

ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENT SEQUENCES IN PARTIAL METRIC SPACES

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ABSTRACT. The present study deals with asymptotically equivalent sequences in partial metric spaces. We define the notions of strongly asymptotically lacunary equivalence, asymptotically statistical equivalence, and asymptotically lacunary statistical equivalence. We theoretically contribute to these notions and investigate some of their basic properties.

1. INTRODUCTION AND PRELIMINARIES

Convergence is one of the fundamental concepts of Analysis and Functional Analysis. Statistical convergence is a generalization of the convergence based on the natural density of positive integers, which is essential in summability. Since Fast [1] and Steinhaus [2] introduced the statistical convergence, applications and some generalizations of this concept have been given by many researchers, especially Buck [3], Salat [4], Fridy [5], and Fridy and Orhan [6].

The notion of convergence of a real sequence was extended to statistical convergence as follows: A sequence (ξ_k) of real numbers is referred to as statistically convergent to α if for any $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |\xi_k - \alpha| \geq \varepsilon\}| = 0.$$

Where the $|\cdot|$ denotes the cardinality of the set $\{k \leq n : |\xi_k - \alpha| \geq \varepsilon\}$. We abbreviate it as $st - \lim \xi_k = \alpha$. The statistical convergence is based on the theory of the notion of natural density [7]. $\delta(K)$ represents the natural density of a set $K \subseteq \mathbb{N}$, which is defined as

$$\delta(K) = \lim_{n \rightarrow \infty} \frac{|\{k \leq n : k \in K\}|}{n}.$$

If K is a finite set, with the help of density definition, $\delta(K) = 0$.

The new concept of convergence is explained by replacing the set $\{k : k < n\}$ along with a set $\{k : k_{r-1} \leq k \leq k_r\}$ for some lacunary sequence (k_r) by Fridy and Orhan [8]. The authors compared this new convergence method with other summability methods and considered certain questions of uniqueness of limit

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value. In the lacunary sequence $\theta = (k_r)$, we take terms with $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ when $r \rightarrow \infty$. Additionally, the interval $I_r = (k_{r-1}, k_r]$ are determined by $\theta = (k_r)$, and its term ratio k_r, k_{r-1} is expressed as q_r , i.e., $q_r = \frac{k_r}{k_{r-1}}$. Lacunary statistical convergence is one of our study's additional primary themes. Fridy and Orhan introduced the concept of lacunary statistical convergence [8] in 1993. The sequence $\xi = (\xi_k)$ is a lacunary statistical convergent to α if for all $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |\xi_k - \alpha| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\theta - \lim \xi_k = \alpha$, and the set of all lacunary statistically convergent sequences is denoted by S_θ .

To compare the convergence rates of the sequences, the definition of an asymptotically regular matrix that preserves the asymptotic equivalence of two non-negative sequences was given by Pobyvanets [9] in 1980. However, due to the zero-valued terms of the sequences, in most cases, it was not possible to compare the terms in $\frac{x}{y}$ form. Therefore, Fridy [10] introduced new methods for comparing convergence rates. Following the work of Fridy [10], Marouf [11] investigated some necessary and sufficient conditions for a matrix to be asymptotically regular. Li [12] studied the asymptotic equivalence and summability of sequences.

In [11], two nonnegative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ are said to be asymptotically equivalent, if

$$\lim_k \frac{\xi_k}{\eta_k} = 1$$

and this is denoted by $\xi \sim \eta$. In 2003, Patterson [13] introduced the notion of asymptotic statistical equivalence by combining asymptotic equivalence and statistical convergence and extended these notions by providing statistical correspondences to the theorems. In Patterson's study, let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ are two nonnegative sequences. These sequences are said to be the asymptotically statistical equivalent of multiple α if, for all $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{\xi_k}{\eta_k} - \alpha \right| \geq \varepsilon \right\} \right| = 0$$

and this is denoted by $\xi \stackrel{S^\alpha}{\sim} \eta$. Simply asymptotically statistical equivalent if $\alpha = 1$. Moreover, let S^α denote the set of ξ and η such that $\xi \stackrel{S^\alpha}{\sim} \eta$.

Patterson and Savas [14] introduced the concept of asymptotic lacunary statistical equivalence, a natural combination of asymptotic equivalence, statistical convergence, and lacunary sequences.

Assume that θ is a lacunary sequence, and consider two nonnegative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$. These sequences are called asymptotically lacunary statistically equivalent concerning the multiple α if, for all $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{\xi_k}{\eta_k} - \alpha \right| \geq \varepsilon \right\} \right| = 0.$$

This relationship is denoted as $\xi \overset{S_\theta^\alpha}{\sim} \eta$ and referred to as simply asymptotically lacunary statistically equivalent when $\alpha = 1$. Moreover, let S_θ^α denote the set of ξ and η such that $\xi \overset{S_\theta^\alpha}{\sim} \eta$.

Assume that θ is a lacunary sequence, and consider two nonnegative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$. These sequences are called strong asymptotically lacunary statistically equivalent concerning the multiple α if for all $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{\xi_k}{\eta_k} - \alpha \right| = 0.$$

This situation is denoted by $\xi \overset{N_\theta^\alpha}{\sim} \eta$ and simply strong asymptotically lacunary equivalent if $\alpha = 1$. Besides, let N_θ^α denote the set of ξ and η such that $\xi \overset{N_\theta^\alpha}{\sim} \eta$.

In 1994, a new definition of partial metric space was introduced by Matthews [15]. Unlike the metric in the usual sense, the partial metric includes the concept of a set whose distance to itself is different from zero. In this context, partial metric is a broader concept. For further details on partial metric spaces, we refer to ([16], [17]) and many others. Convergence and summability in partial metric spaces have recently increased in popularity. Nuray [18] introduced the notions of statistical convergence and strong Cesaro summability in the mentioned spaces and investigated the relations between statistical convergence and strong Cesaro summability. Glle et al. [19] defined the concept of ideal convergence, which is a generalization of ordinary and statistical convergence and deals with relations between newly comprehensive concepts. As a result, studies on generalized convergence concepts in partial metric spaces maintain their popularity, and relevant theories are being developed ([20], [21], [22]).

Let's recall the partial metric space and its related comprehensive notions. In the rest of the study, the set of nonnegative real numbers will be denoted by $\mathbb{R}^{\geq 0}$.

Definition 1.1. [15] Let $\mathbb{V}$ be a non-empty set. A function $p : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}^{\geq 0}$ is said to be a partial metric provided that, for each $\alpha, \beta, \gamma \in \mathbb{V}$, the following conditions are satisfied:

- (P₁) $\alpha = \beta \Leftrightarrow p(\alpha, \alpha) = p(\alpha, \beta) = p(\beta, \beta)$
- (P₂) $p(\alpha, \alpha) \leq p(\alpha, \beta)$
- (P₃) $p(\alpha, \beta) = p(\beta, \alpha)$
- (P₄) $p(\alpha, \beta) \leq p(\alpha, \gamma) + p(\gamma, \beta) - p(\gamma, \gamma)$

In this case, the pair $(\mathbb{V}, p)$ is said to be a partial metric space (it is clear that if $p(\alpha, \beta) = 0$, then from the axioms (P₁) and (P₂), $\alpha = \beta$) besides, every metric space is automatically a partial metric space, but the inverse does not hold.

Unless otherwise stated, partial metric space $(\mathbb{V}, p)$ will be denoted by $\mathbb{V}_p$ in the rest of the paper.

Example 1.2. Let $p : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}^{\geq 0}$ $p(\alpha, \beta) = \max\{\alpha, \beta\}$ be a mapping such that p is partial metric on $\mathbb{R}^{\geq 0}$. However, p is not metric because (P₂) property is not satisfied.

Example 1.3. Let $\mathbb{V} = \{[a, b] : a \leq b\}$ with $a, b \in \mathbb{R}$ define $\mathbf{p} : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}^{\geq 0}$ such that $\mathbf{p}([a_1, a_2], [b_1, b_2]) = \max\{a_2, b_2\} - \min\{a_1, b_1\}$, then $(\mathbb{V}, \mathbf{p})$ is a partial metric space.

Example 1.4. Let $\mathbf{p}^w$ be a function on $\mathbb{V} \times \mathbb{V}$ such that

$$\mathbf{p}^w(\alpha, \beta) = 2\mathbf{p}(\alpha, \beta) - \mathbf{p}(\alpha, \alpha) - \mathbf{p}(\beta, \beta).$$

Then, $(\mathbb{V}, \mathbf{p}^w)$ is a metric space.

With the function defined in Example (1.4), a metric space is obtained from each partial metric space.

Definition 1.5. [15] Let $\alpha \in \mathbb{V}$ and $\varepsilon > 0$. Then, the set

$$\mathcal{B}_p(\alpha, \varepsilon) = \{\beta \in \mathbb{V} : \mathbf{p}(\alpha, \beta) < \mathbf{p}(\alpha, \alpha) + \varepsilon\}$$

is called an open ball of radius ε with center α .

Each partial metric $\mathbf{p}$ on $\mathbb{V}$ generates a $\tau_{\mathbf{p}}$ topology that takes as a base the family of $\mathbf{p}$ -open balls on $\mathbb{V}$ for each element $\alpha \in \mathbb{V}$ and $\varepsilon > 0$ with this $\tau_{\mathbf{p}}$ topology, $(\mathbb{V}, \tau_{\mathbf{p}})$ is a τ_0 space.

Definition 1.6. [15] Let (ξ_k) be a sequence in $\mathbb{V}_{\mathbf{p}}$.

(1) (ξ_k) is referred to as convergent to $\alpha \in \mathbb{V}$, if

$$\lim_{k \rightarrow \infty} \mathbf{p}(\xi_k, \alpha) = \mathbf{p}(\alpha, \alpha)$$

or equivalently, for all $\varepsilon > 0$, there exists a $k_{\varepsilon} \in \mathbb{N}$ such that

$$|\mathbf{p}(\xi_k, \alpha) - \mathbf{p}(\alpha, \alpha)| < \varepsilon$$

whenever $k > k_{\varepsilon}$.

(2) (ξ_k) is properly convergent to $\alpha \in \mathbb{V}$, if $\lim_{k \rightarrow \infty} \mathbf{p}^w(\xi_k, \alpha) = 0$.

(3) A sequence (ξ_k) is said to be bounded in $\mathbb{V}_{\mathbf{p}}$ if for all $k, l \in \mathbb{N}$, there exists a $C > 0$ such that $\mathbf{p}(\xi_k, \xi_l) < C$.

Nuray [18] characterized the following definition of statistical convergence in $\mathbb{V}_{\mathbf{p}}$.

Definition 1.7. A sequence (ξ_k) in $\mathbb{V}_{\mathbf{p}}$ is referred to as statistically convergent to $\alpha \in \mathbb{V}$ if for all $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |\mathbf{p}(\xi_k, \alpha) - \mathbf{p}(\alpha, \alpha)| \geq \varepsilon\}) = 0.$$

The lacunary statistical convergent in $\mathbb{V}_{\mathbf{p}}$ conceptualized by Glle et al. [20] is expressed in the following Definition (1.8):

Definition 1.8. A sequence (ξ_k) in $\mathbb{V}_{\mathbf{p}}$ is said to be lacunary statistically convergent to $\alpha \in \mathbb{V}$ if for all $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |\mathbf{p}(\xi_k, \alpha) - \mathbf{p}(\alpha, \alpha)| \geq \varepsilon\}| = 0.$$

We denoted by $S_{\theta}^{\mathbf{p}} - \lim_{k \rightarrow \infty} \mathbf{p}(\xi_k, \alpha) = \mathbf{p}(\alpha, \alpha)$.

2. MAIN RESULTS

In this section, we present exhaustive definitions and theorems with regard to asymptotic equivalence, asymptotic statistical equivalence, asymptotic lacunary statistical equivalence, and strong asymptotic lacunary equivalence in $\mathbb{V}_p$. In addition, we deal with the relations between these notions.

Definition 2.1. Two nonnegative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ in $\mathbb{V}_p$ are said to be asymptotically equivalent if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{k \rightarrow \infty} p \left(\frac{\xi_k}{\eta_k}, \alpha \right) = 1.$$

We abbreviate it as $\xi \stackrel{p}{\sim} \eta$.

Definition 2.2. Two nonnegative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ in $\mathbb{V}_p$ are referred to as asymptotically statistical equivalent of multiple α if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| p \left(\frac{\xi_k}{\eta_k}, \alpha \right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| = 0.$$

This situation is denoted by $\xi \stackrel{p-S^\alpha}{\sim} \eta$ and simply asymptotically statistical equivalent if $p(\alpha, \alpha) = 1$.

Definition 2.3. Let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ be two nonnegative sequences in $\mathbb{V}_p$. These sequences are called properly asymptotically statistical equivalent of multiple α if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : p^w \left(\frac{\xi_k}{\eta_k}, \alpha \right) \geq \varepsilon \right\} \right| = 0.$$

We abbreviate it as $\xi \stackrel{p^w-S^\alpha}{\sim} \eta$.

Theorem 2.4. Let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ be two nonnegative sequences in $\mathbb{V}_p$. Then,

$$\xi \stackrel{p^w-S^\alpha}{\sim} \eta \Leftrightarrow \xi \stackrel{p-S^\alpha}{\sim} \eta.$$

Proof.

$$\begin{aligned}
\xi \stackrel{p^w - S^\alpha}{\sim} \eta &\Leftrightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : p^w \left(\frac{\xi_k}{\eta_k}, \alpha \right) \geq \varepsilon \right\} \right| = 0 \\
&\Leftrightarrow p^w \left(\frac{\xi_k}{\eta_k}, \alpha \right) < \varepsilon \text{ for almost all } k \\
&\Leftrightarrow \left| 2p \left(\frac{\xi_k}{\eta_k}, \alpha \right) - p \left(\frac{\xi_k}{\eta_k}, \frac{\xi_k}{\eta_k} \right) - p(\alpha, \alpha) \right| < \varepsilon \text{ for almost all } k \\
&\Leftrightarrow \left| p \left(\frac{\xi_k}{\eta_k}, \alpha \right) - p(\alpha, \alpha) \right| < \varepsilon \text{ and } \left| p \left(\frac{\xi_k}{\eta_k}, \frac{\xi_k}{\eta_k} \right) - p(\alpha, \alpha) \right| < \varepsilon \text{ for almost all } k \\
&\Leftrightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| p \left(\frac{\xi_k}{\eta_k}, \alpha \right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| = 0 \\
\text{and } \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| p \left(\frac{\xi_k}{\eta_k}, \frac{\xi_k}{\eta_k} \right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| &= 0 \\
&\Leftrightarrow \xi \stackrel{p - S^\alpha}{\sim} \eta
\end{aligned}$$

□

Definition 2.5. Let θ be a lacunary sequence and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_p$. These sequences are referred to as the asymptotically lacunary statistical equivalent of multiple α if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| p \left(\frac{\xi_k}{\eta_k}, \alpha \right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| = 0,$$

and is denoted by $\xi \stackrel{p - S_\theta^\alpha}{\sim} \eta$. In addition, the set of aforesaid sequences is denoted by $p - S_\theta^\alpha$. If $p(\alpha, \alpha) = 1$, the sequences mentioned are simply asymptotically lacunary statistical equivalent.

Definition 2.6. Let θ be a lacunary sequence and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are non-negative sequences in $\mathbb{V}_p$. These sequences are called properly asymptotically lacunary statistical equivalent of multiple α if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : p^w \left(\frac{\xi_k}{\eta_k}, \alpha \right) \geq \varepsilon \right\} \right| = 0.$$

We abbreviate it as $\xi \stackrel{p^w - S_\theta^\alpha}{\sim} \eta$.

Theorem 2.7. Let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ be two nonnegative sequences in $\mathbb{V}_p$. Then,

$$\xi \stackrel{p^w - S_\theta^\alpha}{\sim} \eta \Leftrightarrow \xi \stackrel{p - S_\theta^\alpha}{\sim} \eta.$$

Proof. This theorem can be proved similarly to the Theorem (2.4). □

Definition 2.8. Let θ be a lacunary sequence and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are nonnegative sequences in $\mathbb{V}_p$. These sequences are called strong asymptotically lacunary

equivalent of multiple α if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| = 0,$$

and is denoted by $\xi \stackrel{p-N_\theta^\alpha}{\sim} \eta$. In addition, the set of aforesaid sequences is denoted by $p - N_\theta^\alpha$. If $p(\alpha, \alpha) = 1$, the sequences mentioned are simply strong asymptotically lacunary equivalent.

Definition 2.9. Let θ be a lacunary sequence and $\xi = (\xi_k)$, $\eta = (\eta_k)$ are nonnegative sequences in $\mathbb{V}_p$. These sequences are called properly strong asymptotically lacunary equivalent of multiple α if for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} p^w\left(\frac{\xi_k}{\eta_k}, \alpha\right) = 0.$$

We abbreviate it as $\xi \stackrel{p^w-N_\theta^\alpha}{\sim} \eta$.

Theorem 2.10. Let $\xi = (\xi_k)$ and $\eta = (\eta_k)$ be two nonnegative sequences in $\mathbb{V}_p$. Then,

$$\xi \stackrel{p^w-N_\theta^\alpha}{\sim} \eta \Leftrightarrow \xi \stackrel{p-N_\theta^\alpha}{\sim} \eta.$$

Proof. This theorem can be proved similarly to the theorem (2.4). $\square$

Theorem 2.11. Let $\theta = (k_r)$ be a lacunary sequence and two nonnegative sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ in $\mathbb{V}_p$. Then,

- (1) $\xi \stackrel{p-N_\theta^\alpha}{\sim} \eta \Rightarrow \xi \stackrel{p-S_\theta^\alpha}{\sim} \eta$,
- (2) $\xi, \eta \in l_\infty$ and $\xi \stackrel{p-S_\theta^\alpha}{\sim} \eta \Rightarrow \xi \stackrel{p-N_\theta^\alpha}{\sim} \eta$,
- (3) $S_\theta^\alpha \cap l_\infty = N_\theta^\alpha \cap l_\infty$.

Proof. (1) Assume that $\xi \stackrel{p-N_\theta^\alpha}{\sim} \eta$, for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\begin{aligned} \sum_{k \in I_r} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| &\geq \sum_{k \in I_r} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \\ &\quad \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \\ &\geq \varepsilon \cdot \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right|. \end{aligned}$$

If both sides of the inequality are multiplied by $\frac{1}{h_r}$ and the limit is taken as $r \rightarrow \infty$ the following expression is obtained:

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right|.$$

Since $\xi \stackrel{p-N_\theta^\alpha}{\sim} \eta$, then $\xi \stackrel{p-S_\theta^\alpha}{\sim} \eta$.

(2) Let $\xi, \eta \in l_\infty$ and $\xi \stackrel{p-S_\theta^\alpha}{\sim} \eta$. Then, for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| = 0$$

and there exists $C > 0$ such that

$$\left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \leq C.$$

For all $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon}} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \\ &\quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| < \varepsilon}} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \\ &\leq \frac{C}{h_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| + \varepsilon. \end{aligned}$$

The following expression is obtained if the limit is taken as $r \rightarrow \infty$ on both sides.

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \leq \lim_{r \rightarrow \infty} \frac{C}{h_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| + \varepsilon.$$

Consequently, $\xi \stackrel{P^{-N_\theta^\alpha}}{\sim} \eta$.

(3) It is clear from 1 and 2. □

Theorem 2.12. Let $\xi = (\xi_k)$, $\eta = (\eta_k)$ are two nonnegative sequences in $\mathbb{V}_p$, θ be a lacunary sequence with $\liminf q_r > 1$. If $\xi \stackrel{P^{-S^\alpha}}{\sim} \eta$, then (ξ_k) and (η_k) are asymptotically lacunary statistical equivalent of multiple α .

Proof. Let $\liminf q_r > 1$. Then there exists a $\ell > 0$ such that $q_r \geq 1 + \ell$ for sufficiently large r , which signify $\frac{h_r}{k_r} \geq \frac{\ell}{1+\ell}$. If $\xi \stackrel{P^{-S^\alpha}}{\sim} \eta$, then for all $\varepsilon > 0$ and $\alpha \in \mathbb{V}$, and for sufficiently large r , we have

$$\begin{aligned} \frac{1}{k_r} \left| \left\{ k \leq k_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| &\geq \frac{1}{k_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &\geq \frac{\ell}{1+\ell} \cdot \frac{1}{h_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right|. \end{aligned}$$

The following expression is obtained if the limit is taken as $r \rightarrow \infty$ on both sides.

$$\lim_{r \rightarrow \infty} \frac{1}{k_r} \left| \left\{ k \leq k_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \geq \frac{\ell}{1+\ell} \lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| p\left(\frac{\xi_k}{\eta_k}, \alpha\right) - p(\alpha, \alpha) \right| \geq \varepsilon \right\} \right|.$$

Since $\xi \stackrel{P^{-S^\alpha}}{\sim} \eta$, then (ξ_k) and (η_k) are asymptotically lacunary statistical equivalent of multiple α . □

Theorem 2.13. Let $\xi = (\xi_k)$, $\eta = (\eta_k)$ are two nonnegative sequences in $\mathbb{V}_p$, θ be a lacunary sequence with $\limsup q_r < \infty$. If $\xi \stackrel{P^{-S_\theta^\alpha}}{\sim} \eta$, then $\xi = (\xi_k)$ and $\eta = (\eta_k)$ are asymptotically statistical equivalent of multiple α .

Proof. Let $\limsup q_r < \infty$, then there exists $M > 0$ such that $q_r < M$, for all $r \geq 1$. Let $\xi \stackrel{\mathbf{p}-S_\theta^\alpha}{\sim} \eta$, $\varepsilon > 0$ and $\alpha \in \mathbb{V}$. There exists $J > 0$ such that, for every $i \geq J$,

$$\gamma_i = \frac{1}{h_i} \left| \left\{ k \in I_i : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| < \varepsilon.$$

We may further find $K > 0$ such that $\gamma_i < K$, for all $i = 1, 2, \dots$. Now, let n be any integer with $k_{r-1} < n < k_r$, where $r > J$. Then we have

$$\begin{aligned} \frac{1}{n} \left| \left\{ k \leq n : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| &\leq \frac{1}{k_{r-1}} \left| \left\{ k \leq k_r : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &= \frac{1}{k_{r-1}} \left| \left\{ k \in I_1 : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &\quad + \frac{1}{k_{r-1}} \left| \left\{ k \in I_2 : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &\quad + \dots + \frac{1}{k_{r-1}} \left| \left\{ k \in I_r : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &= \frac{k_1}{k_{r-1}k_1} \left| \left\{ k \in I_1 : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &\quad + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)} \left| \left\{ k \in I_2 : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &\quad + \dots + \frac{k_J - k_{J-1}}{k_{r-1}(k_J - k_{J-1})} \left| \left\{ k \in I_J : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &\quad + \dots + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})} \left| \left\{ k \in I_r : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \\ &= \frac{k_1}{k_{r-1}} \gamma_1 + \frac{k_2 - k_1}{k_{r-1}} \gamma_2 + \dots + \frac{k_J - k_{J-1}}{k_{r-1}} \gamma_J \\ &\quad + \frac{k_{J+1} - k_J}{k_{r-1}} \gamma_{J+1} + \dots + \frac{k_r - k_{r-1}}{k_{r-1}} \gamma_r \\ &\leq \left\{ \sup_{i \geq 1} \gamma_i \right\} \frac{k_J}{k_{r-1}} + \left\{ \sup_{i \geq J} \gamma_i \right\} \frac{k_r - k_J}{k_{r-1}} \\ &\leq K \frac{k_J}{k_{r-1}} + \varepsilon M. \end{aligned}$$

Since $r \rightarrow \infty$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \mathbf{p} \left(\frac{\xi_k}{\eta_k}, \alpha \right) - \mathbf{p}(\alpha, \alpha) \right| \geq \varepsilon \right\} \right| \leq K \lim_{r \rightarrow \infty} \frac{k_J}{k_{r-1}} + \varepsilon M.$$

Since $\xi \stackrel{\mathbf{p}-S_\theta^\alpha}{\sim} \eta$, then $\xi = (\xi_k)$ and $\eta = (\eta_k)$ are asymptotically statistical equivalent of multiple α . $\square$

Theorem 2.14. Let $\xi = (\xi_k)$, $\eta = (\eta_k)$ are two nonnegative sequences in $\mathbb{V}_\mathbf{p}$, θ be a lacunary sequence with $1 < \liminf q_r < \limsup q_r < \infty$. Then,

$$\xi \stackrel{\mathbf{p}-S_\theta^\alpha}{\sim} \eta = \xi \stackrel{\mathbf{p}-S^\alpha}{\sim} \eta.$$

Proof. It is clear that to obtain the proof using Theorem (2.12) and Theorem (2.13). $\square$

3. CONCLUSION

Partial metric spaces, a generalization of metric spaces in which the distance of an arbitrary element to itself need not be zero, have been studied in many fields, especially in fixed point theory. The notions of convergence and summability theory, which have a vital role in understanding the behavior of sequences, have been little studied in partial metric spaces. Hence, in this study, we first introduced the concepts of asymptotical equivalent and asymptotically statistical equivalent in partial metric spaces. Afterward, the notions of asymptotically lacunary statistical equivalents are introduced, and their relations are analyzed. By defining the notions of properly asymptotic statistical equivalent and properly asymptotic lacunary statistical equivalent in partial metric spaces. Similar studies for double, triple sequences and even sequences of sets can make important contributions to the development of summability theory.

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