Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 21 (2024) 99-122 https://doi.org/10.56947/amcs.v21.265

MULTIPLICITY OF HETEROCLINIC SOLUTIONS FOR THE DISCRETE $p(k)$-LAPLACE KIRCHHOFF TYPE EQUATIONS WITH A PARAMETER

BRAHIM MOUSSA [1], ISMAËL NYANQUINI[2] AND STANISLAS OUARO[3*]

Abstract. In this paper, we prove the existence of at least one and at least two nontrivial heteroclinic solutions for the discrete $p(k)$-Laplace Kirchhoff type equations depending on a parameter. The proof of our main result is based on a local minimum theorem for differentiable functionals due to Ricceri and on the mountain pass theorem of P. Pucci and J. Serrin.

1. Introduction

In this paper, we are interested in the following Kirchhoff type problem with heteroclinic condition at the boundary

$$\begin{cases} -M(\eta[u])[\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))-q(k)|u(k)|^{p(k)-2}u(k)] \\ \quad = \lambda f(k,u(k)), \ \ k\in\mathbb{Z}, \\ \lim\limits_{k\to-\infty} u(k)=-1, \qquad \lim\limits_{k\to\infty} u(k)=1, \end{cases} \tag{1.1}$$

where

$$\eta[u]=\sum_{k\in\mathbb{Z}}\left(A_0(k-1,|\triangle u(k-1)|)+\frac{q(k)}{p(k)}|u(k)|^{p(k)}\right),$$

$\triangle u(k)=u(k+1)-u(k)$ is the forward difference operator, $\mathbb{Z}$ is the set of integers, $\lambda>0$ is a real parameter, $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is a continuous function in the second variable, $f(k,t)=-f(-k,-t)$ for all $k\in\mathbb{Z}$ and $t\in\mathbb{R}$. Moreover, we assume that the function $a(k,\xi):\mathbb{Z}\times[0,\infty)\to[0,\infty)$ is continuous, $A_0:\mathbb{Z}\times[0,\infty)\to[0,\infty)$ which satisfies $A_0(k,t)=\int\limits_0^t a(k,\xi)\xi d\xi$, $p(\cdot):\mathbb{Z}\to(1,\infty)$ is a function such that

$$1<p^-=\inf_{k\in\mathbb{Z}}p(k)\le p(k)\le p^+=\sup_{k\in\mathbb{Z}}p(k)<\infty, \tag{1.2}$$

the function $q:\mathbb{Z}\to\mathbb{R}$ satisfies

$$q(k)\ge q_0>0 \ \text{ for all } \ k\in\mathbb{Z},\ q_0<p^+,\ q(k)\to\infty \ \text{ as } \ |k|\to\infty. \tag{1.3}$$

Date: Received: Jan 18, 2024; Accepted: Feb 4, 2024.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 44B20.
Key words and phrases. Kirchhoff type equation, heteroclinic solutions, nontrivial solution, variational methods, critical point theory.

We suppose that M and a satisfy the following conditions.

$(H1)$ There exist a function $a_1 \in l^{p'(\cdot)}$ with $\dfrac{1}{p(k)} + \dfrac{1}{p'(k)} = 1$ and a nonnegative constant a_2 such that
$$|a(k,|\xi|)\xi| \leq a_1(k) + a_2|\xi|^{p(k)-1},$$
for all $k \in \mathbb{Z}^+$ and $\xi \in \mathbb{R}$.

$(H2)$ There exists a positive constant c such that
$$\min\left\{a(k,|\xi|), |\xi|\frac{\partial a}{\partial \xi}(k,|\xi|) + a(k,|\xi|)\right\} \geq c|\xi|^{p(k)-2},$$
for all $k \in \mathbb{Z}^+$ and $\xi \in \mathbb{R}$.

$(H3)$ $M : (0,\infty) \to (0,\infty)$ is continuous, nondecreasing and there exist two positive constant m_0 and m_1 such that
$$m_0 \leq M(t) \leq m_1 \ \text{ for } \ t > 0.$$

We recall that a solution $u = \{u(k) : k \in \mathbb{Z}\}$ of the problem (1.1) is heteroclinic if $\lim\limits_{k\to-\infty} u(k) = -1$ and $\lim\limits_{k\to\infty} u(k) = 1$.

Let $x = \{x(k) : k \in \mathbb{Z}^+\}$ be a solution to the following problem.
$$\begin{cases} -M(\eta[x])[\triangle(a(k-1,|\triangle x(k-1)|)\triangle x(k-1)) - q(k)|x(k)|^{p(k)-2}x(k)] \\ \quad = \lambda f(k, x(k)+1), \ \ k \in \mathbb{Z}^+, \\ x(0) = -1, \quad \lim\limits_{k\to\infty} x(k) = 0. \end{cases} \tag{1.4}$$

Let $u = \{u(k) : k \in \mathbb{Z}\}$ such that
$$u(k) := \begin{cases} x(k)+1, & \text{for } \ k \in \mathbb{Z}^+, \\ 0, & \text{for } \ k = 0, \\ -x(-k)-1, & \text{for } \ k \in \mathbb{Z}^-, \end{cases} \tag{1.5}$$
then u is a solution to the problem (1.1). Therefore, in this paper, we will study the existence of heteroclinic solutions to the problem (1.4).

The presence of the non-local term $\eta[u]$ is an important feature of this work. Kirchhoff (see [27]) in 1876 established a model defined by the following equation.
$$\rho\frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{Ea}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.6}$$
where $\rho > 0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of the cross section and L is the initial length of the string.

Equation (1.6) takes into account the change of the tension on the string which is caused by the change of its length during the vibration. After that, several physicists also considered such equations for their research in the theory of non-linear vibrations theoretically or experimentally [8, 9, 34, 36]. On the other hand, Kirchhoff equation received great attention only after Lions in 1978 (see [31]) suggested an abstract framework of the problem related to the stationary analogous of the equation of Kirchhoff type. In recent years, many authors have investigated Kirchhoff type equations, we refer the readers to [2, 10]. We also refer the readers to the recent results of the discrete Kirchhoff type problems [11, ?, 28, 35, 37, 42, 43, 44] and the references therein. More recently, in [19] the authors dealt with the discrete problems of the Kirchhoff type by using the minimization method.

The nonhomogeneous differential operator in problem (1.1),

$$\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)), \tag{1.7}$$

generalizes the usual operators with variable exponent. Obviously, when $a(k,t) = t^{p(k)-2}$, the operator (1.7) is the standard discrete $p(\cdot)$-Laplacian operator, that is,

$$\triangle_{p(k-1)}u(k-1) := \triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right).$$

This operator was initiated by Mihailescu et *al.* in [32], where some eigenvalue problems were investigated.

Differential equations represent the discrete counterpart of PDEs and are usually studied in connection with numerical analysis. We refer the reader to the monograph of Rădulescu and Repovš (see [39]) for a comprehensive qualitative analysis of nonlinear PDEs with variable exponent using variational and topological methods. The operator (1.7) represents the discrete counterpart of the anisotropic operator

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,\left|\frac{\partial}{\partial x_i}u\right|\right)\frac{\partial}{\partial x_i}u.$$

This operator was analyzed by I.H. Kim and Y.H. Kim [26]. Consequently, the problem (1.1) can be seen as the discrete counterpart of such PDEs under nonhomogeneous Dirichlet boundary conditions.

Partial differential equations have been widely used for the modeling of many phenomena arising from the study of elastic mechanics [47], electrorheological fluids [40] and image restoration [12].

Heteroclinic solutions of difference equations have been extensively studied by many authors in the literature. We refer the readers to the following papers involving the discrete $p(k)$-Laplacian operator and p-Laplacian operator [7, 17, 18, 29] and references therein. The discrete $p(k)$-Laplacian operator has more complicated properties than that of the discrete p-Laplacian operator, mainly because it is not homogeneous.

In [7], the authors studied the following problem

$$\begin{cases} -\triangle\left(\phi_p\left(\triangle u(k-1)\right)\right) = \lambda f(k,u(k)), & k\in\mathbb{Z},\\ u(-\infty)=-1, \quad u(\infty)=1. \end{cases} \tag{1.8}$$

By using the critical point theory, the authors proved the existence of at least one solution of problem (1.8) for all $\lambda > 2/p$.

In [29], the authors considered the case where $0 < \lambda \le 2/p$ of the problem (1.8). By using the mountain pass lemma, they obtained the existence of at least one nontrivial solution of problem (1.8) in the case where $0 < \lambda \le 2/p$.

For discrete problems with $p(k)$-Laplacian operator, in [18] the authors proved, by using the minimization method, the existence of weak heteroclinic solution for the following problem

$$\begin{cases} -\triangle\left(a(k-1,\triangle u(k-1))\right)+g(k,u(k))=f(k,u(k)), \ \ k\in\mathbb{Z}^*,\\ u(0)=0, \ \ \lim\limits_{k\to-\infty}u(k)=-1, \ \ \lim\limits_{k\to\infty}u(k)=1. \end{cases} \tag{1.9}$$

In [17], the authors studied a slightly more general version of the problem (1.9), that is,

$$\begin{cases} -\triangle\left(a(k-1,\triangle u(k-1))\right)+\alpha(k)g(k,u(k))=\delta(k)f(k,u(k)), \ \ k\in\mathbb{Z}^*,\\ u(0)=0, \ \ \lim\limits_{k\to\infty}u(k)=-1, \ \ \lim\limits_{k\to\infty}u(k)=1, \end{cases} \tag{1.10}$$

by using a minimization method to obtain the existence of heteroclinic solutions.

In this paper, using the mountain pass theorem of Pucci and Serrin (see [38]) and the variational principle of Ricceri (see [41]), we prove the existence of at least one nontrivial heteroclinic solution and two nontrivial heteroclinic solutions of problem (1.4). This work is largely motivated by the papers [17, 29].

The paper is organized as follows. In Section 2, some necessary preliminary results are presented. Section 3 is devoted to auxiliary results. The existence of at least one nontrivial heteroclinic solution of problem (1.4) is investigated in Section 4. Finally, in Section 5, we prove the existence of at least two nontrivial heteroclinic solutions of problem (1.4).

2. Preliminaries

In this section, we first define the functional spaces and some of their useful properties which will be used later. We introduce for each $p(\cdot):\mathbb{Z}^+\to(1,\infty)$ and $q(\cdot):\mathbb{Z}^+\to\mathbb{R}$ the spaces

$$l^{p(\cdot)}_{-1,+}=\left\{x:\mathbb{Z}^+\to\mathbb{R};\ x(0)=-1 \text{ and } \rho_{p_+(\cdot)}(x):=\sum_{k=1}^{\infty}|x(k)|^{p(k)}<\infty\right\},$$

$$l^{p(\cdot)}_{-1,+,q(\cdot)}=\left\{x:\mathbb{Z}^+\to\mathbb{R};\ x(0)=-1 \text{ and } \rho_{q(\cdot),p_+(\cdot)}(x):=\sum_{k=1}^{\infty}\frac{q(k)}{p(k)}|x(k)|^{p(k)}<\infty\right\}$$

and

$$W^{1,p(\cdot)}_{-1,+,q(\cdot)}=\left\{x:\mathbb{Z}^+\to\mathbb{R};\ x(0)=-1 \text{ and } \rho_{1,q(\cdot),p_+(\cdot)}(x):=\sum_{k=1}^{\infty}\frac{q(k)}{p(k)}|x(k)|^{p(k)}+\sum_{k=1}^{\infty}\frac{|\triangle x(k)|^{p(k)}}{p(k)}<\infty\right\}.$$

On $l^{p(\cdot)}_{-1,+}$ and $l^{p(\cdot)}_{-1,+,q(\cdot)}$, we introduce the Luxemburg norms

$$\|x\|_{p_+(\cdot)}=\inf\left\{\nu>0;\sum_{k=1}^{\infty}\left|\frac{x(k)}{\nu}\right|^{p(k)}\leq 1\right\},$$

$$\|x\|_{q(\cdot),p_+(\cdot)} = \inf\left\{\nu > 0; \sum_{k=1}^{\infty} \frac{q(k)}{p(k)}\left|\frac{x(k)}{\nu}\right|^{p(k)} \leq 1\right\}$$

and, on the space $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$, the Luxemburg norm

$$\|x\|_{1,q(\cdot),p_+(\cdot)} = \inf\left\{\nu > 0; \sum_{k=1}^{\infty} \frac{q(k)}{p(k)}\left|\frac{x(k)}{\nu}\right|^{p(k)} + \sum_{k=1}^{\infty} \frac{1}{p(k)}\left|\frac{\triangle x(k)}{\nu}\right|^{p(k)} \leq 1\right\}.$$

We also consider the spaces

$$l^1_{-1,+} = \left\{x : \mathbb{Z}^+ \to \mathbb{R};\ x(0) = -1 \text{ and } \|x\|_{l^1_{-1,+}} := \sum_{k=1}^{\infty} |x(k)| < \infty\right\}$$

and

$$l^\infty_{-1,+} = \left\{x : \mathbb{Z}^+ \to \mathbb{R};\ x(0) = -1 \text{ and } \|x\|_\infty := \sup_{k\in\mathbb{Z}^+} |x(k)| < \infty\right\}.$$

Put

$$K := \sup_{k\in\mathbb{Z}^+} \left(\frac{p(k)}{q(k)}\right)^{\frac{1}{p(k)}}.$$

Note that $K < \infty$.

We use several inequalities which are connected to the above Luxemburg norms.

$$\|x\|_\infty \leq K\|x\|_{1,q(\cdot),p_+(\cdot)}. \tag{2.1}$$

Indeed, since $\|x\|_{1,q(\cdot),p_+(\cdot)} \leq \nu$, there exists $\epsilon > 0$ such that

$$\|x\|_{1,q(\cdot),p_+(\cdot)} \leq \nu \leq \|x\|_{1,q(\cdot),p_+(\cdot)} + \epsilon \text{ for all } x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}.$$

Hence, for all $k \in \mathbb{Z}^+$, we get

$$\frac{q(k)}{p(k)}\left|\frac{x(k)}{\nu}\right|^{p(k)} \leq 1.$$

So,

$$|x(k)| \leq \nu\left(\frac{p(k)}{q(k)}\right)^{\frac{1}{p(k)}},\ \ k \in \mathbb{Z}^+,$$

which imply that

$$\|x\|_\infty \leq K\left(\|x\|_{1,q(\cdot),p_+(\cdot)} + \epsilon\right).$$

Letting $\epsilon = 0$, one has $\|x\|_\infty \leq K\|x\|_{1,q(\cdot),p_+(\cdot)}$. Therefore, we obtain that (2.1) holds.

As in [13, 30], we have the following result.

Lemma 2.1. *The embedding* $W^{1,p(\cdot)}_{-1,+,q(\cdot)} \hookrightarrow l^{p(\cdot)}_{-1,+,q(\cdot)}$ *is compact.*

We also have the following results (see [16, 17]).

Lemma 2.2. *Assume that the condition (1.2) is fulfilled. Then the following properties hold.*
(a) $\rho_{q(\cdot),p_+(\cdot)}(x+y) \leq 2^{p^+}\left(\rho_{q(\cdot),p_+(\cdot)}(x) + \rho_{q(\cdot),p_+(\cdot)}(y)\right)$ *for all* $x, y \in l^{p(\cdot)}_{-1,+,q(\cdot)}$;
(b) for $x \in l^{p(\cdot)}_{-1,+,q(\cdot)}$, *if* $\gamma > 1$, *then*

$$\begin{aligned}\rho_{q(\cdot),p_+(\cdot)}(x) &\leq \gamma\rho_{q(\cdot),p_+(\cdot)}(x) \leq \gamma^{p^-}\rho_{q(\cdot),p_+(\cdot)}(x)\\ &\leq \rho_{q(\cdot),p_+(\cdot)}(\gamma x) \leq \gamma^{p^+}\rho_{q(\cdot),p_+(\cdot)}(x)\end{aligned}$$

and if $0 < \gamma < 1$, *then*

$$\begin{aligned}\gamma^{p^+}\rho_{q(\cdot),p_+(\cdot)}(x) &\leq \rho_{q(\cdot),p_+(\cdot)}(\gamma x) \leq \gamma^{p^-}\rho_{q(\cdot),p_+(\cdot)}(x)\\ &\leq \gamma\rho_{q(\cdot),p_+(\cdot)}(x) \leq \rho_{q(\cdot),p_+(\cdot)}(x);\end{aligned}$$

(c) for every fixed $x \in l^{p(\cdot)}_{-1,+,q(\cdot)} \setminus \{0\}$, $\rho_{q(\cdot),p_+(\cdot)}(\gamma x)$ *is continuous convex even function in* γ, *and it increases strictly when* $\gamma \in [0,\infty)$.

Proposition 2.3. *If* $x, x_n \in l^{p(\cdot)}_{-1,+}$ *and* $p^+ < \infty$, *then the following properties hold.*

1. $\|x\|_{p_+(\cdot)} < 1\ (=1; >1)$ *if and only if* $\rho_{p_+(\cdot)}(x) < 1\ (=1; >1)$,
2. $\|x\|_{p_+(\cdot)} > 1$ *implies* $\|x\|^{p^-}_{p_+(\cdot)} \leq \rho_{p_+(\cdot)}(x) \leq \|x\|^{p^+}_{p_+(\cdot)}$,
3. $\|x\|_{p_+(\cdot)} < 1$ *implies* $\|x\|^{p^+}_{p_+(\cdot)} \leq \rho_{p_+(\cdot)}(x) \leq \|x\|^{p^-}_{p_+(\cdot)}$,
4. $\|x_n\|_{p_+(\cdot)} \to 0$ *if and only if* $\rho_{p_+(\cdot)}(x_n) \to 0$ *as* $n \to \infty$.

Proposition 2.4. *Let* $x \in l^{p(\cdot)}_{-1,+,q(\cdot)} \setminus \{0\}$. *Then,*

$$\|x\|_{q(\cdot),p_+(\cdot)} = \beta \ \text{ if and only if } \ \rho_{q(\cdot),p_+(\cdot)}\left(\frac{x}{\beta}\right) = 1.$$

Proposition 2.5. *If* $x, x_n \in l^{p(\cdot)}_{-1,+,q(\cdot)}$ *and* $p^+ < \infty$, *then the following properties hold.*

1. $\|x\|_{q(\cdot),p_+(\cdot)} < 1\ (=1; >1)$ *if and only if* $\rho_{q(\cdot),p_+(\cdot)}(x) < 1\ (=1; >1)$,
2. $\|x\|_{q(\cdot),p_+(\cdot)} > 1$ *imply* $\|x\|^{p^-}_{q(\cdot),p_+(\cdot)} \leq \rho_{q(\cdot),p_+(\cdot)}(x) \leq \|x\|^{p^+}_{q(\cdot),p_+(\cdot)}$,
3. $\|x\|_{q(\cdot),p_+(\cdot)} < 1$ *imply* $\|x\|^{p^+}_{q(\cdot),p_+(\cdot)} \leq \rho_{q(\cdot),p_+(\cdot)}(x) \leq \|x\|^{p^-}_{q(\cdot),p_+(\cdot)}$,
4. $\|x_n\|_{q(\cdot),p_+(\cdot)} \to 0$ *if and only if* $\rho_{q(\cdot),p_+(\cdot)}(x_n) \to 0$ *as* $n \to \infty$.

Proposition 2.6. *Let* $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)} \setminus \{0\}$. *Then,*

$$\|x\|_{1,q(\cdot),p_+(\cdot)} = \theta \ \text{ if and only if } \ \rho_{1,q(\cdot),p_+(\cdot)}\left(\frac{x}{\theta}\right) = 1.$$

Proposition 2.7. *If* $x, x_n \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ *and* $p^+ < \infty$, *then the following properties hold.*

1. $\|x\|_{1,q(\cdot),p_+(\cdot)} < 1\ (=1; >1)$ *if and only if* $\rho_{1,q(\cdot),p_+(\cdot)}(x) < 1\ (=1; >1)$,
2. $\|x\|_{1,q(\cdot),p_+(\cdot)} > 1$ *imply* $\|x\|^{p^-}_{1,q(\cdot),p_+(\cdot)} \leq \rho_{1,q(\cdot),p_+(\cdot)}(x) \leq \|x\|^{p^+}_{1,q(\cdot),p_+(\cdot)}$,
3. $\|x\|_{1,q(\cdot),p_+(\cdot)} < 1$ *imply* $\|x\|^{p^+}_{1,q(\cdot),p_+(\cdot)} \leq \rho_{1,q(\cdot),p_+(\cdot)}(x) \leq \|x\|^{p^-}_{1,q(\cdot),p_+(\cdot)}$,

(4) $\|x_n\|_{1,q(\cdot),p_+(\cdot)} \to 0$ *if and only if* $\rho_{1,q(\cdot),p_+(\cdot)}(x_n) \to 0$ *as* $n \to \infty$.

One also has the following lemma (see [18]).

Lemma 2.8. *(Discrete Hölder type inequality) Let* $x \in l_+^{p(\cdot)}$ *and* $y \in l_+^{p'(\cdot)}$, *where* $l_+^{p'(\cdot)}$ *denote the conjugate space of* $l_+^{p(\cdot)}$, *with* $\dfrac{1}{p(k)} + \dfrac{1}{p'(k)} = 1$ *for all* $k \in \mathbb{Z}^+$. *Then,*

$$\sum_{k=1}^{\infty} |xy| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|x\|_{p_+(\cdot)} \|y\|_{p'_+(\cdot)} \leq 2\|x\|_{p_+(\cdot)} \|y\|_{p'_+(\cdot)}.$$

Our first tool and approach is based on the variational principle of Ricceri [41, Theorem 2.1], which is recalled below (see [3, Theorem 2.1(a)] or [4, Theorem 3.1 and Remark 3.1] for a different version of the result). We refer to the works [1, 5, 6, 14, 15, 20, 21, 22, 33] in which Theorem 2.9 below has been successfully applied to show the existence of at least one nontrivial solution for boundary value problems.

Theorem 2.9. *(see [3]). Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is (strongly) continuous, sequentially weakly lower semicontinuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$ and for each $r > \inf_{u \in X} \Phi(u)$, let ψ be the function defined as*

$$\psi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}.$$

Then, for each $r > \inf_{u \in X} \Phi(u)$ and each $\lambda \in (0, \frac{1}{\psi(r)})$, the restriction of I_λ to $\Phi^{-1}((-\infty, r))$ admits a global minimum (precisely a local minimum), which is a critical point of I_λ in X.

Our second main tool and approach is based on the critical point theory developed by Pucci and Serrin (see [38]). We refer to the papers [24, 25] in which Theorem 2.10 below has been applied to obtain the multiple solutions for the discrete problems involving the p-Laplacian operator.

Next, if X is a reflexive Banach space and X^* is its dual, we recall that a function $J \in C^1(X)$ is said to satisfy the Palais-Smale condition ((PS) for short) if every sequence (u_n) in X such that $(J(u_n))$ is bounded and $J'(u_n) \to 0$ in X^* has a convergent subsequence.

We recall the mountain pass theorem, due to Pucci and Serrin (see [38]).

Theorem 2.10. *([38, Theorem 1]). If X is a reflexive Banach space, $J \in C^1(X)$ satisfies (PS), and there exist $\check{u} \in X$ and real numbers $0 < r_1 < r_2$ such that $\|\check{u}\| > r_2$ and*

$$\inf_{r_1 \leq \|u\| \leq r_2} J(u) = \alpha \geq \max\{J(0), J(\check{u})\},$$

then there exists a critical point $\hat{u} \in X$ *of* J *such that* $J(\hat{u}) \geq \alpha$. *Moreover, if* $J(\hat{u}) = \alpha$ *then* $r_1 \leq \|\hat{u}\| \leq r_2$.

3. Variational structure and auxiliary results

In this part, we establish a variational structure and auxiliary results for problem (1.4), which will be used in the next section. We connect solutions to (1.4) with critical points of the following functional.

$$I_\lambda(x) = \widehat{M}(\eta[x]) - \lambda \sum_{k=1}^{\infty} F(k, x(k)+1), \tag{3.1}$$

where $\widehat{M}(t) = \int\limits_0^t M(s)ds$ and $F(k,t) = \int\limits_0^t f(k,s)ds$.

Let us define the functionals $\Phi, \Psi : W^{1,p(\cdot)}_{-1,+,q(\cdot)} \to \mathbb{R}$ by

$$\Phi(x) = \widehat{M}(\eta[x]) \tag{3.2}$$

and

$$\Psi(x) = \sum_{k=1}^{\infty} F(k, x(k)+1). \tag{3.3}$$

Then, for every $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$, we have

$$I_\lambda(x) = \Phi(x) - \lambda\Psi(x), \;\; \lambda > 0. \tag{3.4}$$

It is easy to show that Φ and Ψ are two functionals of class $C^1(W^{1,p(\cdot)}_{-1,+,q(\cdot)}, \mathbb{R})$ whose Gâteaux derivatives at the point $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ are given by

$$\begin{aligned} \langle \Phi'(x), y \rangle &= M\left(\eta[x]\right) \left[\sum_{k=1}^{\infty} a(k-1, |\triangle x(k-1)|) \triangle x(k-1) \triangle y(k-1) \right. \\ &+ \left. \sum_{k=1}^{\infty} q(k)|x(k)|^{p(k)-2} x(k) y(k) \right], \end{aligned} \tag{3.5}$$

$$\langle \Psi'(x), y \rangle = \sum_{k=1}^{\infty} f(k, x(k)+1) y(k), \tag{3.6}$$

for all $x, y \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$.

Then, the functional I_λ is of class $C^1(W^{1,p(\cdot)}_{-1,+,q(\cdot)}, \mathbb{R})$ with the derivative given by

$$\langle I'_\lambda(x), y \rangle = \langle \Phi'(x), y \rangle - \lambda \langle \Psi'(x), y \rangle,$$

for all $x, y \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$.

A critical point to I_λ is a point $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ such that

$$M(\eta[x])\left[\sum_{k=1}^{\infty} a(k-1,|\triangle x(k-1)|)\triangle x(k-1)\triangle y(k-1) + \sum_{k=1}^{\infty} q(k)|x(k)|^{p(k)-2}x(k)y(k)\right] = \lambda\sum_{k=1}^{\infty} f(k,x(k)+1)y(k), \tag{3.7}$$

which in turn is a weak solution to the problem (1.4) for any $y \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. Hence, to solve the problem (1.4), we need to find critical points of problem (1.4) and further investigate their multiplicity.

Example 2.1 As examples of functions A_0 and a satisfying assumptions (1.2), $(H1)$ and $(H2)$, we can give the following.

- If $A_0(k,|\xi|) = \dfrac{1}{p(k)}|\xi|^{p(k)}$, then $a(k,|\xi|) = |\xi|^{p(k)-2}$, for all $(k,\xi) \in \mathbb{Z}^+ \times \mathbb{R}$.
- If $A_0(k,|\xi|) = \dfrac{1}{p(k)}\left[\left(1+|\xi|^2\right)^{\frac{p(k)}{2}} - 1\right]$, then $a(k,|\xi|) = (1+|\xi|^2)^{\frac{p(k)-2}{2}}$, for all $(k,\xi) \in \mathbb{Z}^+ \times \mathbb{R}$.
- If $A_0(k,|\xi|) = \dfrac{1}{p(k)}|\xi|^{p(k)} + \dfrac{\sqrt{1+|\xi|^{2p(k)}}}{p(k)}$, then $a(k,|\xi|) = \left(1 + \dfrac{|\xi|^{p(k)}}{\sqrt{1+|\xi|^{2p(k)}}}\right)|\xi|^{p(k)-2}$, for all $(k,\xi) \in \mathbb{Z}^+ \times \mathbb{R}$.

Lemma 3.1. *Suppose that the hypotheses* (1.3) *and* $(H3)$ *hold. Then, every critical point* $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ *of* I_λ *is a solution to the problem* (1.4) *for all* $\lambda > 0$.

Proof. Fix $\lambda > 0$. Assume that

$$I'_\lambda(x) = \Phi'(x) - \lambda\Psi'(x) = 0 \text{ in } (W^{1,p(\cdot)}_{-1,+,q(\cdot)})^*.$$

From relations (3.5) and (3.6), we see that

$$M(\eta[x])\sum_{k=1}^{\infty}\Big[a(k-1,|\triangle x(k-1)|)\triangle x(k-1)\triangle y(k-1) + q(k)|x(k)|^{p(k)-2}x(k)y(k) - \lambda f(k,x(k)+1)y(k)\Big] = 0, \tag{3.8}$$

for all $y \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$.

For each $m \in \mathbb{Z}$, we consider $e_m \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ setting by

$$e_m(k) := \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \in \mathbb{Z}^+\backslash\{m\}, \end{cases} \tag{3.9}$$

for all $k \in \mathbb{Z}^+$.
Thus, it follows from (3.8), with $y = e_m$ that
$-M(\eta[x])[\triangle(a(m-1, |\triangle x(m-1)|)\triangle x(m-1)) - q(m)|x(m)|^{p(m)-2}x(m)]$
$-\lambda f(m, x(m)+1) = 0$.
Therefore, x is a solution of (1.4).

Proposition 3.2. *Assume that the hypothesis $(H2)$ is fulfilled. Then, the following estimate*

$$\langle a(k,|x|)x - a(k,|y|)y, x-y\rangle$$
$$\geq \begin{cases} c\,(|x|+|y|)^{p(k)-2}\,|x-y|^2 & \text{if } 1 < p(k) < 2 \\ 4^{2-p^+}c|x-y|^{p(k)} & \text{if } \quad p(k) \geq 2. \end{cases}$$

holds true for all $x, y \in \mathbb{R}$ and $k \in \mathbb{Z}^+$ with $(x,y) \neq (0,0)$.

Proof. Let $x, y \in \mathbb{R}$ such that $(x,y) \neq (0,0)$. Define $\varphi(k,x) = a(k,|x|)x$.
By $(H2)$, we can write, for all $x \in \mathbb{R}\backslash\{0\}$,

$$\frac{\partial\varphi(k,x)}{\partial x} = |x|\frac{\partial a}{\partial x}(k,|x|) + a(k,|x|) \geq c|x|^{p(k)-2}. \tag{3.10}$$

Note that

$$\varphi(k,x) - \varphi(k,y) = \int_0^1 \frac{\partial\varphi(k, y+t(x-y))}{\partial x}(x-y)dt. \tag{3.11}$$

Consider $k \in \mathbb{Z}^+$ such that $p(k) \geq 2$. Then, from (3.10) and (3.11), one has

$$\begin{aligned} \langle a(k,|x|)x - a(k,|y|)y, x-y\rangle &= \int_0^1 \frac{\partial\varphi}{\partial x}(k, y+t(x-y))(x-y)(x-y)dt \\ &\geq \int_0^1 c|y+t(x-y)|^{p(k)-2}|x-y|^2 dt. \end{aligned}$$

Without loss of generality, we can assume that $|x| \leq |y|$. Then, for any $t \in [0, 1/4]$, we obtain

$$|y+t(x-y)| \geq |y| - \frac{1}{4}|x-y| \geq \frac{1}{4}|x-y|.$$

Thus,

$$\begin{aligned} \langle a(k,|x|)x - a(k,|y|)y, x-y\rangle &\geq \int_0^1 c|y+t(x-y)|^{p(k)-2}|x-y|^2 dt \\ &\geq 4^{2-p^+}c|x-y|^{p(k)}. \end{aligned}$$

Next, let $k \in \mathbb{Z}^+$ with $1 < p(k) < 2$. As before, condition $(H2)$ implies that

$$\begin{aligned} \frac{\partial\varphi(k,x)}{\partial x} &= |x|\frac{\partial a}{\partial x}(k,|x|) + a(k,|x|) \\ &\geq c|x|^{p(k)-2}, \end{aligned}$$

for all $x \in \mathbb{R} \setminus \{0\}$. By the fact that $|tx + (1-t)y| \le |x| + |y|$, we conclude that

$$\begin{aligned}\langle a(k,|x|)x - a(k,|y|)y, x-y\rangle &\ge \int_0^1 c|y + t(x-y)|^{p(k)-2}|x-y|^2 dt \\ &\ge c\left(|x|+|y|\right)^{p(k)-2}|x-y|^2.\end{aligned}$$

The proof of Proposition 3.2 is complete. □

Lemma 3.3. *Assume that $(H1)$-$(H3)$ hold true. Then, the operator $\Phi' : W^{1,p(\cdot)}_{-1,+,q(\cdot)} \to (W^{1,p(\cdot)}_{-1,+,q(\cdot)})^*$ is strictly monotone on $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ and verify the (S_+) condition, i.e., for every sequence $\{x_n\} \subset W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ such that $x_n \rightharpoonup x$ in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle \Phi'(x_n) - \Phi'(x), x_n - x\rangle \le 0$, one has $x_n \to x$ in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ as $n \to \infty$. Here, $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ and its dual $(W^{1,p(\cdot)}_{-1,+,q(\cdot)})^*$.*

Proof. One shows that Φ' is a strict monotone operator.
Let us consider the functional $\phi : W^{1,p(\cdot)}_{-1,+,q(\cdot)} \to \mathbb{R}$ is given by

$$\phi(x) = \eta[x] = \sum_{k=1}^{\infty}\left(\int_0^{|\triangle x(k-1)|} a(k-1,s)sds + \frac{q(k)}{p(k)}|x(k)|^{p(k)}\right), \text{ for all } x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}.$$

Then, $\phi \in C^1(W^{1,p(\cdot)}_{-1,+,q(\cdot)}, \mathbb{R})$ and its Gâteaux derivative given by

$$\langle \phi'(x), y\rangle = \sum_{k=1}^{\infty} a(k-1, |\triangle x(k-1)|)\triangle x(k-1)\triangle y(k-1) + \sum_{k=1}^{\infty} q(k)|x(k)|^{p(k)-2}x(k)y(k),$$

for all $x, y \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$.
For all $x, y \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ such that $x \ne y$, we see that

$$\begin{aligned}&\langle \phi'(x) - \phi'(y), x-y\rangle \\ &= \sum_{k=1}^{\infty}\left(a(k-1,|\triangle x(k-1)|)\triangle x(k-1) - a(k-1,|\triangle y(k-1)|)\triangle y(k-1)\right)\triangle(x-y)(k-1) \\ &+ \sum_{k=1}^{\infty} q(k)\left(|x(k)|^{p(k)-2}x(k) - |y(k)|^{p(k)-2}y(k)\right)(x(k)-y(k)).\end{aligned}$$

From Proposition 3.2 and the following well-known inequality, for all $x, y \in \mathbb{R}$,

$$\begin{aligned}&\left(|x|^{p(k)-2}x - |y|^{p(k)-2}y\right)(x-y) \\ &\ge \begin{cases} c_1\left(|x|+|y|\right)^{p(k)-2}|x-y|^2 & \text{if } 1 < p(k) < 2, \\ c_2|x-y|^{p(k)} & \text{if } \quad p(k) \ge 2.\end{cases}\end{aligned} \tag{3.12}$$

Therefore,

$$\langle \phi'(x)-\phi'(y), x-y\rangle \geq \begin{cases} \min\{c,c_1\}\sum_{k=1}^{\infty} \breve{x}(k)^{p(k)-2}|\triangle x(k)-\triangle y(k)|^2>0 & \text{if } 1<p(k)<2,\\ \min\left\{4^{2-p^+}c, c_2\right\}\sum_{k=1}^{\infty}|\triangle x(k)-\triangle y(k)|^{p(k)}>0 & \text{if } p(k)\geq 2,\end{cases}$$

where $\breve{x}(k)=|\triangle x(k)|+|\triangle y(k)|$. This implies that ϕ' is strictly monotone. Then, from Proposition 25.10 of [45], it follows that ϕ is strictly convex. However, as M is nondecreasing, then $\widehat{M}$ is convex in $(0,\infty)$. Therefore, for all $x,y\in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ with $x\neq y$ and any $\tau_1,\tau_2\in(0,1)$ such that $\tau_1+\tau_2=1$, one has

$$\widehat{M}\left(\phi(\tau_1 x+\tau_2 y)\right)<\widehat{M}\left(\tau_1\phi(x)+\tau_2\phi(y)\right)\leq \tau_1\widehat{M}\left(\phi(x)\right)+\tau_2\widehat{M}\left(\phi(y)\right).$$

This proves that Φ is strictly convex and therefore Φ' is strictly monotone in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$.

Next, let $\{x_n\}\subset W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ be a sequence such that $x_n\rightharpoonup x$ in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ as $n\to\infty$ and

$$\limsup_{n\to\infty}\langle\Phi'(x_n)-\Phi'(x), x_n-x\rangle\leq 0.$$

By the above inequality and the strictly monotonicity of Φ', we know that

$$\lim_{n\to\infty}\langle\Phi'(x_n)-\Phi'(x), x_n-x\rangle=0.$$

Thus,

$$\lim_{n\to\infty}\langle\Phi'(x_n), x_n-x\rangle=0,$$

hence,

$$\lim_{n\to\infty} M\left(\eta[x_n]\right)\left[\sum_{k=1}^{\infty}a(k-1,|\triangle x_n(k-1)|)\triangle x_n(k-1)\triangle(x_n-x)(k-1)\right.$$
$$\left.+\sum_{k=1}^{\infty}q(k)|x_n(k)|^{p(k)-2}x_n(k)(x_n-x)(k)\right]=0. \tag{3.13}$$

Furthermore, condition $(H1)$ combined with Lemma 2.8 and Proposition 2.7, imply that

$$\begin{aligned}\eta[x_n]&=\sum_{k=1}^{\infty}\left(\int_0^{|\triangle x_n(k-1)|}a(k-1,s)ds+\frac{q(k)}{p(k)}|x_n(k)|^{p(k)}\right)\\ &\leq\sum_{k=1}^{\infty}a_1(k-1)|\triangle x_n(k-1)|+a_2\sum_{k=1}^{\infty}\frac{|\triangle x_n(k-1)|^{p(k-1)}}{p(k-1)}+\sum_{k=1}^{\infty}\frac{q(k)}{p(k)}|x_n(k)|^{p(k)}\\ &\leq 2\|\triangle x_n\|_{p_+(\cdot)}\|a_1\|_{p'_+(\cdot)}+\max\{a_2,1\}\rho_{1,q(\cdot),p_+(\cdot)}(x_n)\\ &\leq K_1\|x_n\|_{1,q(\cdot),p_+(\cdot)}+K_2\max\left\{\|x_n\|^{p^+}_{1,q(\cdot),p_+(\cdot)},\|x_n\|^{p^-}_{1,q(\cdot),p_+(\cdot)}\right\}.\end{aligned}$$

So, it follows that $(\eta[x_n])_{n\geq 1}$ is bounded.
As M is continuous, up to a subsequence, there is $t_0 \geq 0$ such that

$$M\left(\eta[x_n]\right) \to M(t_0) \geq m_0 \text{ as } n \to \infty. \tag{3.14}$$

Relations (3.13) and (3.14) imply that

$$\lim_{n\to\infty} \Bigg[\sum_{k=1}^{\infty} a(k-1, |\triangle x_n(k-1)|)\triangle x_n(k-1)\triangle(x_n - x)(k-1) + \sum_{k=1}^{\infty} q(k)|x_n(k)|^{p(k)-2}x_n(k)(x_n - x)(k)\Bigg] = 0.$$

Since the embedding $W^{1,p(\cdot)}_{-1,+,q(\cdot)} \hookrightarrow l^{p(\cdot)}_{-1,+,q(\cdot)}$ is compact, one has

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} q(k)|x_n(k)|^{p(k)-2}x_n(k)(x_n - x)(k) = 0$$

and so

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a(k-1, |\triangle x_n(k-1)|)\triangle x_n(k-1)\triangle(x_n - x)(k-1) = 0.$$

Therefore,

$$\lim_{n\to\infty} \langle \phi'(x_n), x_n - x\rangle = 0.$$

Then,

$$\lim_{n\to\infty} \langle \phi'(x_n) - \phi'(x), x_n - x\rangle = 0. \tag{3.15}$$

On the other hand, by Proposition 3.2 and using (3.12), we deduce that

$$\langle \phi'(x_n) - \phi'(x), x_n - x\rangle \geq \begin{cases} \min\{c, c_1\} \sum_{k=1}^{\infty} \tilde{x}(k)^{p(k)-2}|\triangle x_n(k) - \triangle x(k)|^2 > 0 & \text{if } 1 < p(k) < 2, \\ \min\left\{4^{2-p^+}c, c_2\right\} \sum_{k=1}^{\infty} |\triangle x_n(k) - \triangle x(k)|^{p(k)} > 0 & \text{if } \quad p(k) \geq 2, \end{cases} \tag{3.16}$$

where $\tilde{x}(k) = |\triangle x_n(k)| + |\triangle x(k)|$.

Therefore, by using discrete Hölder type inequality (see Lemma 2.8), we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} |\triangle x_n(k) - \triangle x(k)|^{p(k)} \\ &= \sum_{k=1}^{\infty} \tilde{x}(k)^{\frac{p(k)(2-p(k))}{2}} \left(\tilde{x}(k)^{\frac{p(k)(p(k)-2)}{2}}|\triangle x_n(k) - \triangle x(k)|^{p(k)}\right) \\ &\leq 2\|\tilde{x}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}} \|\tilde{x}^{\frac{p(\cdot)(p(\cdot)-2)}{2}}|\triangle x_n(k) - \triangle x(k)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}} \\ &\leq 2\|\tilde{x}\|^{\alpha}_{p_+(\cdot)} \left(\sum_{k=1}^{\infty} \tilde{x}(k)^{p(k)-2}|\triangle x_n(k) - \triangle x(k)|^2\right)^{\beta}, \end{aligned} \tag{3.17}$$

where α is either $p^-(2-\overline{p})/2$ or $\overline{p}(2-p^-)/2$ and β is either $p^-/2$ or $\overline{p}/2$ with $\overline{p} = \sup\limits_{\{k\in\mathbb{Z}^+:1<p(k)<2\}} p(k)$. Hence, by using (3.15)-(3.17), we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}|\triangle x_n(k)-\triangle x(k)|^{p(k)}=0. \tag{3.18}$$

Using Proposition 2.3(4), we deduce by relation (3.18) that

$$\|\triangle x_n-\triangle x\|_{p_+(\cdot)}\to 0 \ \text{ as } \ n\to\infty$$

and then

$$\|x_n-x\|_{1,q(\cdot),p_+(\cdot)}\to 0 \ \text{ as } \ n\to\infty.$$

Therefore Φ' is of type (S_+). Thus, the proof of Lemma 3.3 is complete. □

Lemma 3.4. *Assume that* $(H1)$*-*$(H3)$ *and* (1.2) *hold. Then,* Φ *is weakly lower semi-continuous, i.e.,* $x_n \rightharpoonup x$ *in* $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ *as* $n\to\infty$ *imply that* $\Phi(x)\leq \liminf\limits_{n\to\infty}\Phi(x_n)$.

Proof. Indeed, assuming that $x_n \rightharpoonup x$ in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ as $n\to\infty$. From (3.5) and Lemma 3.3, it follows that Φ is convex (see [46, Proposition 42.6]),so we deduce that for each $n\in\mathbb{N}$,

$$\Phi(x_n)\geq\Phi(x)+\langle\Phi'(x),x_n-x\rangle.$$

Then,

$$\liminf_{n\to\infty}\Phi(x_n)\geq\Phi(x)+\liminf_{n\to\infty}\langle\Phi'(x),x_n-x\rangle.$$

Therefore,

$$\liminf_{n\to\infty}\Phi(x_n)\geq\Phi(x).$$

Hence, the proof of Lemma 3.4 is complete. □

4. Existence of a heteroclinic solution

In this section, we establish the existence of at least one nontrivial heteroclinic solution of problem (1.4), by using Theorem 2.9.
We make the following additional assumptions.

$(H4)$ $\displaystyle\sup_{\varepsilon>0}\frac{\varepsilon^{p^+}}{\sum_{k=1}^{\infty}\max_{|t|\leq\varepsilon}F(k,t+1)}<\infty.$

$(H5)$ $\displaystyle\sum_{k=1}^{\infty}\max_{|t|\leq T}f(k,t+1)<\infty$ for all $T>0$.

$(H6)$ There exists $k_0\in\mathbb{Z}^+$ such that

$$\liminf_{|t|\to\infty}\inf_{k_0\in\mathbb{Z}^+}\frac{F(k_0,t+1)}{|t|^{p^-}}=\infty.$$

Theorem 4.1. *Suppose that the conditions* $(H1)$-$(H6)$ *are satisfied. Then, there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *problem* (1.4) *has at least one non-trivial solution* x_λ *in* $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. *Moreover,* $1/\rho_{1,q(\cdot),p_+(\cdot)}(x_\lambda) \to 0$ *as* $\lambda \to \infty$.

Proof. We aim is to apply Theorem 2.9 to the problem (1.4). Take $X = W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ and put Φ, Ψ, I_λ as in (3.2), (3.3) and (3.4), respectively. By Lemma 3.1, the critical point $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ of I_λ is a solution to the problem (1.4). The functional Φ is C^1 and is sequentially weakly lower semicontinuous in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. The functional Ψ is also C^1 and is sequentially weakly upper semicontinuous in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$.

We will prove that the functional Φ is coercive. Let $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ with $\|x\|_{1,q(\cdot),p_+(\cdot)} > 1$. Using the condition $(H3)$, we get

$$\Phi(x) = \widehat{M}(\eta[x]) \geq m_0 \int_0^{\eta[x]} d\xi.$$

For x in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ with $\|x\|_{1,q(\cdot),p_+(\cdot)} > 1$, from the above inequality, by using $(H2)$, one has

$$\begin{aligned}\Phi(x) &\geq m_0 \left(c \sum_{k=1}^{\infty} \frac{|\triangle x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{\infty} \frac{q(k)}{p(k)} |x(k)|^{p(k)} \right) \\ &\geq m_0 \min\{1, c\} \rho_{1,q(\cdot),p_+(\cdot)}(x). \end{aligned} \tag{4.1}$$

Hence, it follows from Proposition 2.7(2) that

$$\Phi(x) \geq m_0 \min\{1, c\} \|x\|^{p^-}_{1,q(\cdot),p_+(\cdot)}.$$

Therefore, Φ is coercive.

Choosing $K > \varepsilon$ and put $r := m_0 \min\{1, c\} \left(\frac{\varepsilon}{K}\right)^{p^+}$. One has $r > \inf_{x \in X} \Phi(x)$. Moreover, for all $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ such that $\Phi(x) < r$, by (4.1), we get

$$\rho_{1,q(\cdot),p_+(\cdot)}(x) \leq \frac{1}{m_0 \min\{1, c\}} \Phi(x) \leq \frac{r}{m_0 \min\{1, c\}} = \left(\frac{\varepsilon}{K}\right)^{p^+} < 1$$

and so $\|x\|_{1,q(\cdot),p_+(\cdot)} < 1$. Taking (3) of Proposition 2.7 and (4.1) into account, one has

$$\|x\|_{1,q(\cdot),p_+(\cdot)} \leq \left(\rho_{1,q(\cdot),p_+(\cdot)}(x)\right)^{\frac{1}{p^+}} \leq \left(\frac{\Phi(x)}{m_0 \min\{1, c\}}\right)^{\frac{1}{p^+}} \leq \left(\frac{r}{m_0 \min\{1, c\}}\right)^{\frac{1}{p^+}}. \tag{4.2}$$

Then, by (2.1) and (4.2), we obtain

$$\|x\|_\infty \leq K \|x\|_{1,q(\cdot),p_+(\cdot)} \leq K \left(\frac{r}{m_0 \min\{1, c\}}\right)^{\frac{1}{p^+}} = \varepsilon.$$

It follows that

$$\Phi^{-1}(-\infty, r) \subseteq \left\{ x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)} \text{ such that } \|x\|_\infty \leq \varepsilon \right\}.$$

Therefore,

$$\sup_{x\in\Phi^{-1}(-\infty,r)} \Psi(x) = \sup_{\Phi(x)\leq r} \sum_{k=1}^{\infty} F(k, x(k)+1) \leq \sum_{k=1}^{\infty} \max_{|t|\leq\varepsilon} F(k, t+1).$$

Since $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\psi(r) &\leq \frac{\sup\limits_{x\in\Phi^{-1}(-\infty,r)} \Psi(x)}{r} \leq \frac{\sum\limits_{k=1}^{\infty} \max\limits_{|t|\leq\varepsilon} F(k,t+1)}{r} \\
&\leq \frac{K^{p^+}}{m_0 \min\{1, c\}} \frac{\sum\limits_{k=1}^{\infty} \max\limits_{|t|\leq\varepsilon} F(k,t+1)}{\varepsilon^{p^+}}.
\end{aligned}$$

Hence,

$$\frac{m_0 \min\{1, c\}}{K^{p^+}} \frac{\varepsilon^{p^+}}{\sum\limits_{k=1}^{\infty} \max\limits_{|t|\leq\varepsilon} F(k,t+1)} \leq \frac{1}{\psi(r)}.$$

On the other hand, by $(H4)$, one has

$$\frac{\varepsilon^{p^+}}{\sum\limits_{k=1}^{\infty} \max\limits_{|t|\leq\varepsilon} F(k,t+1)} \leq \sup_{\varepsilon>0} \frac{\varepsilon^{p^+}}{\sum\limits_{k=1}^{\infty} \max\limits_{|t|\leq\varepsilon} F(k,t+1)} < \infty.$$

Thus, if we choose

$$\lambda^* = \frac{m_0 \min\{1, c\}}{K^{p^+}} \sup_{\varepsilon>0} \frac{\varepsilon^{p^+}}{\sum\limits_{k=1}^{\infty} \max\limits_{|t|\leq\varepsilon} F(k,t+1)},$$

then by the above inequality, $\lambda^* > 0$. Therefore, owing to Theorem 2.9, for any $\lambda \in (0, \lambda^*)$, the functional I_λ admits one critical point $x_\lambda \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ such that $\Phi(x_\lambda) < r$.

Next, we prove that x_λ is nontrivial. From $(H6)$, there exists a sequence $\{c_n\}$ in $W^{1,p(\cdot)}_{-1,+,q(\cdot)} \setminus \{0\}$ with $c_n \to 0$ as $n \to \infty$, such that

$$\liminf_{n\to\infty} \inf_{k_0\in\mathbb{Z}^+} \frac{F(k_0, c_n+1)}{|c_n|^{p^-}} = \infty. \tag{4.3}$$

Let us fix $K_1 > 0$. Choosing L such that

$$L > \frac{K_1 m_1 \left(p^- a_1^+ |c_n b| + (c_n b)^{p^-} (a_2 + q(k_0))\right)}{p^- |c_n b|^{p^-}} \tag{4.4}$$

and using relation (4.3), we deduce that there exists a constant $K_2 > 0$ such that for any $k_0 \in \mathbb{Z}^+$, with $|c_n| > K_2$, one has

$$\frac{F(k_0, c_n + 1)}{|c_n|^{p^-}} > L.$$

Moreover, there exists $0 < b < 1$ such that

$$\frac{F(k_0, c_n b + 1)}{|c_n b|^{p^-}} > L \text{ for all } |c_n| > K_2 \text{ and } k_0 \in \mathbb{Z}^+. \tag{4.5}$$

On the other hand, set $x_n = c_n \overline{x}$ for all $n \in \mathbb{N}$. Now, pick $\overline{x} \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$, defined as follows.

$$\overline{x}(k) := \begin{cases} b & \text{if } k = k_0, \\ 0 & \text{if } k \in \mathbb{Z}^+ - \{k_0\}. \end{cases}$$

So, by (3.2), (3.3), (4.5), $(H1)$ and $(H3)$, we get $\Psi(x_n) = F(k_0, c_n b+1) > L|c_n b|^{p^-}$ and

$$\begin{aligned} \Phi(x_n) &\leq m_1 \left(a_1^+ |c_n b| + \frac{a_2}{p(k_0)} |c_n b|^{p(k_0)} + \frac{q(k_0)}{p(k_0)} |c_n b|^{p(k_0)} \right) \\ &\leq m_1 \left(a_1^+ |c_n b| + \frac{(c_n b)^{p^-}}{p^-} (a_2 + q(k_0)) \right). \end{aligned}$$

Consequently,

$$\frac{\Psi(x_n)}{\Phi(x_n)} \geq \frac{p^- L |c_n b|^{p^-}}{m_1 \left(p^- a_1^+ |c_n b| + (c_n b)^{p^-} (a_2 + q(k_0)) \right)} \to \infty \text{ as } n \to \infty.$$

Therefore, there exists a sequence $\{y_n\} \subset W^{1,p(\cdot)}_{-1,+,q(\cdot)}$, $y_n \to 0$ as $n \to \infty$ such that, for n large enough, $y_n \in \Phi^{-1}(-\infty, r)$ and for any $\lambda > 0$, one has $\dfrac{\Phi(y_n)}{\Psi(y_n)} < \lambda$. Then, $\Phi(y_n) - \lambda \Psi(y_n) < 0$.
So, it follows that

$$I_\lambda(y_n) = \Phi(y_n) - \lambda \Psi(y_n) < 0.$$

Since x_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we obtain

$$I_\lambda(x_\lambda) \leq I_\lambda(y_n) < 0.$$

Hence, we conclude that x_λ is a nontrivial solution to the problem (1.4).

Finally, we claim that $1/\rho_{1,q(\cdot),p^+(\cdot)}(x_\lambda) \to 0$ as $\lambda \to \infty$. From $I_\lambda = \Phi - \lambda \Psi$, we get

$$\langle \Phi'(x_\lambda), x_\lambda \rangle = \langle I'_\lambda(x_\lambda), x_\lambda \rangle + \lambda \sum_{k=1}^{\infty} f(k, x_\lambda(k) + 1) x_\lambda(k).$$

Since x_λ is a critical point of I_λ for any $\lambda \in (0, \lambda^*)$, one has

$$\langle I'_\lambda(x_\lambda), x_\lambda \rangle = 0 \text{ for any } \lambda \in (0, \lambda^*).$$

It follows that

$$\langle \Phi'(x_\lambda), x_\lambda \rangle = \lambda \sum_{k=1}^{\infty} f(k, x_\lambda(k)+1)x_\lambda(k) \text{ for any } \lambda \in (0, \lambda^*). \tag{4.6}$$

On the other hand, condition $(H5)$ implies the existence of a constant $K_3 > 0$ such that

$$\begin{aligned} \sum_{k=1}^{\infty} |f(k, x_\lambda(k)+1)||x_\lambda(k)| &\leq \sum_{k=1}^{\infty} \max_{|t|\leq T} f(k, t+1)|x_\lambda(k)| \\ &\leq K_3 \|x_\lambda\|_{1,q(\cdot),p_+(\cdot)} \text{ for any } \lambda \in (0, \lambda^*). \end{aligned} \tag{4.7}$$

Since I_λ is coercive, it follows that the sequence $\{x_\lambda\}$ is bounded in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. Then,

$$\|x_\lambda\|_{1,q(\cdot),p_+(\cdot)} \leq C^* \text{ for any } \lambda \in (0, \lambda^*), \tag{4.8}$$

where $C^* > 0$ is a constant number.
Moreover, by (3.5) and $(H2)$, one has

$$\begin{aligned} \langle \Phi'(x_\lambda), x_\lambda \rangle &\geq M\left(\eta[x_\lambda]\right) \min\{1, c\} \left[\sum_{k=1}^{\infty} |\triangle x_\lambda(k-1)|^{p(k-1)} + \sum_{k=1}^{\infty} q(k)|x_\lambda(k)|^{p(k)} \right] \\ &\geq M\left(\eta[x_\lambda]\right) p^- \min\{1, c\} \rho_{1,q(\cdot),p_+(\cdot)}(x_\lambda). \end{aligned} \tag{4.9}$$

Hence, by (4.6)-(4.9), for any $\lambda \in (0, \lambda^*)$, we get

$$\frac{1}{\rho_{1,q(\cdot),p_+(\cdot)}(x_\lambda)} \geq \frac{M\left(\eta[x_\lambda]\right) p^- \min\{1, c\}}{\lambda K_3 C^*}. \tag{4.10}$$

Thus, it follows from (3.14) and (4.10) that, as $\lambda \to \infty$,

$$1/\rho_{1,q(\cdot),p_+(\cdot)}(x_\lambda) \to 0.$$

The proof of Theorem 4.1 is complete. □

5. Existence of two nontrivial heteroclinic solutions

To prove that problem (1.4) has at least two nontrivial heteroclinic solutions, we apply Theorem 2.10.
We assume the following additional assumptions.

$(H7)$ $\lim\limits_{t\to 0} \dfrac{f(k, t+1)}{|t|^{p^+-1}} = 0$ uniformly for all $k \in \mathbb{Z}^+$.

$(H8)$ $\sup\limits_{|t|\leq\rho} |F(\cdot, t+1)| \in l^1_{-1,+}$ for all $\rho > 0$.

$(H9)$ $\limsup\limits_{|t|\to\infty} \dfrac{F(k, t+1)}{|t|^{p(k)}} \leq 0$ uniformly for all $k \in \mathbb{Z}^+$.

$(H10)$ $F(m, d+1) > 0$ for some $m \in \mathbb{Z}^+$, $d \in \mathbb{R}$.

One has the following result.

Theorem 5.1. *Assume that the hypotheses $(H1)$-$(H3)$ and $(H7)$-$(H10)$ are fulfilled. Then, the problem (1.4) has at least two nontrivial solutions for all $\lambda > 0$.*

For the proof of Theorem 5.1, one needs the following lemma.

Lemma 5.2. *Assume that* $(H2)$-$(H3)$ *and* $(H8)$-$(H9)$ *hold true. Then, the functional* I_λ *is coercive and satisfies the Palais-Smale condition for all* $\lambda > 0$.

Proof. Let us fix $\lambda > 0$. We first show that I_λ is coercive. By $(H9)$, for all $0 < \epsilon < m_0 q_0/(2^{\bar{p}}\lambda p^+)$, there exists $\rho > 0$ such that

$$F(k,t+1) \le \epsilon |t|^{p(k)} \le \frac{m_0 q_0}{2^{\bar{p}}\lambda p^+}|t|^{p(k)} \le \frac{m_0 q(k)}{2^{\bar{p}}\lambda p(k)}|t|^{p(k)},$$

for all $k \in \mathbb{Z}^+$ and all $t \in \mathbb{R}$, with $|t| > \rho$.
Besides, by $(H8)$ there exists $\mu \in l^1_{-1,+}$ such that

$$|F(k,t+1)| \le \mu(k), \text{ for all } k \in \mathbb{Z}^+ \text{ and all } t \in [-\rho,\rho].$$

So, it follows that

$$F(k,t+1) \le \frac{m_0 q(k)}{2^{\bar{p}}\lambda p(k)}|t|^{p(k)} + \mu(k), \text{ for all } (k,t) \in \mathbb{Z}^+ \times \mathbb{R}.$$

Let $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ with $\|x\|_{1,q(\cdot),p_+(\cdot)} > 1$. The above inequality, Proposition 2.7(2) and assumptions $(H2)$-$(H3)$ yield

$$\begin{aligned}
I_\lambda(x) \quad & \ge m_0 \int_0^{\eta[x]} d\xi - \lambda \sum_{|x(k)|>\rho} F(k,x(k)+1) - \lambda \sum_{|x(k)|\le\rho} F(k,x(k)+1) \\
& \ge m_0 \left(c\sum_{k=1}^{\infty} \frac{|\triangle x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{\infty} \frac{q(k)}{p(k)}|x(k)|^{p(k)} \right) - \frac{m_0}{2^{\bar{p}}} \sum_{k=1}^{\infty} \frac{q(k)}{p(k)}|x(k)|^{p(k)} \\
& -\lambda \sum_{k=1}^{\infty} \mu(k) \\
& \ge m_0 \min\left\{ c, 1 - \frac{1}{2^{\bar{p}}} \right\} \|x\|^{p^-}_{1,q(\cdot),p_+(\cdot)} - \lambda \|\mu\|_{l^1_{-1,+}} \to \infty \text{ as } \|x\|_{1,q(\cdot),p_+(\cdot)} \to \infty,
\end{aligned}$$

since $p^- > 1$. Therefore, I_λ is coercive.

Now, we prove that I_λ satisfies the Palais-Smale condition. Let $\{x_n\} \subset W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ be a (PS)-sequence such that $\{I_\lambda(x_n)\}$ is bounded and $\|I'_\lambda(x_n)\|_{\left(W^{1,p(\cdot)}_{-1,+,q(\cdot)}\right)^*} \to 0$. Since I_λ is coercive, we get that the sequence $\{x_n\}$ is bounded in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. Up to a subsequence, $x_n \rightharpoonup x$ weakly in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ and $x_n \to x$ strongly in $l^{p(\cdot)}_{-1,+,q(\cdot)}$. From $I_\lambda = \Phi - \lambda\Psi$, we see that

$$\langle \Phi'(x_n), x_n - x\rangle = \langle I'_\lambda(x_n), x_n - x\rangle + \lambda \langle \Psi'(x_n), x_n - x\rangle,$$

therefore

$$\langle \Phi'(x_n) - \Phi'(x), x_n - x\rangle = \langle I'_\lambda(x_n), x_n - x\rangle + \lambda \langle \Psi'(x_n), x_n - x\rangle. \tag{5.1}$$

Since $\|I'_\lambda(x_n)\|_{(W^{1,p(\cdot)}_{-1,+,q(\cdot)})^*} \to 0$ and $\{x_n - x\}$ is bounded in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$, we deduce by the inequality $|\langle I'_\lambda(x_n), x_n - x\rangle| \leq \|I'_\lambda(x_n)\|_{\left(W^{1,p(\cdot)}_{-1,+,q(\cdot)}\right)^*}\|x_n - x\|$ that

$$\langle I'_\lambda(x_n), x_n - x\rangle \to 0 \ \text{ as } \ n \to \infty.$$

Next, using the compact embedding $W^{1,p(\cdot)}_{-1,+,q(\cdot)} \hookrightarrow l^{p(\cdot)}_{-1,+,q(\cdot)}$, it follows that, as $n \to \infty$,

$$\langle \Psi'(x_n), x_n - x\rangle = \sum_{k=1}^{\infty} f(k, x_n(k) + 1)(x_n - x)(k) \to 0.$$

Hence, by relation (5.1), one has

$$\lim_{n\to\infty} \langle \Phi'(x_n) - \Phi'(x), x_n - x\rangle = 0.$$

By the strict monotonicity of Φ', we get

$$\limsup_{n\to\infty} \langle \Phi'(x_n) - \Phi'(x), x_n - x\rangle = \lim_{n\to\infty} \langle \Phi'(x_n) - \Phi'(x), x_n - x\rangle = 0.$$

We conclude that

$$\limsup_{n\to\infty} \langle \Phi'(x_n) - \Phi'(x), x_n - x\rangle \leq 0$$

and then from condition (S_+), $x_n \to x$ strongly in $W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. □

Proof. of Theorem 5.1
Note that $I_\lambda(0) = 0$. Moreover, from Lemma 5.2, I_λ satisfies the Palais-Smale condition. The rest of the proof is divided into two steps.

Step 1. We first show that 0 is a strict local minimizer of I_λ for all $\lambda > 0$. Fix $0 < \epsilon < m_0 q_0/(2^{\bar{p}}\lambda p^+)$. Condition $(H7)$ implies that there exists $\sigma \in (0, 1)$ such that

$$F(k, t + 1) \leq \epsilon|t|^{p^+} \leq \epsilon|t|^{p(k)} \ \text{ for all } k \in \mathbb{Z}^+ \text{ and all } |t| \leq \sigma. \tag{5.2}$$

Set $r_2 = \dfrac{1}{K}\sigma$ and $r_1 = \dfrac{1}{2\{\min\{1, c\}\}^{1/\bar{p}}} r_2$. For $x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ with $r_1 \leq \|x\|_{1,q(\cdot),p_+(\cdot)} \leq r_2$, by (2.1) and Lemma 2.1, it follows that

$$\|x\|_\infty < \sigma \ \text{ for all } x \in B_\sigma,$$

where $B_\sigma = \left\{x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)} \text{ such that } \|x\|_\infty < \sigma\right\}$.
Using (5.2), $(H2)$ and $(H3)$, we deduce that

$$\begin{aligned}
I_\lambda(x) &\geq m_0\left(c\sum_{k=1}^{\infty}\frac{|\triangle x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{\infty}\frac{q(k)}{p(k)}|x(k)|^{p(k)}\right) - \lambda\epsilon\sum_{k=1}^{\infty}|x(k)|^{p(k)} \\
&\geq m_0\min\{1, c\}r_1^{\bar{p}} - \lambda\epsilon\frac{p^+}{q_0}\rho_{q(\cdot),p_+(\cdot)}(x) \\
&\geq \left(\frac{m_0}{2^{\bar{p}}} - \frac{\lambda\epsilon p^+}{q_0}\right) r_2^{\bar{p}}.
\end{aligned}$$

Therefore,

$$\inf_{r_1 \leq \|x\|_{1,q(\cdot),p_+(\cdot)} \leq r_2} I_\lambda(x) = \alpha \geq \left(\frac{m_0}{2^{\bar{p}}} - \frac{\lambda \epsilon p^+}{q_0} \right) r_2^{\bar{p}} > 0,$$

where $r_2^{\bar{p}} = \min\left\{ r_2^{p^-}, r_2^{p^+} \right\}$.

Now, show that 0 is not a global minimizer for I_λ. By $(H10)$, we get

$$F(m, d+1) > 0 \text{ for some } m \in \mathbb{Z}^+,\ d \in \mathbb{R}. \tag{5.3}$$

On the other hand, set $x_0 = de_m$ with e_m in (3.9). Now, choose

$$\lambda > \frac{m_1 \left(a_1^+ |d| + \frac{d^{p^-}}{p^-} \left(a_2 + q(m) \right) \right)}{F(m, d+1)}. \tag{5.4}$$

Then, by (3.2)-(3.4), $(H1)$ and $(H3)$, one has

$$\begin{aligned} I_\lambda(x_0) &\leq m_1 \left(a_1^+ |d| + \frac{a_2}{p(m)} |d|^{p(m)} + \frac{q(m)}{p(m)} |d|^{p(m)} \right) - \lambda F(m, d+1) \\ &\leq m_1 \left(a_1^+ |d| + \frac{d^{p^-}}{p^-} \left(a_2 + q(m) \right) \right) - \lambda F(m, d+1). \end{aligned}$$

Therefore, $I_\lambda(x_0) < 0$ for all λ satisfying (5.4). Consequently, 0 is not a global minimizer for I_λ.

Step 2. We prove that for any λ satisfying (5.4), I_λ has a global minimizer. Choose $\varrho \in \mathbb{R}$ such that $I_\lambda(x_0) < \varrho < 0$. Let

$$M = \left\{ x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)} \text{ such that } I_\lambda(x) \leq \varrho \right\}.$$

One has $M \neq \varnothing$. By Lemma 5.2, it follows that I_λ is coercive and satisfies the Palais-Smale condition, hence I_λ is bounded from below. We prove that I_λ is bounded below on M. For that, assume by contradiction that there exists a sequence $\{x_n\}$ in M such that

$$I_\lambda(x_n) \to -\infty \text{ as } n \to \infty.$$

Since $\{x_n\}$ is a bounded sequence in M, then there exists a subsequence still denoted by $\{x_n\}$ such that

$$x_n \rightharpoonup x \text{ in } W^{1,p(\cdot)}_{-1,+,q(\cdot)} \text{ and } x_n \to x \text{ in } l^{p(\cdot)}_{-1,+,q(\cdot)}.$$

Using (3.6) and Lemma 3.4, we assert that

$$\liminf_{n\to\infty} I_\lambda(x_n) \geq I_\lambda(x).$$

On the other hand, set $\varsigma = \inf_{x \in M} I_\lambda(x) = \inf_{x \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}} I_\lambda(x)$. Now, let $\{x_n\}$ be a sequence in M such that

$$I_\lambda(x_n) \to \varsigma \text{ as } n \to \infty.$$

As before, we obtain that up to a subsequence

$$x_n \rightharpoonup \check{x} \text{ in } W^{1,p(\cdot)}_{-1,+,q(\cdot)} \text{ and } x_n \to \check{x} \text{ in } l^{p(\cdot)}_{-1,+,q(\cdot)},$$

for some $\check{x} \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$. Consequently, $I_\lambda(\check{x}) = \varsigma < 0$ and thus $\check{x}$ is a critical point of I_λ. By Theorem 2.10, there exists a critical point $\hat{x} \in W^{1,p(\cdot)}_{-1,+,q(\cdot)}$ of I_λ such that $I_\lambda(\hat{x}) \geq \alpha$. Hence, it follows that $\hat{x} \neq \check{x}$. Thus, our problem (1.4) has at least two nontrivial solutions. □

Acknowledgement. Acknowledgements could be placed at the end of the text but precede the references.

References

1. G.A. Afrouzi, M. Bohner, G. Caristi, S. Heidarkhani and S. Moradi; *An existence result for impulsive multi-point boundary value systems using a local minimisation principle*. J. Optim. Theory Appl. **177**, 1-20 (2018). https://doi.org/10.1007/s10957-018-1253-1
2. A. Arosio and S. Pannizi; *On the well-posedness of the Kirchhoff string*. Trans. Amer. Math. Soc. **348**, 305-330 (1996). https://doi.org/2155178
3. G. Bonanno and G. M. Bisci; *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*. Bound. Value Probl. **2009**, 1-20 (2009). https://doi.org/10.1155/2009/670675
4. G. Bonanno and P. Candito; *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*. J. Differ. Equ. **244**, 3031-3059 (2008). https://doi.org/10.1016/j.jde.2008.02.025
5. M. Bohner, G. Caristi, S. Heidarkhani and S. Moradi; *Existence of at least one homoclinic solution for a nonlinear second-order difference equation*. Int. J. Nonlinear Sci. Numer. Simulat. **20**, 433-439 (2019). https://doi.org/10.1515/ijnsns-2018-0223
6. M. Bohner, G. Caristi, S. Heidarkhani and S. Moradi; *A critical point approach to boundary value problems on the real line*. Appl. Math. Lett. **76**, 215-220 (2018). https://doi.org/10.1016/j.aml.2017.08.017
7. A. Cabada and S. Tersian; *Existence of heteroclinic solutions for a discrete p-Laplacian problems with a parameter*. Nonlinear Anal. RWA **12**, 2429-2434 (2011). https://doi.org/10.1016/j.nonrwa.2011.02.022
8. G. F. Carrier; *A note on the vibrating string*. Quart. Appl. Math. **7**, 97-101 (1949). https://www.jstor.org/stable/43633708
9. G. F. Carrier; *On the nonlinear vibration problem of the elastic string*. Quart. Appl. Math. **3**, 157-165 (1945). https://www.jstor.org/stable/43633501
10. M.M. Cavalcanti, V.N. Cavalcanti and J.A. Soriano; *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*. Adv. Differ. Equ. **6**, 701-730 (2001). DOI:10.57262/ade/1357140586
11. O. Chakrone, E.M. Hssini, M. Rahmani and O. Darhouche; *Multiplicity results for a p-Laplacian discrete problems of Kirchhoff type*. Appl. Math. Comput. **276**, 310-315 (2016). https://doi.org/10.1016/j.amc.2015.11.087
12. Y. Chen, S. Levine and M. Rao; *Variable exponent, linear growth functionals in image processing*. SIAM J. Appl. Math. **66**(4), 1383-1406 (2006). DOI.10.1137/050624522
13. P. Chen, H. Tang and R. P. Agarwal; *Existence of homoclinic solutions for $p(n)$-Laplacian Hamiltonian systems on Orlicz sequence spaces*. Math. Comput. Model **55**, 989-1002 (2012). https://doi.org/10.1016/j.mcm.2011.09.025
14. Z. El Allali, L. Kong and M. Ousbika; *Existence of homoclinic solutions for the discrete $p(k)$-Laplacian operator*. Qual. Theory Dyn. Syst. **21**, 37 (2022).https://doi.org/10.1007/s12346-022-00568-z
15. M. Galewski and G.M. Bisci; *Existence results for one-dimensional fractional equations*. Math. Methods. Appl. Sci. **39**, 1480-1492 (2016). https://doi.org/10.1002/mma.3582

16. A. Guiro, B. Koné and S. Ouaro; *Weak homoclinic solutions of anisotropic difference equation with variable exponents*. Adv. Differ. Equ. **154**, 1-13 (2012). https://doi.org/10.1186/1687-1847-2012-154
17. A. Guiro, B. Koné and S. Ouaro; *Weak heteroclinic solutions and competition phenomena to anisotropic difference equations with variable exponents*. Opuscula. Math. **34**(4), 733-745 (2014). http://dx.doi.org/10.7494/OpMath.2014.34.4.733
18. A. Guiro, B. Koné and S. Ouaro; *Weak heteroclinic solutions of anisotropic difference equations with variable exponent*. Electron. J. Differ. Equ. **225**, 1-9 (2013). http://ejde.math.txstate.edu
19. A. Guiro, I. Ibrango and S. Ouaro; *Weak heteroclinic solutions of discrete nonlinear problems of Kirchhoff type with variable exponents*. Nonlinear. Dyn. Syst. Theory. **18**(1), 67-79 (2018). http://e-ndst.kiev.ua67
20. S. Heidarkhani, G.A. Afrouzi and S. Moradi; *An existence result for discrete anisotropic equations*. Taiwanese J. Math. **22**(3), 725-739 (2018). DOI:10.11650/tjm/170801
21. S. Heidarkhani, S. Moradi and D. Barilla; *Existence results for second-order boundary value problems with variable exponents*. Nonlinear Anal. RWA **44**, 40-53 (2018). https://doi.org/10.1016/j.nonrwa.2018.04.003
22. S. Heidarkhani, Y. Zhou, G. Caristi, G.A. Afrouzi and S. Moradi; *Existence results for fractional differential systems through a local minimisation principle*. Comput. Math. Appl. 2016. https://doi.org/10.1016/j.camwa.2016.04.012
23. S. Heidarkhani, G. Caristi and A. Salari; *Perturbed Kirchhoff-type p-Laplacian discrete problems*. Collect. Math. **68**, No. 3, 401-418 (2017). https://doi.org/10.1007/s13348-016-0180-4
24. A. Iannizzotto and V. Rădulescu; *Positive homoclinic solutions for the discrete p-Laplacian with a coercive weight function*. Differ. Int. Equ. **27**(1-2), 35-44 (2014). https://doi.org/10.57262/die/1384282852
25. A. Iannizzotto and S. Tersian; *Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory*. J. Math. Anal. Appl. **403**, 173-182 (2013). https://doi.org/10.1016/j.jmaa.2013.02.011
26. I.H. Kim and Y.H. Kim; *Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents*. manuscripta math. 147, 169–191 (2015). https://doi.org/10.1007/s00229-014-0718-2
27. G. Kirchhoff; *Vorlesungen uber mathematische Physik*. Mechanik, Teubner, Leipzig, (1876).
28. B. Koné, I. Nyanquini and S. Ouaro; *Weak solutions to discrete nonlinear two-point boundary value problems of Kirchhoff type*. Electron. J. Differ. Equ. **2015**(105), 1-10 (2015). http://ejde.math.txstate.edu
29. J. Kuang and Z. Guo; *Heteroclinic solutions for a class of $p(k)$-Laplacian difference equations with a parameter*. Appl. Math. Lett. **100**, Article ID 106034 (2020). https://doi.org/10.1016/j.aml.2019.106034
30. J. Kuang; *On ground states of discrete $p(k)$-Laplacian systems in generalized Orlicz sequence spaces*. Abstr. Appl. Anal. **2014**, Article ID 808102 (2014).https://doi.org/10.1155/2014/808102
31. J.L. Lions; *On some questions in boundary value problems of mathematical physics. In Contemporary Developpements in Continuum Mechanics and Partial Differential Equations*. Proc. Internat. Sympos. Inst. Mat. Univ. Fed. Rio de Janeiro, North-Holland Math Stud. **30**, 284-345 (1978). https://doi.org/10.1016/S0304-0208(08)70870-3
32. M. Mihailescu, V. Radulescu and S. Tersian; *Eigenvalue problems for anisotropic discrete boundary value problems*. J. Differ. Equ. Appl. **15**(6), 557-567 (2009). https://doi.org/10.1080/10236190802214977
33. M. K. Moghadam and R. Wieteska; *Existence and uniqueness of positive solution for nonlinear difference equations involving $p(k)$-Laplacian*. An. St. Univ. Ovidius Constanta. Methods Nonlinear Anal. **27**(1), 141-167 (2019). https://doi.org/10.2478/auom-2019-0008

34. R. Narashima; *Nonlinear vibration of an elastic string*. J. Sound Vibration. **8**, 134-146 (1968). https://doi.org/10.1016/0022-460X(68)90200-9
35. S. Ouaro and M. Zoungrana; *Multiplicity of solutions to discrete inclusions with the p(k)-Laplace Kirchhoff type equations*. Asia Pacif. J. Math. **5**(1), 27-49 (2018). DOI:10.1080/10236198.2015.1056177
36. D.W. Oplinger; *Frequency response of a nonlinear stretched string*. J. Acoustic Soc. Amer. **32**, 1529-1538 (1960). https://doi.org/10.1121/1.1907948
37. A. Ourraoui and A. Ayoujil; *On a class of non-local discrete boundary value problem*. Arab J. Math. Sci. **28**, No. 2, 130-141 (2022). https://doi.org/10.1108/AJMS-06-2020-0003
38. P. Pucci and J. Serrin; *A mountain pass theorem*. J. Differ. Equ. **60**, 142-149 (1985). https://doi.org/10.1016/0022-0396(85)90125-1
39. V. Rădulescu and D. Repovš; *Partial Differential Equations with Variable exponents*. Variational Methods and Qualitative Analysis, CRC Press, Boca Raton FL, 2015. https://doi.org/10.1201/b18601
40. M. Ruzicka; *Electrorheological Fluids: Modelling and Mathematical theory*. Lecture Notes in Mathematics 1748. Springer, Berlin (2002). https://doi.org/10.1007/BFb0104029
41. B. Ricceri; *A general variational principle and some of its applications*. J. Comput. Appl. Math. **113**, 401-410 (2000). https://doi.org/10.1016/S0377-0427(99)00269-1
42. J. Yang and J. Liu; *Nontrivial solutions for discrete Kirchhoff type problems with resonance via critical groups*. Adv. Differ. Equ. **2013**(1), 1-14 (2013). https://doi.org/10.1186/1687-1847-2013-308
43. Z. Yucedag; *Solutions for a discrete boundary value problem involving Kirchhoff type equation via variational methods*. TWMS J. App. Eng. Math. **8**(1), 144-154 (2018). http://jaem.isikun.edu.tr/web/index.php/archive/97-vol8no1/330
44. Z. Yucedag; *Existence of solutions for anisotropic discrete boundary value problems of Kirchhoff type*. Int. J. Differ. Equ. Appl. **13**(1), 1-15 (2014). http://dx.doi.org/10.12732/ijdea.v13i1.1364
45. E. Zeidler; *Nonlinear Functional Analysis and its Applications II/B*. Berlin-Heidelberg-New York, (1990). https://doi.org/10.1007/978-1-4612-0981-2
46. E. Zeidler; *Nonlinear Functional Analysis and its Applications III*. Springer, New York (1985). https://doi.org/10.1007/978-1-4612-5020-3
47. V. Zhikov; *Averaging of functionals in the calculus of variations and elasticity*. Math. USSR. Izv. **29**, 33-66 (1987). DOI:10.1070/IM1987v029n01ABEH000958

[1] Laboratoire de Mathématiques et Informatique, UFR/SEA, Université Nazi BONI, 01 BP 1091 Bobo-Dioulasso 01, Burkina Faso.
Email address: moussa13brahim.moussa@gmail.com

[2] Laboratoire de Mathématiques et Informatique, UFR/SEA, Université Nazi BONI, 01 BP 1091 Bobo-Dioulasso 01, Burkina Faso
Email address: nyanquis@gmail.com

[3] Laboratoire de Mathématiques et Informatique, UFR/SEA, Université Joseph KI-ZERBO, BP 7021 Ouagadougou 03, Burkina Faso
Email address: ouaro@yahoo.fr