Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 21 (2024) 146-161 https://doi.org/10.56947/amcs.v21.264

SOME RESULTS ON ULTRADIFFERNTIAL CONVOLUTION PROPERTY OF ULTRADISTRIBUTIONS

ANSLEM UCHE AMAONYEIRO[1] AND KENNETH IFEANYI ISIFE[2]*

Abstract. We consider some sequential conditions for the convolvability of two ultradistributions, specifically the Roumieu ultradistributions defined on the space of all ultradifferentiable functions in the space of ultradistributions and show the similarities between these conditions in the sense of Mincheva-Kaminska S. The sequential conditions are based on the use of approximate identities which satisfy certain conditions to be a Banach algebra for the space. We present an equivalent result involving the convolution of the differentiability of two Roumieu ultradistributions with equivalent norm condition.

1. Introduction and preliminaries

The extended form of generalized functions called ultradistributions has become one of the top analysis areas of mathematics. Several investigations concerning the tensor product and convolutions of different classes (such as Beurling and Roumieu type) in the non-quasianalytic form have been carried, with refinements, by Dimovski et al. in [7]. In the work, the authors established some accepted definitions and presented some fundamental results of convolution of a class of ultradistributions called Roumieu ultradistributions in a way analogous to the known general approaches of Chevalley and Schwartz in case of distributions. Also, several general sequential conditions for the convolvability of Roumieu ultradistributions on $\mathbb{R}^n$ have been carried out by Svetlana in [2]. The author presented that the two classes are equivalent in the sense of Pilipovic and Prangoski, and the conditions presented were based on the use of approximate units. For more details, see recent concerning the quasianalytic case in [2, 4].

The purpose of this work is to present net conditions with its significant role in the tensor product and convolution of Roumieu ultradistributions, and present some equivalent results to those used in sequential approach to ultradistributions (see [2])and Beurling type ultradistributions (see [10]). The conditions to be considered are basically established with the use of approximate identities in

Date: Received: Jan 14, 2024; Accepted: Feb 11, 2024.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 46F05; Secondary 46F10, 46E10.

Key words and phrases. Ultradistribution, approximate identity, convolution, ultradifferential operator.

equivalence to the types of $\mathcal{R}$-approximate units as counterparts of the used approximate units in the sense of Dierolf and Voigt (see [6]). The two classes $\mathbb{U}^{\{M_p\}}$ and $\overline{\mathbb{U}}^{\{M_p\}}$ of the approximate identities are used in the equivalent sequential characterizations of the integrable ultradistributions, analogous to that proved by Pilipovic in [12]. As a result, we give in this work sequential definitions of the tensor product and convolutions of Roumieu ultradistributions. We prove in many results that sequential approaches are equivalent to the ones defined using the approximate identities like the Fejer kernel.

An important of aspect of this work is presented in the refinements of the conditions placed on the defining sequences, and an inverse norm has been established analogous to the one used in [2].

It is worth to recall that Mincheva-Kamiňnska in [2] and Pilipovic in [12] used different classes of approximate units to characterize the same property in the cases of convolution of Roumieu ultradistributions and Beurling ultradistributions, respectively. Our proofs presented is based on equivalent notions with essential concern on the Roumieu ultradistributions via the definition of the approximate identities such as the Fejer kernel. The motivation for this work is derived on based on the convolution of ultradistributions through the sequential approach as seen in [2].

We present some vital definitions. In this sequel, we consider the C^∞-functions and Roumieu ultradistributions defined on n-dimensional Euclidean space $\mathbb{R}^n$ with the use of the multi-dimensional notation in $\mathbb{R}^n$.
With consideration given to the dimension of $\mathbb{R}^n$, we denote the relevant space of test functions and the corresponding space of ultrdistributions of the Roumieu type by simply adding the index n at the end of the respective symbol. The constant function 1 defined on $\mathbb{R}^n$ can also be denoted by 1_n and the value of the ultradistribution $T \in \mathcal{D}_n^{\prime\{M_p\}}$ defined on the space of test functions $\varphi \in \mathcal{D}_n^{\{M_p\}}$ by $\langle T, \varphi \rangle$. In the sequel, we may consider the respective spaces without n as with $\mathcal{D}^{\{M_p\}}$.

Definition 1.1. The definitions of the spaces of the test functions and Roumieu ultradistributions are given by the sequence $(M_p)_{p\in\mathbb{N}_0}$ of positive numbers. The extended conditions or the equivalent conditions to the conditions given in [2] imposed on the sequence (M_p) are given below:

(M.1) $M_{p+1}^2 \leq M_p M_{p+2}$ or $\frac{M_p^2}{M_{p+1}} \leq M_{p-1}$ provided $M_{p+1} \neq 0$ for all $p \in \mathbb{N}_0$
(M.2) $\frac{M_p}{M_q} \leq AH^p M_{\frac{p}{q}}$ for $\frac{p}{q} \geq 1$
(M.2') $\frac{M_{p+1}}{M_p} \leq AH^{p+1}$ provided $M_p \neq 0$
(M.3) $\displaystyle\sum_{p=q-1}^{\infty} \frac{M_p}{M_{p+1}} \leq AqM_qM_{q+1}$ for $q \in \mathbb{N} \setminus \{0,1\}$

(M.3') $\displaystyle\sum_{p=0}^{\infty} \frac{M_p}{M_{p+1}} < \infty$

for some constants $A > 0$ and $H > 0$. Conditions (M.2') and (M.3') are equivalent to the conditions (M.2) and (M.3) respectively.

Remark 1.2. For clarity sake, we will assume that the sequence stated above satisfies the extended conditions and their equivalents as presented above. It follows that some of the conditions defined can be proved using the induction hypothesis (see [12]).

It will be convenient to extend the defined sequence $(M_p)_{p\in\mathbb{N}_0}$ to the sequence with dimension n as $(M_k)_{k\in\mathbb{N}_0^n}$. We define the length of k as contained in the formula:

$$M_{|k|} := M_{k_1+k_2+\cdots+k_n}, \quad k = (k_1, \cdots, k_n) \in \mathbb{N}_0^n$$

Definition 1.3. The associated function of the sequence (M_p) as defined in [2] can be refined in an equivalent form as

$$M(\rho) = \sup_{p\in\mathbb{N}_0} \Big(\log_+\Big(\frac{\rho^p}{M_p}\Big)\Big) = \inf_{p\in\mathbb{N}_0} \Big(\frac{1}{\log_+(\frac{\rho^p}{M_p})}\Big) = \inf_{p\in\mathbb{N}_0} \Big(\log_+\Big(\frac{M_p}{\rho^p}\Big)\Big)$$

for $\rho > 0$ and $\rho^p \neq 0$.

We denote the differential operator $D^{|k|}$ of the form, for the arbitrary $k = (k_1, \cdots, k_n) \in \mathbb{N}_0^n$,

$$D^{|k|} = D^{k_1+k_2+\cdots+k_n} = D_1^{k_1} \cdot D_2^{k_2} \cdots D_n^{k_n} = \Big(\frac{1}{i}\frac{\partial}{\partial x_1}\Big)^{k_1} \cdots \Big(\frac{1}{i}\frac{\partial}{\partial x_n}\Big)^{k_n}.$$

The monotonously increasing numerical sequences to infinity are considered. The class of such sequence is considered in this work for clear understanding.

Definition 1.4. We define the product sequence (R_p) corresponding to the class of sequences $(r_p)_{p\in\mathbb{N}_0} \in \mathcal{R}$ with the initial condition as $r_0 = 1$ as the elements of the form: $(R_p) := \Big\{R_p : R_p = \displaystyle\prod_{i=0}^{p} r_i\Big\}$.

Recall the following similar form which establishes the symmetry of two assertions as presented below:

Lemma 1.5. *Let $(a_k)_{k\mathbb{N}_0}$ be a sequence of nonnegative numbers.*

(I) *The following two conditions are equivalent:*

(A_1) $\exists h > 0 \quad \displaystyle\sup_{k\in\mathbb{N}_0} \frac{a_k}{h^k} < \infty,$

(B_1) $\exists (r_k) \in \mathcal{R} \quad \displaystyle\sup_{k\in\mathbb{N}_0} \frac{a_k}{R_k} < \infty;$

(II) *The following two conditions are equivalent:*

(A_2) $\exists h > 0 \quad \displaystyle\sup_{k\in\mathbb{N}_0} (h^k a_k) < \infty,$

(B_2) $\exists (r_k) \in \mathcal{R} \quad \displaystyle\sup_{k\in\mathbb{N}_0} (R_k a_k) < \infty;$

where (R_k) is the product sequence corresponding to the sequence $(r_k) \in \mathcal{R}$.

We now present the equivalent form of the Lemma 1.5 which emphasizes the symmetry of the two assertions. Before we present the equivalent result to the Lemma, we shall use one of the properties of supremum and infimum for sequences below:

Proposition 1.6. *Let S be a subset of the real numbers which is bounded from below such that $\inf S > 0$. Let $\frac{1}{S}$ denote the subset of the real numbers defined as follows: $\frac{1}{S} = \left\{\frac{1}{x} : x \in S\right\}$. Then we have*

$$\sup(\frac{1}{S}) = \frac{1}{\inf S}$$

Lemma 1.7. *Let $(a_n)_{n\mathbb{N}_0}$ be a sequence of nonnegative numbers.*

(I) *The following two conditions are equivalent:*

(A_1^*) $\exists h > 0 \quad \inf\limits_{n\in\mathbb{N}_0} \dfrac{h^n}{a_n} < \infty,$

(B_1^*) $\exists (r_n) \in \mathcal{R} \quad \inf\limits_{n\in\mathbb{N}_0} \dfrac{R_n}{a_n} < \infty;$

(II) *The following two conditions are equivalent:*

(A_2^*) $\exists h > 0 \quad \inf\limits_{n\in\mathbb{N}_0} (\dfrac{1}{h^n a_n}) < \infty$ *provided* $h^n a_n \neq 0,$

(B_2^*) $\exists (r_n) \in \mathcal{R} \quad \inf\limits_{n\in\mathbb{N}_0} (\dfrac{1}{R_n a_n}) < \infty;$

where (R_n) is the product sequence corresponding to the sequence $(r_n) \in \mathcal{R}$.

Remark 1.8. Recall that Lemma 1.5 delivers the two characterizations which is dual to each other by using the concept of Lemma 1.7. The sequence defined above is of the slow increasing property satisfying condition (A_1^*), and of the rapid decreasing property satisfying condition (A_2^*). The conditions are expressed via the basic properties of the sequences of numbers, also described by product sequences corresponding to sequences of the class $\mathcal{R}$.

With the concept of majorization technique, one can prove the following inequality.

Lemma 1.9. *For every $(r_p) \in \mathcal{R}$, the following inequality holds:*

$$\begin{aligned} &\Big[(r_0 r_1)\cdot(r_0 r_1 r_2)\cdots(r_0\cdots r_k)\Big]\cdot\Big[(r_0 r_1)\cdot(r_0 r_1 r_2)\cdots(r_0\cdots r_l)\Big] \\ &\qquad \le \Big[(r_{0+0} r_{1+1})\cdot(r_{0+0} r_{1+1} r_{2+2})\cdots(r_{0+0}\cdots r_{k+l})\Big] \end{aligned} \tag{1.1}$$

where the product sequence (r_p) is the corresponding sequence to the product sequence (R_p) defined previously.

We can define some of the properties of the sequence $(r_p) \in \mathcal{R}$ using the following notation:

$$\lambda(r_p) = (\bar{r_p}), \quad \text{where} \quad \bar{r_p} = 1 \quad \text{and} \quad \bar{r_p} = \lambda r_p \quad \text{for} \quad p \in \mathbb{N} \tag{1.2}$$

2. The Notion of Ultradistributions and Ultradifferentiable Functions

In this section, we consider some introductory concepts about ultradifferentiable functions with some terminologies. We give the inverse norm defined for the space of ultradistributions.
In the sequel, take φ be a complex-valued function defined on $\mathbb{R}^n$, and a compact subset K in $\mathbb{R}^n$. We define previous norms as

$$\|\varphi\|_\infty := \sup_{x\in\mathbb{R}^n} |\varphi(x)|; \quad \|\varphi\|_K := \sup_{x\in K} |\varphi(x)|$$

Equivalently, we can re-define the norms as follow:

$$\|\varphi\|_\infty := \inf_{x\in\mathbb{R}^n} \Big(\frac{1}{|\varphi(x)|}\Big); \quad \|\varphi\|_K := \inf_{x\in K} \Big(\frac{1}{|\varphi(x)|}\Big)$$

provided $\inf \varphi < \infty$.
Now, for a given sequence (M_p), compact subset $K \in \mathbb{R}^n$ and a constant $h > 0$, the locally convex space of all C^∞-functions φ on $\mathbb{R}^n$, denoted by $\mathcal{E}^{\{M_p\}}_{K,n,h}$ with its topology can be defined by the family of semi-norms defined by

$$q_{n,K}(\varphi) = \inf_{n\in\mathbb{N}_0^n} \Big(\frac{h^{|n|}M_n}{\|D^n\varphi\|_K}\Big) \tag{2.1}$$

such that it is finite. The definition is valid provided $|D^n\varphi| \neq 0$ since φ has support in K.

Definition 2.1. Let the sequence (M_p) be fixed. The locally convex space of ultradifferentiable functions defined on $\mathbb{R}^n$ is defined by the following topologies

$$\mathcal{D}^{\{M_p\}}_{K,n} := \varinjlim_{h\to\infty} \mathcal{D}^{\{M_p\}}_{K,h,n} \qquad \mathcal{D}^{\{M_p\}}_{n} := \varinjlim_{K\subset\mathbb{R}^n} \mathcal{D}^{\{M_p\}}_{K,n,h}; \tag{2.2}$$

$$\mathcal{E}^{\{M_p\}}_{n} := \varprojlim_{K\subset\mathbb{R}^n} \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\}}_{K,h,n} \tag{2.3}$$

We also define the Banach space of all C^∞-functions φ defined on $\mathbb{R}^n$ with the following inductive limit topology and the equivalent norm to the norm defined in [2] as follow:

$$\mathcal{D}^{\{M_p\}}_{L^\infty,n} := \varinjlim_{h\to\infty} \mathcal{D}^{\{M_p\}}_{L^\infty,h,n}$$

such that the following equivalent norm is finite

$$\|\varphi\|_{\infty,h} := \inf\Big\{\Big(\|D^k\varphi\|\Big)^{-1} h^k M_k : k \in \mathbb{N}_0^n\Big\} \tag{2.4}$$

With the consideration of the sequences (M_p) and $(r_p) \in \mathbb{R}$, The Banach space, denoted by $\mathcal{D}^{\{M_p\}}_{K,(r_p),n}$, is the space of all infinitely continuously differentiable functions defined on $\mathbb{R}^n$ with support in the compact subset K such that

$$\|\varphi\|_{K,(r_p)} := \sup_{n\in\mathbb{N}_0^n} \frac{\|D^k\varphi\|_K}{R_{|k|}M_k} = \inf_{n\in\mathbb{N}_0^n} \frac{R_{|k|}M_k}{\|D^k\varphi\|_K} < \infty \tag{2.5}$$

The following result is an equivalent result to the result due to Mincheva-Kaminska [2].

Proposition 2.2. *The following equality is established*

$$\mathcal{D}_{K,n}^{\{M_p\}} := \varprojlim_{(r_p \in \mathcal{R})} \mathcal{D}_{K,(r_p),n}^{\{M_p\}}$$

For a given sequence (M_p), the following projective description of the space $\mathcal{D}_{L^\infty,n}^{\{M_p\}}$ as the smallest topology generated by the family of the seminorms given by

$$\mathcal{D}_{L^\infty,n}^{\{M_p\}} = \bigcap \mathcal{D}_{L^\infty,(r_p),n}^{\{M_p\}}$$

where the equality holds in the sense of locally convex space. We denote by $\mathcal{B}_n^{\{M_p\}}$ the completion of $\mathcal{D}_n^{\{M_p\}}$ in $\mathcal{D}_{L^\infty,n}^{\{M_p\}}$.

The following assertion concerns the product of functions in $\mathcal{D}_{L^\infty,(r_p),n}^{\{M_p\}}$ which is analogous to the result proved in [11].

Proposition 2.3. *Let $\varphi_1, \varphi_2 \in \mathcal{D}_{L^\infty,n}^{\{M_p\}}$. Then $\varphi_1 \cdot \varphi_2 \in \mathcal{D}_{L^\infty,n}^{\{M_p\}}$. Moreover, for every $(r_p) \in \mathcal{R}$ such that $r_1 > 2$ the inequality holds:*

$$\|\varphi_1 \cdot \varphi_2\|_{2(r_p)} \leq \|\varphi_1\|_{(r_p)} \cdot \|\varphi_2\|_{(r_p)} \tag{2.6}$$

Similarly, we have following important assertion is as result of Minkowski's inequality

Proposition 2.4. *Let $\varphi_1, \varphi_2 \in \mathcal{D}_{L^\infty,n}^{\{M_p\}}$. Then $\varphi_1 + \varphi_2 \in \mathcal{D}_{L^\infty,n}^{\{M_p\}}$. Moreover, for every $(r_p) \in \mathcal{R}$ such that $r_1 > 2$ the inequality holds:*

$$\|\varphi_1 + \varphi_2\|_{(r_p)} \leq \|\varphi_1\|_{(r_p)} + \|\varphi_2\|_{(r_q)} \tag{2.7}$$

with $\frac{1}{(r_p)} + \frac{1}{(r_q)} = 1$ for $\frac{1}{\bar{r}_p} + \frac{1}{\bar{r}_q} = 1$ and it is finite.

We now introduce the notion of Roumieu ultradistributions

3. Roumieu Ultradistributions

In this section, we introduce the concept of a special type of ultradistributions called Roumieu ultradistributions having dicussed the ultradifferentiable functions defined on finite dimensional Euclidean space $\mathbb{R}^n$.

Definition 3.1. The strong dual of the space $\mathcal{D}_n^{\{M_p\}}$, denoted by $\mathcal{D}_n^{'\{M_p\}}$, is called the space of Roumieu ultradistributions.

The space of Roumieu integrable ultradistributions, denoted by $\mathcal{D}_{L^1}^{'\{M_p\}}$, is the strong dual of the space $\mathcal{B}_n^{\{M_p\}}$.

Remark 3.2. The respective inclusion mapping between the spaces $\mathcal{D}^{\{M_p\}}$ and $\mathcal{B}^{\{M_p\}}$ since the space $\mathcal{D}^{\{M_p\}}$ is dense in $\mathcal{B}^{\{M_p\}}$. Furthermore, the space of Roumieu integrable ultradistributions $\mathcal{D}_{L^1}^{'\{M_p\}}$ is the subspace of the space of Roumieu ultradistributions $\mathcal{D}^{'\{M_p\}}$

In replace of the $\mathcal{R}$-approximate unit, we define the approximate identity in the space of ultradifferentiable functions.

Definition 3.3. A sequence (u_n) of ultradifferentiable functions $(u_n) \in \mathcal{D}_n^{\{M_p\}}$ is called an $\mathcal{R}$-approximate identity in $\mathcal{D}_n^{\{M_p\}}$ if and only if

$$\lim_{n\to\infty} \|u_n * f - f\| = 0 \quad \forall f \in \mathcal{D}_n^{\{M_p\}}$$

Equivalently, the sequence is called an $\mathcal{R}$-approximate identity in $\mathcal{D}_n^{\{M_p\}}$ if it converges to 1 in $\mathcal{E}_n^{\{M_p\}}$ such that for every product sequence $(r_p) \in \mathcal{R}$ we have the following:

$$\sup \|u_n\|_{(r_p)} = \sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}_0} \Big(R_{|k|} M_k (\|D^k u_n\|)^{-1} \Big) < \infty \tag{3.1}$$

Definition 3.4. The $\mathcal{R}$-approximate identity is called a special $\mathcal{R}$-approximate identity if $\forall$ compact $K \subset \mathbb{R}^n$ and $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$$|u_n(x) - 1| < \varepsilon \quad \forall n \neq n_0 \quad \text{and} \quad x \in K.$$

Denote the class of all $\mathcal{R}$-approximate identities defined on $\mathbb{R}^n$ and class of all special $\mathbb{R}$-approximate identities defined on $\mathbb{R}^n$ by $\mathbb{U}_n^{\{M_p\}}$ and $\overline{\mathbb{U}}_n^{\{M_p\}}$ respectively.

The defined approximate identity can satisfies the following properties as depicted in the result below:

Theorem 3.5. *Suppose that the sequence $(u_n) \in \mathcal{D}_n^{\{M_p\}}$ has the following properties*

(i) $\sup\limits_{n\in\mathbb{N}} \|u_n\|_{(r_p)} < \infty$
(ii) $\frac{1}{2\pi}\int_0^{2\pi} u_n(x)dx = 1$
(iii) $0 < \delta < \pi \implies \int_0^{2\pi-\delta} |u_n(x)|dx = 0$

Then (u_n) is an $\mathcal{R}$-approximate identity for the Banach algebra $\mathcal{D}_n^{\{M_p\}}$.

Remark 3.6. An example of the approximate identity is given by the Fejer kernels defined by

$$F_n(x) = \sum_{|j|\leq n} \Big(1 - \frac{|j|}{n+1}\Big) e^{ijx}$$

The above approximate identity can also be shown that

$$F_n(x) = \frac{1}{n+1}\Big(\frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{1}{2}x)}\Big)^2.$$

4. Notion of Integrable Roumieu Ultradistributions

We formulate some characterizations of integrable Roumieu ultradistributions in different equivalent forms using the concept of the $\mathcal{R}$-approximate identities, which are analogous to the results of Svetlana, and Dierolf and Voigt involving integrable ultradistributions and integrable distributions, respectively. We also use the concept of the Beurling class of ultradistributions as given in [12]. The proofs of the respective results are as well given in [2] and [11].
Now we present the equivalent result of the theorem 5.1 in [2].

Proposition 4.1. *Given $T \in \mathcal{D}_n^{'\{M_p\}}$. The we have the following equivalent statements:*

(A) *$T \in \mathcal{D}_{,L^1,n}^{'\{M_p\}}$ is continuous on $\mathcal{D}_n^{\{M_p\}}$ in the topology induced by $\mathcal{B}_n^{'\{M_p\}}$ if there are a sequence $(r_p) \in \mathcal{R}$ and a constant $C > 0$ such that the following inequality holds for all ultradifferentiable functions $\varphi \in \mathcal{D}_n^{\{M_p\}}$;*

$$-C\|\varphi\|_{(r_p)} \leq \langle T, \varphi\rangle \leq C\|\varphi\|_{(r_p)} \tag{4.1}$$

Similarly, we have the satisfying condition imposed on T

$$C = \sup_{\varphi \neq 0}\Big(\frac{\langle T, \varphi\rangle}{\|\varphi\|_{(r_p)}}\Big) \simeq \inf_{\varphi \neq 0}\Big(\frac{\|\varphi\|_{(r_p)}}{\langle T, \varphi\rangle}\Big)$$

(B) *there is a sequence $(r_p) \in \mathcal{R}$ such that for every $\varepsilon > 0$, there exists a compact subset K in $\mathbb{R}^n$ such that the following condition is satisfied:*

$$|\langle T, \varphi\rangle| = \begin{cases} \varepsilon\|\varphi\|_{(r_p)} & \varepsilon > 0 \\ -\varepsilon\|\varphi\|_{(r_p)} & -\varepsilon < 0 \end{cases}$$

with supp$\varphi \cap K = \emptyset$;

(C) *for every $(u_n) \in \mathbb{U}_n^{\{M_p\}}$, the sequence $(\langle T, u_n\rangle)_{n\in\mathbb{N}}$ is Cauchy;*

(D) *for every $(\overline{u_n}) \in \overline{\mathbb{U}}_n^{\{M_p\}}$, the sequence $(\langle T, \bar{u_n}\rangle)_{n\in\mathbb{N}}$ is Cauchy;*

(E) *there are a sequence $(r_p) \in \mathcal{R}$, a constant $C > 0$ and a regular compact K such that inequality (5.1) holds for $\varphi \in \mathcal{D}_n^{\{M_p\}}$ with supp$\varphi \cap K = \emptyset$.*

Moreover, if T satisfies any of the stated conditions (A)-(E), then

$$\lim_{n\to\infty} \|u_n * T - T\|_{(r_p)} = \lim_{n\to\infty} \|\bar{u_n} * T - T\|_{(r_p)} = 0$$

for arbitrary $(u_n) \in \mathbb{U}_n^{\{M_p\}}$ and $(\bar{u_n}) \in \overline{\mathbb{U}}_n^{\{M_p\}}$.

In the next section, we formulate some extended and equivalent conditions of convolvability of Roumieu ultradistributions.

5. Equivalent Sequential Convolutions of Roumieu Ultradistributions

Having introduced the concept of $\mathcal{R}$-approximate identity, it makes it easy to consider several definitions of the convolutions of Roumieu ultradistributions with reference to the conditions of convolutions. These conditions require some level of correspondence of Roumieu ultradistributions via certain approximate identities such as the Fejer kernels such that they are Cauchy sequences.

We will present some result on equivalence of convolvability of Roumiue ultradistributions as proved in [13], and prove that all the sequential definitions in the form of approximate identities are equivalent to the general convolution of Roumieu ultradistributions in the sense of Pilipovic and Prangoski [13]. The proof will be established based on the integrability result presented previously. The study of the convolution of Roumieu ultradistributions was actually based on the ε tensor product of the concerned spaces of test functions with some results proved in [2] and [13].

The locally convex space $\mathcal{D}^{L^\infty,n}_{\{M_p\}}$ equipped with the topology defined by the family $\{q_{g,(r_p)} : g \in C_0, (r_p) \in \mathcal{R}\}$ of seminorms defined by

$$q_{g,(r_p)}(\varphi) = \sup_{k\in\mathbb{N}_0^n} \sup_{x\in\mathbb{R}^n} \frac{|g(x)D^k\varphi(x)|}{R_{|k|}M_k}$$

is denoted by $\tilde{\mathcal{D}}^{L^\infty,n}_{\{M_p\}}$ and the strong dual of the space $\tilde{\mathcal{D}}^{L^\infty,n}_{\{M_p\}}$ is given by $\left(\tilde{\mathcal{D}}^{L^\infty,n}_{\{M_p\}}\right)'_n$ which is equal to $\mathcal{D}'^{\{M_p\}}_{L^1,n}$.
The definition of the convolution of two Roumieu ultradistributions is given below as contained in [13].

Definition 5.1. The two Roumieu ultradistributions S and T are convolvable if the following condition is satisfied

$$(C) \qquad V_\varphi := (S \otimes T)\varphi^\triangle \quad \forall \quad \varphi \in \mathcal{D}^{\{M_p\}}_n$$

where $\varphi^\triangle$ is the function of the class $\mathcal{E}^{\{M_p\}}_{2n}$ defined by

$$\varphi^\triangle(x,y) := \varphi(x+y), \quad x, y \in \mathbb{R}^n \tag{5.1}$$

If S, T are convolvable, then the convolution $S*T$ of S and T in $\mathcal{D}'^{\{M_p\}}_n$ is defined by

$$\langle S*T, \varphi\rangle := \langle V_\varphi, 1_{2n}\rangle_{2n}, \quad \varphi \in \mathcal{D}^{\{M_p\}}_n \tag{5.2}$$

Definition 5.2. Let (u_n) be an approximate identity. Let $S, T \in \mathcal{D}'^{\{M_p\}}_n$. Then for every $\varphi \in \mathcal{D}^{\{M_p\}}_n$, the convolution of S and T is attained in the sense of the following conditions below:

(P) $\left(\langle S\otimes T, u_n\varphi^\triangle\rangle_{2n}\right)$ is Cauchy for all $(u_n) \in \mathbb{U}^{\{M_p\}}_{2n}$;
(Q) $\left(\langle (u_n^1 S)\otimes (u_n^2 T), \varphi^\triangle\rangle_{2n}\right)$ is Cauchy for all $(u_n^1) \in \mathbb{U}^{\{M_p\}}_n$;
(R) $\left(\langle (u_n S)\otimes T, \varphi^\triangle\rangle_{2n}\right)$ is Cauchy for all $(u_n) \in \mathbb{U}^{\{M_p\}}_n$;
(S) $\left(\langle S\otimes (u_n T), \varphi^\triangle\rangle_{2n}\right)$ is Cauchy for all $(u_n) \in \mathbb{U}^{\{M_p\}}_n$;

respectively.

Consider the following definition of convolution of tow Roumieu ultradistributions in the sense of the conditions (P-S) as stated in the definition above.

Definition 5.3. Assume that $S, T \in \mathcal{D}'^{\{M_p\}}_n$ are convolvable in the sense of the conditions (P-S) respectively, then the convolution of the two ultradistributions in the sense of the stated respective conditions is defined by the corresponding formular below:

$$\langle S \overset{P}{*} T, \varphi\rangle_n = \left(\langle S\otimes T, u_n\varphi^\triangle\rangle_{2n}\right) \text{ is Cauchy for all } (u_n) \in \mathbb{U}^{\{M_p\}}_{2n};$$
$$\langle S \overset{Q}{*} T, \varphi\rangle_n = \left(\langle (u_n^1 S)\otimes (u_n^2 T), \varphi^\triangle\rangle_{2n}\right) \text{ is Cauchy for all } (u_n^1) \in \mathbb{U}^{\{M_p\}}_n;$$
$$\langle S \overset{R}{*} T, \varphi\rangle_n = \left(\langle (u_n S)\otimes T, \varphi^\triangle\rangle_{2n}\right) \text{ is Cauchy for all } (u_n) \in \mathbb{U}^{\{M_p\}}_n;$$
$$\langle S \overset{S}{*} T, \varphi\rangle_n = \left(\langle S\otimes (u_n T), \varphi^\triangle\rangle_{2n}\right) \text{ is Cauchy for all } (u_n) \in \mathbb{U}^{\{M_p\}}_n;$$

respectively.

Remark 5.4. In this case defined above, the modified conditions of the additional definitions of the convolution in $\mathcal{D}_n^{'\{M_p\}}$ replaces the definitions 7.1 and 7.2 in [2] with the introduction of the $\mathcal{R}$-approximate identities by the respective classes. This modifies conditions, basically using the Fejer kernels, lead to equivalent sequential conditions of the convolution in $\mathcal{D}_n^{'\{M_p\}}$ which also serve as counterparts of the definitions of convolution in distributions. These are equivalent.
Also, the sequential definitions of convolution of S and T as presented in Definition 5.3 under the respective corresponding conditions do not necessarily depend on the choice of the approximate identity even though we gave an example of approximate identity as Fejer kernel.
The next theorem will be an equivalent result to the one proved in [2] as Theorem 7.1, which states that all the conditions for the convolutions of two Roumieu ultradistributions are all the same.

Consider the following result.

Theorem 5.5. *Let $S, T \in \mathcal{D}_n^{'\{M_p\}}$. Each of the following two conditions of tensor product and convolutions is equivalent to condition (C) of convolvability of S and T:*

(C1) $(\check{S} \otimes T) * \varphi = S(\check{T} * \varphi) \in \mathcal{D}_{L^1,n}^{'\{M_p\}}$ *for all* $\varphi \in \mathcal{D}_n^{\{M_p\}}$;

(C2) $(\check{T} \otimes S) * \varphi = (\check{S} * \varphi)T \in \mathcal{D}_{L^1,n}^{'\{M_p\}}$ *for all* $\varphi \in \mathcal{D}_n^{\{M_p\}}$.

Furthermore, if any of the conditions (C), (C1), (C2) hold, then we have

$$\langle S * T, \varphi \rangle_n = \langle (\check{S} \otimes T) * \varphi, 1_n \rangle_n = \langle (\check{T} \otimes S) * \varphi, 1_n \rangle \tag{5.3}$$

Theorem 5.6. *Given that all the conditions formulated in Definition 5.3 are equivalent to the condition (C) of the convolvability of ultradistributions. Assume that one of the conditions is true, then all corresponding convolutions of Roumieu ultradistributions S and T exist in $\mathcal{D}_n^{'\{M_p\}}$. Then the equalities hold*

$$S \overset{P}{*} T = T \overset{Q}{*} S = S \overset{R}{*} T = S \overset{S}{*} T$$

with commutative law satisfied.

Proof. Let $S, T \in \mathcal{D}_n^{'\{M_p\}}$ be two Roumieu ultradistributions. We will prove that all the convolution definitions are equivalent in the sense of the $\mathcal{R}$-approximate identities. To do this, we will follow the scheme of implications below:

$$\begin{tikzcd}
 & (Q) \arrow[r] & (C1) \arrow[dr] & & & \\
(P) \arrow[ur] \arrow[dr] & & & (C) \arrow[r] & (S) \arrow[r] & (P) \\
 & (R) \arrow[r] & (C2) \arrow[ur] & & &
\end{tikzcd}$$

Assume that the first condition (P) is satisfied, that is, for a fixed $\varphi \in \mathcal{D}_n^{\{M_p\}}$, then there a number $\alpha \in \mathbb{C}$ such that

$$\lim_{n\to\infty} \langle S \otimes T, u_n \varphi^{\triangle} \rangle_{2n} = \langle S \otimes T, \varphi^{\triangle} \rangle = \alpha \tag{5.4}$$

since $u_n \longrightarrow 1$ as $n \to \infty$ for all $(u_n) \in \mathbb{U}_n^{\{M_p\}}$.
We show that (P) implies both (Q) and (R). To see this, fix $(u_n^1), (u_n^2) \in \mathbb{U}_n^{\{M_p\}}$. Then $(u_n)\varphi^{\triangle} \in \mathbb{U}_{2n}^{\{M_p\}}$ for $n \in \mathbb{N}$. From the simple property of the tensor product in $\mathcal{D}_n^{\{M_p\}}$ and $\mathcal{D}_n'^{\{M_p\}}$, we get the following equality,

$$\langle (u_n^1 S) \otimes (u_n^2 T), \varphi^{\triangle} \rangle_{2n} = \langle S \otimes T, u_n \varphi^{\triangle} \rangle_{2n}$$

By taking the double limit, we have

$$\lim_{n,m\to\infty} \langle (u_n^1 S) \otimes (u_m^2 T), \varphi^{\triangle} \rangle_{2n} = \langle S \otimes T, \varphi^{\triangle} \rangle_{2n}$$

for all $\varphi \in \mathcal{D}_n^{\{M_p\}}$ and $n \in \mathbb{N}$. Then $(P) \implies (Q)$ and thus, we have

$$\langle S \overset{P}{*} T, \varphi \rangle_n = \langle S \overset{Q}{*} T, \varphi \rangle_n = \langle T \overset{Q}{*} S, \varphi \rangle_n \tag{5.5}$$

since $*$ is commutative in the sense of the conditions of the approximate identities.
Next, we show that $(P) \implies (R)$. Similarly, fix $(u_n) \in \mathbb{U}_n^{\{M_p\}}$. Then $u_n \varphi^{\triangle} \in \mathbb{U}_{2n}^{\{M_p\}}$ for $n \in \mathbb{N}$. From the notion of the ε-tensor product in $\mathcal{D}_n$ and $\mathcal{D}_{2n}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \langle (u_n S) \otimes T, \varphi^{\triangle} \rangle_{2n} &= \lim_{n\to\infty} \langle (T \otimes u_n S), \varphi^{\triangle} \rangle_{2n} \\
&= \lim_{n\to\infty} \langle T \otimes u_n S, \varphi^{\triangle} \rangle_{2n} \\
&= \langle T \otimes \lim_{n\to\infty} u_n S, \varphi^{\triangle} \rangle_{2n} \\
&= \langle T \otimes S, \varphi^{\triangle} \rangle_n = \langle S \otimes T, \varphi^{\triangle} \rangle_n
\end{aligned}$$

Next, we assume (Q) and (R), and show that they imply (C1) and (C2), respectively. To see this, fist assume (Q), that is, $(\langle (u_n^1 S) \otimes (u_n^2 T), \varphi^{\triangle} \rangle_{2n})$ is Cauchy. Let $(u_n^1), (u_n^2) \in \mathbb{U}_n^{\{M_p\}}$. Then for $\alpha \in \mathbb{C}$ and fix $n \in \mathbb{N}$, then for the limit, we have

$$\lim_{n\to\infty} \langle (u_m^2 T) \otimes (u_n^1 S), \varphi^{\triangle} \rangle_{2n} = \langle (u_m^2 T) \otimes S, \varphi^{\triangle} \rangle_{2n}$$

Then since $u_m^2 T \to T$ in $\mathcal{D}_n'^{\{M_p\}}$ as $m \to \infty$ and

$$\operatorname{supp}\Big[((u_m^2 T) \otimes (u_n^1 S)) \cdot \varphi^{\triangle}\Big] \subset \operatorname{supp}\Big[((u_m^2 T) \otimes S) \cdot \varphi^{\triangle}\Big] \quad \forall \quad n, m \in \mathbb{N}$$

since the support is compact.
Consequently, we have

$$\begin{aligned}
\lim_{m\to\infty} \langle (u_m T) \otimes S, \varphi^{\triangle} \rangle_{2n} &= \lim_{m\to\infty} \langle S \otimes (u_m T), \varphi^{\triangle} \rangle_{2n} \\
&= \langle S \otimes T, \varphi^{\triangle} \rangle_n = \langle S(\breve{T} * \varphi), 1_n \rangle_n
\end{aligned}$$

for all $(u_m) \in \mathbb{U}_m^{\{M_p}$ and $\varphi \in \mathcal{D}_n$. That is,

$$\langle T \overset{Q}{*} S, \varphi \rangle_n = \langle S(\check{T} * \varphi), 1_n \rangle_n$$

Thus we have shown that (Q) implies (C1).
by symmetry, we conclude that $(R) \implies (C2)$ as we have

$$\langle T \overset{R}{*} S, \varphi \rangle_n = \langle S \overset{P}{*} T, \varphi \rangle_n, \quad \varphi \in \mathcal{D}_n^{\{M_p\}}$$

Thus

$$\langle T * S, \varphi \rangle_n = \langle S * T, \varphi \rangle_n = \langle (\check{S} * \varphi)T, 1_n \rangle_n \tag{5.6}$$

which are true under conditions (Q) and (S), respectively.
Finally, we show that $(C) \implies (P)$. Assume that (C) holds, that is, $(S \otimes T)\varphi^{\triangle} \in \mathcal{D}_{L^1,2n}^{\prime\{M_p\}}$. To do this, fix $(u_n) \in \mathbb{U}_n^{\{M_p\}}$ and take the limit

$$\begin{aligned} \lim_{n\to\infty} \langle S \otimes T, u_n \varphi^{\triangle} \rangle_{2n} &= \langle S \otimes T, \lim_{n\to\infty} u_n \varphi^{\triangle} \rangle_{2n} \\ &= \langle S \otimes T, \varphi^{\triangle} \rangle_n \end{aligned}$$

since $\operatorname{supp}[(S \otimes T)u_n\varphi^{\triangle}] \subset \operatorname{supp}[(S \otimes T)\varphi^{\triangle}]$. This completes the proof that all the conditions in Definition 5.3 are equivalent via the use of the $\mathcal{R}$-approximate identities. ☐

6. Main Results

In this section, we will present an equivalent result involving the convolution of the operation of the differentiability of two ultradistributions. We start with the definition of the ultradifferentiable operator.

Definition 6.1. We define an ultradifferential operator of class $\{M_p\}$ of order p as an operator of the form

$$P(D) = \sum_{|k|\le p} c_k D^k, \quad c_k \in \mathbb{C} \tag{6.1}$$

if for every $L > 0$ there is a constant depending on L, C_L such that the following inequality holds

$$|N_L c_k| \le L^k M_k^{-1}, \quad k \in \mathbb{N}_0^n$$

where $C_L^{-1} = N_L > 0$ and $D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$.

The condition in Definition 6.1 can be presented in an equivalent form as

$$\inf_k (L^k (M_k N_L)^{-1}) < \infty$$

In other words, an ultradifferential operator of the form (6.1) is of class $\{M_p\}$ if there are $C > 0$ and $(u_p) \in \mathcal{R}$ such that

$$(U_{|k|} M_k)^{-1} |c_k| \le C, \quad k \in \mathbb{N}_0^n \tag{6.2}$$

The above operator $P(D)$ defines the respective continuous mappings $\mathcal{D}_n^{\{M_p\}} \longrightarrow \mathcal{D}_n^{\{M_p\}}$ and $\mathcal{D}_n^{\prime\{M_p\}} \longrightarrow \mathcal{D}_n^{\prime\{M_p\}}$. The series $P(D)S$ converges absolutely in the following sense:

$$\begin{aligned}\langle P(D)S, u_n\varphi^{\triangle}\rangle &= \Big\langle \sum_{|k|\le p} c_k D^k S, u_n\varphi^{\triangle}\Big\rangle \\ &= \Big\langle S, \sum_{|k|\le p} (-1)^{|k|} D^k (c_k\varphi^{\triangle}) u_n Big\rangle\end{aligned}$$

Applying the limit as $n \to \infty$ and absolute value condition, we have $\langle S, \sum_{|k|\le p} D^k(c_k\varphi^{\triangle})\rangle$.
Before we present the important result, we state the useful result in an alternative form as in [13]:

Lemma 6.2. *For every approximate identity $(s_p) \in \mathcal{R}$, there exists a sequence $(u_p) \in \mathcal{R}$ such that $u_p \le s_p$ for $p \in \mathbb{N}$ and*

$$\frac{R_{p+q}}{R_p R_q} \le 2^{p+q}, \quad \textit{provided} \quad R_p R_q \ne 0, \quad \forall \quad p, q \in \mathbb{N}_0 \quad u_0 = 1. \tag{6.3}$$

Theorem 6.3. *Let $P(D)$ be an ultradifferential operator of class $\{M_p\}$. For every $S, T \in \mathcal{D}_n'^{\{M_p\}}$, then convolution holds in both $P(D)$ with T and S with $P(D)T$ in the following form:*

$$\sum_{|k|\le p} \frac{\alpha!}{\beta!(\alpha-\beta)!} (\partial^\beta S) * (\partial^{\alpha-\beta} T) = \sum_{|k|\le p} c_k D^k S * T = S * \sum_{|k|\le p} c_k D^k T \tag{6.4}$$

Proof. We show that both $P(D)S$ and T, and as well as S and $P(D)T$ are convolvable.
To see this, let $S, T \in \mathcal{D}_n'^{\{M_p\}}$ be convolvable. Using the definition of the ultradifferential operator of order p and $S * T$ and then, we have

$$\begin{aligned}\Big\langle \sum_{|k|\le p} c_k D^k (S*T), \varphi^{\triangle}\Big\rangle &= \Big\langle (S*T), \sum_{|k|\le p} (-1)^{|k|} D^k (c_k\varphi^{\triangle})\Big\rangle \\ &= \Big\langle (S\otimes T)\sum_{|k|\le p} (-1)^{|k|} D^k (c_k\varphi^{\triangle}), 1_{2n}\Big\rangle \\ &= \lim_{n\to\infty} \Big\langle (S\otimes T), u_n \Big[\sum_{|k|\le p} (-1)^{|k|} D^k (c_k\varphi^{\triangle})\Big]\Big\rangle_{2n}\end{aligned}$$

for all $(u_n) \in \overline{\mathbb{U}}_{2n}^{\{M_p\}}$ and $\varphi \in \mathcal{D}_n^{\{M_p\}}$.
We show that the sequence $\Big(\langle P(D)S\otimes T, u_n\varphi^{\triangle}\rangle\Big)_{n\in\mathbb{N}}$ is convergent. Let $\langle P(D)S\otimes T, u_n\varphi^{\triangle}\rangle$ be given. Then $\forall\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\Big|\langle P(D)S\otimes T, u_n\varphi^{\triangle}\rangle - |\langle P(D)S\otimes T, u\varphi^{\triangle}\rangle\Big| \\ &= \Big|\Big\langle (S\otimes T)\sum_{|k|\le p} D^k c_k S\otimes T, u_n\varphi^{\triangle}\Big\rangle - \Big\langle \sum_{|k|\le p} D^k c_k S\otimes T, u\varphi^{\triangle}\Big\rangle\Big| \\ &= \Big|\Big\langle S\otimes T, \sum_{|k|\le p} (-1)^{|k|} D^k (c_k u_n\varphi^{\triangle})\Big\rangle - \Big\langle S\otimes T, \sum_{|k|\le p} (-1)^{|k|} D^k (c_k u\varphi^{\triangle})\Big\rangle\Big| < \varepsilon\end{aligned}$$

The above confirms the absolute convergence of the series. We also the inequalities

$$P(-D)(u_n\varphi^{\triangle}) = u_n P(-D)\varphi^{\triangle} + \nu_n \tag{6.5}$$

on $\mathbb{R}^{2n}$ for all $n \in \mathbb{N}$ where ν_n are certain functions in $\mathcal{D}_n^{\{M_p\}}$ such that

$$\lim_{n\to\infty} \langle S \otimes T, \nu_n \rangle = 0 \tag{6.6}$$

Using the Leibnitz rule in [1], then we get, $\forall \quad n \in \mathbb{N}$, the equalities

$$\begin{aligned}
P(-D_x)(u_n\varphi^{\triangle}) &= \sum_{|\alpha|\le p} (-1)^{|\alpha|} c_k \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (D^\beta u_n)(D^{\alpha-\beta}\varphi)^{\triangle} \\
&= \sum_{\beta\le\alpha} D^\beta u_n \sum_{|\alpha|\le p} (-1)^{|\alpha+\beta|} \binom{\alpha+\beta}{\beta} c_{\alpha+\beta} (D^{\alpha-\beta}\varphi)^{\triangle} \\
&= u_n P(-D)\varphi^{\triangle} + \nu_n
\end{aligned}$$

on $\mathbb{R}^{2n}$ with the function ν_n defined for $x, y \in \mathbb{R}^n$ by

$$\nu_n(x+y) = \sum_{\beta\mathbb{N}\setminus\{0\}} D^\beta u_n(x+y) \sum_{|\alpha|\le p} (-1)^{|\alpha+\beta|} \binom{\alpha+\beta}{\beta} c_{\alpha+\beta} D^{\alpha-\beta}\varphi(x+y) \tag{6.7}$$

Clearly, this shows that ν_n depends on the initial sequence (u_n).
Now, it suffices to show that $|(S\otimes T)\nu_n - (S\otimes T)\nu| < \varepsilon$ for all $n \in \mathbb{N}$. Let $\varphi \in \mathcal{D}_n^{\{M_p\}}$ be given such that $\varphi(x) = 1$ for all $x \in \operatorname{supp}\varphi$. By (6.7), we have

$$\langle S \otimes T, \nu_n \rangle = \langle (S\otimes T)\varphi^{\triangle}, \nu_n \rangle, \quad n \in \mathbb{N}$$

We show that for every $(r_p) \in \mathcal{R}$

$$\inf_{n\in\mathbb{N}} \|\nu_n\| < \infty \tag{6.8}$$

Since $\overline{(u_n)} \in \overline{\mathbb{U}}_{2n}^{\{M_p\}}$ and 6.8 implies

$$\inf_{n\in\mathbb{N}} \|\overline{u}_n + \nu_n\| < \infty$$

Fix an arbitrary $(r_p) \in \mathcal{R}$. Putting $s_k := \max\{r_k, u_k\}$ for $k \in \mathbb{N}$, we have $(s_p) \in \mathcal{R}$. By Lemma 6.2, there exists $(r_p) \in \mathcal{R}$ such that $r_k \le s_k$ and inequality (6.3) hold. We have

$$r_p > 16H^2 \quad \text{for} \quad p \in \mathbb{N} \tag{6.9}$$

where $H > \frac{1}{4}$ is a constant from condition (M.2)
Let $\alpha \in \mathbb{N}_0^n$ be fixed arbitrarily. Fix $n \in \mathbb{N}$ and $(r_p) \in \mathcal{R}$ as provided above, then using the representation of $\nu_n(x+y)$, we have

$$\begin{aligned}
(R_{|\alpha|}M_\alpha)^{-1}\|D_x^\alpha \nu_n\|_\infty \le &\sum_{i\in\mathbb{N}\setminus\{0\}} \sum_{j\le\alpha} \binom{\alpha}{j} (R_{|\alpha+i-j|}M_{\alpha+i-j})^{-1}\|D_x^{\alpha+i-j}u_n\|_\infty \\
&\sum_{\beta\in\mathbb{N}_0^n} \binom{\beta+i}{i} |c_{\beta+i}| \cdot (R_{|\beta+j|}M_{\beta+j})^{-1}\|D_x^{\beta+j}\varphi^{\triangle}\|_\infty \\
&\cdot (R_{|\alpha|})^{-1}R_{|\alpha+i-j|}R_{\beta+j} \cdot (M_\alpha)^{-1}M_{|\alpha+i-j|}M_{\beta+j}.
\end{aligned} \tag{6.10}$$

According to assumption 6.9 and the part of the proof in [2], we have

$$\inf_{n\to\mathbb{N}} \|\nu_n\|_{(r_p)} = \inf_{n\to\mathbb{N}} \inf_{\alpha\in\mathbb{N}_0^n} \frac{R_{|\alpha|}M_\alpha}{\|D_x^\alpha \nu_n} < \infty$$

and, since $r_k \leq s_k$, we have

$$\inf_{n\to\mathbb{N}} \|\nu_n\|_{(s_p)} \leq \inf_{n\to\mathbb{N}} \|\nu_n\|_{(r_p)} < \infty$$

Hence the proof. □

Acknowledgement. The authors would like to gratefully appreciate those who participated in the preparation of the manuscript for their remarks and comments. This work was sponsored by the authors involved.

References

1. Amaonyeiro, A. U. (2019). On The Extended Generalized Leibnitz Rule for Differentiability of Distributions. International Journal of Science and Research. 8(6). Paper ID: ART20197976. ISSN: 2319-7064. 1797-1799.
2. Mincheva-Kaminska, S. (2021). A sequential approach to the convolution of Roumieu ultradistributions. Advances in Operator Theory 26(6). https://doi.org/10.48550/arXiv.1911.00503
3. Debrouwere, A., Vindas, J. (2019). On weighted inductive limits of spaces of ultradifferentiable functions and their duals. Math. Nachir. 292(3), 573-602. https://doi.org/10.1002/mana.201700395
4. Debrouwere, A., Vindas, J. (2019). A projective descriptive of generalized Gelfand-Shilov spaces of Roumieu type. Analysis, Probability, Applications, and Computation. Trends in Mathematics, pp. 407-417. Birhäuser, Cham https://doi.org/10.1007/978-3-030-04459-6_39
5. Debrouwere, A., Prangoski, B.., Vindas, J. (2019). On the projective description of spaces of ultradifferentiable functions of Roumieu type. Preprint arXiv: 1912.11938 [math.FA] https://doi.org/10.1007/978-3-030-87502-2_37
6. Dierolf, P., Voigt, J. (1976). Convolution and S^1-convolution of distributions. Collect. Math. 29, 185-196. http://eudml.org/doc/43721
7. Dimovski, P., Pilipovič, S., Prangoski, Vindas, J. (2016) Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces. Kyoto J. Math. 56(2), 401-440. https://doi.org/10.1215/21562261-3478916
8. Dimovski, P., Pilipovic, S., Prangoski, Vindas, J.(2015). On a class of translation-invariant spaces of quasianalytic ultradistributions. Novi Sad J. Math. 45(1), 143-175. https://doi.org/10.30755/nsjom.dans14.07
9. Kamiňski, A. (1982). Convolution, product and Fourier transform of distributions. Stud. Math. 74, 83-86. https://doi.org/10.4064/sm-74-1-83-96
10. Kamiňski, A., Perisic, D., Pilipovic, S. (1995). Existence theorems for convolution of ultradistributions. Dissertationes Math. 340, 93-114. https://dx.doi.org/10.18778/7525-971-1.05
11. Mincheva-Kamiňski, S. (2019). Sequential conditionsof integrability of Roumieu ultradistributions. Preprint arXiv.1910.14173v2 [math.FA] https://doi.org/arXiv:1910.14173
12. Pilipovič, S. (1991). On the convolution in the space of Beurling ultradistributions. Comment. Math. Univ. St. Paul 40, 15-27. MR1104777
13. Pilipovic, S., Prangoski, B. (2014). On the convolution of Roumieu ultradistributions through the ε tensor product. Monatsh. Math, 173, 83-105. https://doi.org/10.1007/s00605-013-0503-4

[1] Department of Mathematics, University of Agriculture, Makurdi, Nigeria.
Email address: anslemamaonyeiro@uam.edu.ng

[2] Department of Mathematics, University of Agriculture, Makurdi, Nigeria.
Email address: kenneth.isife@uam.edu.ng