hand, if g is a homomorphism of R such that $g \neq 1$; then $f = g - 1$ is a semiderivation which is not a derivation. In case R is prime and $f \neq 0$; it has been shown by Chang [4] that g must necessarily be a ring endomorphism.

Following the idea in [10], a symmetric bi-additive function $D : R \times R \longrightarrow R$ is called a symmetric bi-semiderivation associated with a function $f : R \longrightarrow R$ (or simply a symmetric bi-semiderivation of a ring R) if

$$D(ab, c) = D(a, c)f(b) + aD(b, c) = D(a, c)b + f(a)D(b, c)$$

and $d(f(a)) = f(d(a))$ for all $a, b, c \in R$.

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Example 1.1. Let $R = \left\{ \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$. R is a ring under matrix addition and matrix multiplication. Define mapping $D : R \times R \to R$ by $D\left(\begin{bmatrix} 0 & 0 & a_0 \\ 0 & 0 & b_0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & b_1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_0 b_1 \\ 0 & 0 & 0 \end{bmatrix}$ and $f : R \to R$ by $f\left(\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We observe that D is a bi-semiderivation associated with f.

Number of mathematicians established a relationship between the structure of prime (semiprime) rings and their subsets including the behavior of mappings satisfying algebraic identities involving such mappings. Herstein [6] determined the structure of a prime ring R which has a derivation $d \neq 0$ such that the values of d commute, that is, for which $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$. In [3], author characterized the biadditive mappings $B : R \times R \to R$ where R is a prime ring with certain additional properties, satisfying $B(x, x)x = xB(x, x)$ for all $x \in R$. In fact, author determine the structures of commutativity preserving mappings on prime rings.

Later on Deng and Ashraf [5] proved that if there exist a derivation d of a semiprime ring R and a mapping $f : I \longrightarrow R$ defined on a nonzero ideal I such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$. Another generalization strict to multipliers only found in [1]. Infact, authors proved that: Let R be a prime ring and I be a nonzero ideal of R. If R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Then the following conditions are equivalent:

(1) $H([x, y]) - [x, y] = 0$, for all $x, y \in I$;
(2) $H([x, y]) + [x, y] = 0$, for all $x, y \in I$;
(3) For all $x, y \in I$, either $H([x, y]) - [x, y] = 0$ or $H([x, y]) + [x, y] = 0$;
(4) R is commutative.

Many direct or indirect generalizations of these kind of results can be found in the literature [3, 4, 9 ,11, 12] and references there in. A recent generalization of the commutative structure of prime ring with the help of multipliers has been shown in [8]. In this paper author obtained the following:

Theorem 1.2. *Let R be a prime ring. If $H : R \to R$ is a nonzero multiplier of R and $H(R) \subseteq Z(R)$, then R is commutative.*

In the present note, we establish the commutativity conditions of prime (semiprime) rings admitting bi-semiderivations with associated function f (taking it surjective everywhere) and multipliers of R satisfying some identities.

2. Main results

We begin with the following preliminary lemmas:

Lemma 2.1. [7] *If R is a semiprime ring, the center of a nonzero one sided ideal is contained in the center of R. As an immediate consequence, any commutative one sided ideal is contained in the center of R.*

Lemma 2.2. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and D be a bi-semiderivation on R with associated function f such that $D(x,x) \subseteq Z(R)$ for all $x \in R$, then either $D = 0$ or R contains a nonzero central ideal.*

Proof. We have given that $D(x,x) \subseteq Z(R)$, for all $x \in R$. That is

$$[D(x,x), y] = 0 \text{ for all } x, y \in I. \tag{2.1}$$

On linearization of above equation in x and using torsion freeness of R, we observe that

$$[D(x,z), y] = 0 \text{ for all } x, y, z \in I. \tag{2.2}$$

Now, replacing z by zu in (2.2), we find

$$D(x,z)[f(u), y] + [z,y]D(x,u) = 0 \text{ for all } x, y, u, z \in I. \tag{2.3}$$

In particular, we have $D(x,z)[f(u), z] = 0$ for all $x, u, z \in I$. Surjectivity of f allowed us to write $D(x,z)R[s,z] = (0)$ for all $x, z \in I$, $s \in R$. Semiprimeness of R clinch that to have a family $\{P_k\}$ of prime ideals of R such that $\bigcap_k P_k = 0$. For any P_k, either

$$[s,z] \in P_k \text{ for all } z \in I, s \in R. \tag{2.4}$$

Or

$$D(x,z) \in P_k \text{ for all } x, z \in I. \tag{2.5}$$

There are some $P_k's$ for which satisfying (2.4), we call it P_{k_1} and satisfying (2.5), we call it P_{k_2} respectively. Recall that $P_{k_2} \cap P_{k_1} = (0)$. From (2.5) we find that

$$RD(x,z) \subseteq P_k \text{ for all } x, z \in I. \tag{2.6}$$

Now construct an ideal $\mathcal{J} = RD(x,z)$ and claim that $\mathcal{J} = (0)$. Since $[rD(x,z), sD(y,u)]$ belong to P_{k_1} by (2.4) and P_{k_2} by (2.6). But we have $P_{k_2} \cap P_{k_1} = 0$. This intrigued that $\mathcal{J} = (0)$ in both the cases, that is, $D(x,z)RD(x,z) = (0)$ for all $x, z \in I$. Semiprimeness of R yields that $D = 0$. Arguing in the similar manner as above , for $\mathcal{J} \neq (0)$, we arrive at a contradiction. Next making use of Lemma 2.1, R contains a nonzero central ideal. This completes the proof. □

Proposition 2.3. *Let R be a prime ring of characteristic not 2, I be a nonzero ideal of R and D be a bi-semiderivation on R with associated function f such that $D(x,x) \subseteq Z(R)$ for all $x \in R$, then either $D = 0$ or R is commutative.*

Proof. Consider the equation (2.3) along with surjectivity of f in particular, we have $D(x,z)[r,y] = 0$ for all $x, y, z \in I$ and $r \in R$. This implies that either $D(x,z) = 0$ or $[r,z] = 0$ for all $z \in I$ as R is prime. Making use of Brauer's trick we get either $D = 0$, or R is commutative.

□

Theorem 2.4. *Let R be a 2 torsion free semiprime ring and H be an additive multiplier. If D is a symmetric bi-semiderivation on R with associated function f such that $[D(x,x),y] \mp H([x,y]) = 0$ for all $x, y \in R$, then either $D = 0$ or R contains a nonzero central ideal.*

Proof. We have given that

$$[D(x,x),y] \mp H([x,y]) = 0 \text{ for all } x, y \in I. \tag{2.7}$$

Now, replacing y by yz in (2.1), we find

$$y[D(x,x),z] \mp H(y)[x,z] = 0 \text{ for all } x, y, z \in I. \tag{2.8}$$

In particular, take $x = z$ in above equation we notice that $y[D(x,x),z] = 0$ for all $x, y, z \in R$. Semiprimeness of R implicit that $[D(x,x),z] = 0$ for all $x, z \in R$. This complete the proof claimed by Lemma 2.2. □

Theorem 2.5. *Let R be a 2-torsion free semiprime ring and H be an additive multiplier. If D is a symmetric bi-semiderivation on R with associated function f such that $D(x,x) \circ y \mp H([x,y]) = 0$ for all $x, y \in R$, then either $D = 0$ or R contains a nonzero central ideal.*

Proof. By our hypothesis, we are given that

$$D(x,x) \circ y \mp H([x,y]) = 0 \text{ for all } x, y \in R. \tag{2.9}$$

Putting yx in place of y in above equation and compare it with (2.9) to find $y[D(x,x),x] = 0$ for all $x, y \in R$. Using the same arguments as in theorem 2.4, we can conclude as desired. □

Theorem 2.6. *Let R be a prime ring of characteristic not 2 and H be a nonzero additive multiplier. If D is a symmetric bi-semiderivation on R with associated function f such that $[D(x,x),y]+H(x \circ y) = 0$ for all $x, y \in R$, then either $D = 0$ or R is commutative.*

Proof. We have given that

$$[D(x,x),y] + H(x \circ y) = 0 \text{ for all } x, y \in R. \tag{2.10}$$

Now, replacing y by yz in (2.10), we find

$$y[D(x,x),z] + H(y)[z,x] = 0 \text{ for all } x, y, z \in R. \tag{2.11}$$

Reinstate py for y in (2.11) and applying (2.11) to have

$$(pH(y) - H(y)p)[z,x] = 0 \text{ for all } p, x, y, z \in R. \tag{2.12}$$

Particularly, we observe the condition $[H(y),y]R[z,x] = 0$ for all $x, y, z \in R$. Primeness of R implies that $[H(y),y] = 0$ or $[z,x] = 0$ for all $x, y, z \in R$. Second case is obvious, so consider the first condition $[H(y),y] = 0$ for all $y \in R$. In this case R is commutative by applying Theorem 1.2.

□

Theorem 2.7. *Let R be a prime ring of characteristic not 2 and H be a nonzero additive multiplier. If D is a symmetric bi-semiderivation on R with associated function f such that $D(x,x) \circ y \mp H(x \circ y) = 0$ for all $x, y \in R$, then either $D = 0$ or R is commutative.*

Proof. By our hypothesis, we are given that

$$D(x,x)\circ y \mp H(x\circ y)=0 \text{ for all } x,y\in R. \tag{2.13}$$

Putting yz for y in (2.14) and make use of (2.14) to obtain

$$y[D(x,x),z]\mp H(y)[x,z]=0 \text{ for all } x,y,z\in R. \tag{2.14}$$

Now using all arguments as in theorem 2.6, we get the required result. □

Theorem 2.8. *Let R be a prime ring of characteristic not 2 and 3 and H be a nonzero additive multiplier. If D is a symmetric bi-semiderivation on R with associated function f such that $[D(x,x),x]\mp[H(x),x]=0$ for all $x\in R$, then either $D=0$ or R is commutative.*

Proof. First consider the condition below

$$[D(x,x),x]+[H(x),x]=0 \text{ for all } x\in R. \tag{2.15}$$

Linearization of (2.15) yields that

$$\begin{aligned}&[D(y,y),x]+2[D(x,y),x]+[D(x,x),y]+2[D(x,y),y]\\&+[H(x),y]+[H(y),x]=0 \text{ for all } x,y\in R.\end{aligned} \tag{2.16}$$

Rewrite above equation by putting $-y$ in place of y to find

$$\begin{aligned}&[D(y,y),x]-2[D(x,y),x]-[D(x,x),y]\\&+2[D(x,y),y]-[H(x),y]-[H(y),x]=0 \text{ for all } x,y\in R.\end{aligned} \tag{2.17}$$

Comparing the last two equation to obtain

$$[D(y,y),x]+2[D(x,y),y]=0 \text{ for all } x,y\in R. \tag{2.18}$$

Particularly, this implies that $3[D(y,y),y]=0$ for all $y\in R$. Using characteristic condition, we get $[D(y,y),y]=0$ for all $y\in R$. Hence we observe that $[H(y),y]=0$ for all $y\in R$ by(2.15). An application of Theorem 1.2 completes the proof. We can find the required result in similar way, if we consider the condition $[D(x,x),x]-[H(x),x]=0$ for all $x\in R$.

□

Theorem 2.9. *Let R be a prime ring and D be a symmetric bi-semiderivation with associated function f. If H is a nonzero additive left multiplier such that $D(H(x),x)=0$ for all $x\in R$, then either $D=0$ or R is commutative.*

Proof. We have given that $D(H(x),x)=0$ for all $x\in R$, linearization yields that

$$D(H(x),y)+D(H(y),x)=0 \text{ for all } y,x\in R. \tag{2.19}$$

Substitute yr in place of y in (2.19) to find

$$D(H(x),y)f(r)+yD(H(x),r)+D(H(y),x)f(r)+H(y)D(r,x)=0 \text{ for all } x,y,r\in R. \tag{2.20}$$

In view of (2.19) and (2.20), one can get

$$yD(H(x),r)+H(y)D(r,x)=0 \text{ for all } x,y,r\in R. \tag{2.21}$$

Putting ty for y in (2.21) and using (2.21), we obtain

$$[H(y),t]D(r,x)=0 \text{ for all } y,x,t,r\in R. \tag{2.22}$$

Primeness of R yields that the two condition either $D = 0$ or $[H(x), x] = 0$ for all $x \in R$. An application of Theorem 1.2 in second condition complete the proof. $\square$

References

1. Ashraf, M. and Ali, S., *On left multipliers and the commutativity of prime rings*, Demonstratio Mathematica XLI **(4)** (2008), 763-771.
2. Bergen, J., *Derivations in Prime Rings,* Canad. Math. Bull. **26**(1983), 267-270.
3. Bresar, M., *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings,* Trans. Amer. Math. Soc. **335 (2)**(1993), 525–546.
4. Chang, J. C., *On semiderivations of prime rings,* Chinese J. Math. **12**(1984), 255-262.
5. Deng, Q. and Ashraf, M., *On strong commutativity preserving mappings,* Results Math. **30** (3-4) (1996), 259–263.
6. Herstein, I. N., *A note on derivations II,* Canad. Math. Bull. **22**(1979), 509-511.
7. Herstein, I.N., *Rings with Involution,* University of Chicago Press, Chicago (1976).
8. Kim, K. H., *A note on multipliers and commutativity of prime rings*, Gulf J. Math. **6(3)**(2018), 70-77.
9. Maksa, G., *A remark on symmetric biadditive functions having non-negative diagonalization*, Glasnik. Mat., **15 (35)**(1980), 279-282.
10. Reddy, C.G. and Bharathi, D., *On the trace of Symmetric bi-Semiderivations on near-rings,* Intern. J. Adv. Sci. Res. Man. **1(8)** (2018), 237-242.
11. Rehman, N. and Ansari, A. Z., *On lie ideals with symmetric bi-additive maps in rings,* Palestine J. Math. **2** (2013), 14-21.
12. Shujat, F., *Symmetric generalized biderivations of prime rings*, Bol. Soc. Paran. Mat. **39 (4)**(2021), 65-72.

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