

L^r INEQUALITIES FOR POLYNOMIALS WITH RESTRICTED ZEROS

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ABSTRACT. Let $p(z)$ be a polynomial of degree n .
For $r > 0$, we denote

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^r dt \right\}^{\frac{1}{r}},$$

and a well-known fact from analysis [21] gives

$$\lim_{r \rightarrow \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^r dt \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|.$$

If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then Govil and Rahman [10] extended Malik's inequality [13] into L^r version by proving

$$\|p'\|_r \leq \frac{n}{\|z+k\|_r} \|p\|_r, \quad r > 0.$$

In this paper, we prove improved and generalized versions of the above inequality.

1. INTRODUCTION AND PRELIMINARIES

Let $p(z)$ be a polynomial of degree n . We define

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad 0 < r < \infty. \quad (1.1)$$

If we let $r \rightarrow \infty$ in the above equality and make use of the well-known fact from analysis [21] that

$$\lim_{r \rightarrow \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, one can define

$$\|p\|_0 = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right\}$$

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and show that

$$\lim_{r \rightarrow 0^+} \|p\|_r = \|p\|_0.$$

It would be of further interest that by taking limit as $r \rightarrow 0^+$ that the stated results on L^r inequalities holding for $r > 0$, hold for $r = 0$ as well.

A famous result due to Bernstein [15, 22] states that if $p(z)$ is a polynomial of degree n , then

$$\|p'\|_\infty \leq n\|p\|_\infty. \quad (1.2)$$

Inequality (1.2) can be obtained by letting $r \rightarrow \infty$ in the inequality

$$\|p'\|_r \leq n\|p\|_r, \quad r > 0. \quad (1.3)$$

Inequality (1.3) for $r \geq 1$ is due to Zygmund [23]. Arestov [1] proved that (1.3) remains valid for $0 < r < 1$ as well.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequalities (1.2) and (1.3) can be respectively improved by

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty \quad (1.4)$$

and

$$\|p'\|_r \leq \frac{n}{\|1+z\|_r} \|p\|_r, \quad r > 0. \quad (1.5)$$

Inequality (1.4) was conjectured by Erdős and later verified by Lax [12], whereas, inequality (1.5) was proved by de-Bruijn [5] for $r \geq 1$. Rahman and Schmeisser [19] showed that it remains true for $0 < r < 1$. As a generalization of (1.4), Malik [13] proved that if $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k} \|p\|_\infty. \quad (1.6)$$

Under the same hypothesis of the polynomial $p(z)$, Govil and Rahman [10] extended inequality (1.6) to L^r version by showing that

$$\|p'\|_r \leq \frac{n}{\|z+k\|_r} \|p\|_r, \quad r \geq 1. \quad (1.7)$$

Chan and Malik [3] considered the lacunary polynomial $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$,

$1 \leq \mu \leq n$, and proved the extension of inequality (1.6) as

$$\|p'\|_\infty \leq \frac{n}{1+k^\mu} \|p\|_\infty. \quad (1.8)$$

Under the same assumptions, Qazi [18, Lemma 1] further improved the bound (1.8) by proving

$$\|p'\|_\infty \leq \frac{n}{1 + \left(\frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0| + \mu|a_\mu|k^{\mu+1}} \right)} \|p\|_\infty. \quad (1.9)$$

Inequality (1.8) was extended to the following L^r form by Pukhta [17] for $r \geq 1$ and independently by Rather [20] for $0 < r < 1$. In fact, they proved

$$\|p'\|_r \leq \frac{n}{\|z+k^\mu\|_r} \|p\|_r. \quad (1.10)$$

The L^r analogue of (1.9) was given for $r \geq 1$ by Dewan et al. [6] and recently and independently by Chanam [4] for $r > 0$ by proving

$$\|p'\|_r \leq \frac{n}{\|z + A\|_r} \|p\|_r, \quad (1.11)$$

where $A = \frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0| + \mu|a_\mu|k^{\mu+1}}$.

Nakprasit and Somsuwan [16] proved a generalization and improvement of inequality (1.10) by considering polynomials having a zero of order s at a point in the disc $|z| < 1$ and the rest of the zeros are in $|z| \geq k, k \geq 1$. In fact, they obtained

Theorem 1.1. *If $p(z) = (z - z_0)^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$,*

is a polynomial of degree n having a zero of order s at z_0 with $|z_0| < 1$ and the remaining $n - s$ zeros are in $|z| \geq k, k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \left\{ \frac{s}{(1 - |z_0|)} + \frac{A}{(1 - |z_0|)^s} \right\} \max_{|z|=1} |p(z)| - \frac{A}{(k + |z_0|)^s} \min_{|z|=k} |p(z)|, \quad (1.12)$$

where

$$A = \frac{(1 + |z_0|)^{s+1}(n - s)}{(1 + k^\mu)(1 - |z_0|)}.$$

Although the literature on polynomial inequalities is vast and growing, over the last four decades, many different authors produced a large number of different versions and generalizations of the above inequalities. Many of these generalizations involve the extension of inequalities for ordinary derivative of polynomials into L^r analogues. More information on this topic can be found in the books of Milovanović et al. [15] and Marden [14].

2. LEMMAS

For the proof of the theorems, we require the following lemmas.

Lemma 2.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k > 0$, then*

$$|p(z)| \geq m \text{ for } |z| \leq k, \quad (2.1)$$

where $m = \min_{|z|=k} |p(z)|$.

This lemma is due to Gardner et al. [9].

Lemma 2.2. *The function*

$$f(x) = k^{t+1} \left\{ \frac{\frac{t}{n} \frac{|a_t|}{x} k^{t-1} + 1}{\frac{t}{n} \frac{|a_t|}{x} k^{t+1} + 1} \right\}$$

where $t = 1, 2, 3, \dots$ and $k \geq 1$ is a non-decreasing function of $x > 0$.

Lemma 2.2 is due to Gardner et al. [9].

Lemma 2.3. *If $p(z) = (z - z_0)^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, is a polynomial of degree n having a zero of order s at z_0 with $|z_0| < 1$ and the remaining $n-s$ zeros are in $|z| \geq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| < 1$,*

$$\begin{aligned} S'_0 &= k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \frac{\alpha m}{(k+|z_0|)^s}|} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \frac{\alpha m}{(k+|z_0|)^s}|} k^{\mu+1} + 1} \right\} \\ &\geq k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^{\mu+1} + 1} \right\} = S_0. \end{aligned} \quad (2.2)$$

Proof of Lemma 2.3. Since $\phi(z) = \frac{p(z)}{(z-z_0)^s} = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$, has no zero in $|z| < k$, $k \geq 1$, then by Lemma 2.1, we have

$$\begin{aligned} |\phi(z)| &\geq \min_{|z|=k} |\phi(z)| \text{ for } |z| \leq k \\ &= \min_{|z|=k} \left\{ \frac{|p(z)|}{|z - z_0|^s} \right\} \\ &\geq \frac{1}{(k + |z_0|)^s} \min_{|z|=k} |p(z)| \\ &= \frac{m}{(k + |z_0|)^s}, \end{aligned} \quad (2.3)$$

where $m = \min_{|z|=k} |p(z)|$.

In particular, (2.3) gives for $z = 0$ that

$$|a_0| \geq \frac{m}{(k + |z_0|)^s}. \quad (2.4)$$

Now, for $m > 0$

$$\begin{aligned} \left| a_0 - \alpha \frac{m}{(k + |z_0|)^s} \right| &\geq \left| |a_0| - |\alpha| \frac{m}{(k + |z_0|)^s} \right| \\ &= |a_0| - |\alpha| \frac{m}{(k + |z_0|)^s} > 0. \end{aligned} \quad (2.5)$$

Applying Lemma 2.2 to (2.5), we have the required conclusion of Lemma 2.3. $\square$

Lemma 2.4. *Under the same hypotheses of Lemma 2.3,*

$$\begin{aligned} S_0 &= k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^{\mu+1} + 1} \right\} \\ &\geq 1. \end{aligned} \quad (2.6)$$

Proof of Lemma 2.4. Since $k \geq 1$ and $\mu = 1, 2, 3, \dots$

$$k^\mu \geq 1,$$

which is equivalent to

$$k^\mu - k \geq 1 - k^{\mu+1}. \quad (2.7)$$

Also,

$$\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^\mu > 0. \quad (2.8)$$

Since the L.H.S of (2.8) is > 0 and in (2.7), the L.H.S is ≥ 0 and the R.H.S is ≤ 0 , we have

$$\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^\mu (k^\mu - k) \geq 1 - k^{\mu+1},$$

from which the claim of Lemma 2.4 follows. $\square$

Lemma 2.5. If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then

$$|q'(z)| \leq k^\mu |p'(z)| \text{ on } |z| = 1, \quad (2.9)$$

where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$.

This lemma was established by Chan and Malik [3].

Lemma 2.6. If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then

$$|q'(z)| \leq k^{\mu+1} \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|} k^{\mu-1} + 1}{1 + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|} k^{\mu+1}} |p'(z)| \text{ on } |z| = 1, \quad (2.10)$$

where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$ and

$$\frac{\mu}{n} \frac{|a_\mu|}{|a_0|} k^\mu \leq 1. \quad (2.11)$$

Lemma 2.6 is due to Qazi [18].

Lemma 2.7. Let z be complex and independent of α , where α is real, then for $r > 0$,

$$\int_0^{2\pi} |1 + ze^{i\alpha}|^r d\alpha = \int_0^{2\pi} |e^{i\alpha} + |z||^r d\alpha. \quad (2.12)$$

This lemma is due to Gardner and Govil [7].

Lemma 2.8. If $p(z)$ is a polynomial of degree n and $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then for each α , $0 \leq \alpha < 2\pi$ and $r > 0$,

$$\int_0^{2\pi} \int_0^{2\pi} |q'(e^{i\theta}) + e^{i\alpha} p'(e^{i\theta})|^r d\theta d\alpha \leq 2\pi n^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta. \quad (2.13)$$

The above lemma was obtained by Aziz and Rather [2].

3. MAIN RESULTS

In this paper, we first extend Theorem 1.1 to L^r version. Besides this, we prove an L^r inequality of an improved version of Theorem 1.1 by involving certain coefficients of the polynomial. More precisely, we prove

Theorem 3.1. *If $p(z) = (z - z_0)^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$,*

is a polynomial of degree n having a zero of order s at z_0 with $|z_0| < 1$ and the remaining $n-s$ zeros are in $|z| \geq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| < 1$ and for each $r > 0$,

$$\left\| p'(z) - s \frac{p(z)}{(z - z_0)} \right\|_r \leq \frac{(n-s)(1+|z_0|)^{s+1}}{1-|z_0|} F_r \left\| \frac{p(z)}{(z - z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right\|_r, \quad (3.1)$$

where

$$F_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |e^{i\alpha} + k^\mu|^r d\alpha \right\}^{\frac{-1}{r}}$$

and

$$m = \min_{|z|=k} |p(z)|.$$

Remark 3.2. Taking limit as $r \rightarrow \infty$ on both sides of inequality (3.1) of Theorem 3.1, we have

$$\max_{|z|=1} \left| p'(z) - s \frac{p(z)}{z - z_0} \right| \leq \frac{(n-s)(1+|z_0|)^{s+1}}{(1-|z_0|)(1+k^\mu)} \max_{|z|=1} \left| \frac{p(z)}{(z - z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right|. \quad (3.2)$$

Let z_1 on $|z| = 1$ be such that

$$\max_{|z|=1} \left| \frac{p(z)}{(z - z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right| = \left| \frac{p(z_1)}{(z_1 - z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right|. \quad (3.3)$$

Now, we choose the argument of α such that

$$\begin{aligned} \left| \frac{p(z_1)}{(z_1 - z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right| &= \frac{|p(z_1)|}{|z_1 - z_0|^s} - |\alpha| \frac{m}{(k+|z_0|)^s} \\ &\leq \frac{\max_{|z|=1} |p(z)|}{(1-|z_0|)^s} - |\alpha| \frac{m}{(k+|z_0|)^s}. \end{aligned} \quad (3.4)$$

From (3.3) and (3.4), we have

$$\max_{|z|=1} \left| \frac{p(z)}{(z - z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right| \leq \left\{ \frac{\max_{|z|=1} |p(z)|}{(1-|z_0|)^s} - |\alpha| \frac{m}{(k+|z_0|)^s} \right\}. \quad (3.5)$$

Using (3.5) to (3.2), we have

$$\max_{|z|=1} \left| p'(z) - s \frac{p(z)}{z - z_0} \right| \leq A \left\{ \frac{\max_{|z|=1} |p(z)|}{(1-|z_0|)^s} - |\alpha| \frac{m}{(k+|z_0|)^s} \right\}, \quad (3.6)$$

where

$$A = \frac{(n-s)(1+|z_0|)^{s+1}}{(1-|z_0|)(1+k^\mu)}.$$

Let z_2 on $|z| = 1$ be such that

$$\max_{|z|=1} |p'(z)| = |p'(z_2)|.$$

Then

$$\left| p'(z_2) - s \frac{p(z_2)}{z_2 - z_0} \right| \leq \max_{|z|=1} \left| p'(z) - s \frac{p(z)}{z - z_0} \right|. \quad (3.7)$$

Combining (3.6) and (3.7), we have

$$\left| p'(z_2) - s \frac{p(z_2)}{z_2 - z_0} \right| \leq A \left\{ \frac{\max_{|z|=1} |p(z)|}{(1-|z_0|)^s} - |\alpha| \frac{m}{(k+|z_0|)^s} \right\}. \quad (3.8)$$

From (3.8), we readily obtain

$$\begin{aligned} |p'(z_2)| &\leq s \left| \frac{p(z_2)}{z_2 - z_0} \right| + A \left\{ \frac{\max_{|z|=1} |p(z)|}{(1-|z_0|)^s} - |\alpha| \frac{m}{(k+|z_0|)^s} \right\} \\ &\leq \frac{s}{1-|z_0|} \max_{|z|=1} |p(z)| + A \left\{ \frac{\max_{|z|=1} |p(z)|}{(1-|z_0|)^s} - |\alpha| \frac{m}{(k+|z_0|)^s} \right\}, \end{aligned}$$

which on taking limit as $|\alpha| \rightarrow 1$ gives

$$\max_{|z|=1} |p'(z)| \leq \left\{ \frac{s}{1-|z_0|} + \frac{A}{(1-|z_0|)^s} \right\} \max_{|z|=1} |p(z)| - \frac{Am}{(k+|z_0|)^s},$$

which is inequality (1.12) of Theorem 1.1.

Remark 3.3. Putting $z_0 = 0$, Theorem 3.1 reduces to the following interesting result which gives the L^r extension of a result proved by Kumar and Lal [11].

Corollary 3.4. *If $p(z) = z^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$,*

is a polynomial of degree n with zero of order s at the origin and the remaining $n-s$ zeros are in $|z| \geq k, k \geq 1$, then for every real or complex number α with $|\alpha| < 1$ and for each $r > 0$,

$$\left\| p'(z) - s \frac{p(z)}{z} \right\|_r \leq (n-s) F_r \left\| \frac{p(z)}{z^s} - \alpha \frac{m}{k^s} \right\|_r, \quad (3.9)$$

where F_r is as defined in Theorem 3.1.

Remark 3.5. If we follow the similar arguments of Remark 3.2, we would be able to verify that inequality (3.9) of Corollary 3.4 represents the L^r version due to Kumar and Lal [11].

Remark 3.6. Further, on putting $s = 0$, Corollary 3.4 provides the integral mean inequality of the ordinary inequality (1.10) proved by Pukhta [17].

Remark 3.7. When $s = 0, \alpha = 0$, Corollary 3.4 reduces to the L^r analogue of inequality (1.8) proved by Chan and Malik [3].

Remark 3.8. If we assign $s = 0$ and $\mu = 1$, Corollary 3.4 becomes an improved L^r inequality of (1.7) due to Govil and Rahman [10].

Remark 3.9. When $s = 0, \mu = 1$ and $\alpha = 0$, Corollary 3.4 gives inequality (1.7).

Remark 3.10. Lastly, when $s = 0, k = \mu = 1$ and taking limit as $|\alpha| \rightarrow 1$, inequality (3.9) of Corollary 3.4 provides an improved version of the well-known de-Bruijn's inequality [5].

Next, under the same set of hypotheses, by involving certain co-efficients of the polynomial, we prove the following improved version of Theorem 1.1. In fact, we prove

Theorem 3.11. *If $p(z) = (z - z_0)^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu \right)$, $1 \leq \mu \leq n - s$,*

$0 \leq s \leq n - 1$, is a polynomial of degree n having a zero of order s at z_0 with $|z_0| < 1$ and the remaining $n - s$ zeros are in $|z| \geq k, k \geq 1$, then for every real or complex number α with $|\alpha| < 1$ and for each $r > 0$,

$$\left\| p'(z) - s \frac{p(z)}{(z - z_0)} \right\|_r \leq \frac{(n - s)(1 + |z_0|)^{s+1}}{1 - |z_0|} G_r \left\| \frac{p(z)}{(z - z_0)^s} - \alpha \frac{m}{(k + |z_0|)^s} \right\|_r, \quad (3.10)$$

where

$$G_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |e^{i\alpha} + S_0|^r d\alpha \right\}^{\frac{-1}{r}},$$

$$S_0 = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha|} \frac{m}{(k + |z_0|)^s} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha|} \frac{m}{(k + |z_0|)^s} k^{\mu+1} + 1} \right\}$$

and

$$m = \min_{|z|=k} |p(z)|.$$

Remark 3.12. By Lemma 2.4, we have for $k \geq 1$,

$$S_0 \geq k^\mu \geq 1,$$

then

$$G_r \leq F_r, \quad (3.11)$$

where G_r and F_r are as defined in Theorems 3.1 and 3.11.

By inequality (3.11), it is evident that the bound of Theorem 3.11 improves over that of Theorem 3.1.

Remark 3.13. If we put $z_0 = 0$ in Theorem 3.11, we have the following result which has some important implications to the earlier known results.

Corollary 3.14. *If $p(z) = z^s \left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, is a polynomial of degree n with zero of order s at the origin and the remaining $n-s$ zeros are in $|z| \geq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| < 1$ and for each $r > 0$,*

$$\left\| p'(z) - s \frac{p(z)}{z} \right\|_r \leq (n-s) G_r \left\| \frac{p(z)}{z^s} - \alpha \frac{m}{k^s} \right\|_r, \quad (3.12)$$

where G_r is as defined in Theorem 3.11.

Remark 3.15. Corollary 3.14 yields a generalized L^r version of an inequality due to Gardner et al. [8].

Remark 3.16. Putting $s = 0$, $\alpha = 0$, Corollary 3.14 reduces to an L^r inequality proved by Chanam [4] and Dewan et al. [6].

Remark 3.17. Assigning $s = 0$ and taking limit as $|\alpha| \rightarrow 1$, inequality (3.12) of Corollary 3.14 assumes an improved and generalized version of de-Bruijn inequality [5].

4. PROOF OF THEOREMS

First, we prove Theorem 3.11

Proof of Theorem 3.11. Let $p(z) = (z - z_0)^s \phi(z)$ where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$, $1 \leq \mu \leq n-s$, is a polynomial of degree $n-s$ having no zero in $|z| < k$, $k \geq 1$. Now let

$$\begin{aligned} m' &= \min_{|z|=k} |\phi(z)| \\ &= \min_{|z|=k} \left\{ \frac{1}{|z - z_0|^s} |p(z)| \right\} \\ &\geq \frac{1}{(k + |z_0|)^s} \min_{|z|=k} |p(z)| \\ &= \frac{1}{(k + |z_0|)^s} m, \end{aligned}$$

where

$$m = \min_{|z|=k} |p(z)|.$$

If $\phi(z)$ has a zero on $|z| = k$, then $m' = 0$. From now on, we assume that all the $n-s$ zeros of $\phi(z)$ lie in $|z| \geq k$, $k \geq 1$. Therefore, for every real or complex number α such that $|\alpha| < 1$, it follows from Rouché's Theorem that all the zeros of the polynomial

$$\psi(z) = \phi(z) - \alpha \frac{m}{(k + |z_0|)^s}, \quad (4.1)$$

where $m = \min_{|z|=k} |p(z)|$, of degree $n-s$ lie in $|z| \geq k$, $k \geq 1$.

Applying Lemma 2.6 to $\psi(z)$, we have for $|z| = 1$,

$$S'_0 |\psi'(z)| \leq |\eta'(z)|, \quad (4.2)$$

where

$$S'_0 = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \frac{\alpha m}{(k+|z_0|)^s}|} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0 - \frac{\alpha m}{(k+|z_0|)^s}|} k^{\mu+1} + 1} \right\} \geq 1 \quad (4.3)$$

and

$$\eta(z) = z^{n-s} \overline{\psi\left(\frac{1}{\bar{z}}\right)}.$$

We can easily verify that for every real or complex number β and $R \geq r' \geq 1$,

$$|R + e^{i\beta}| \geq |r' + e^{i\beta}|.$$

This implies for each $r > 0$,

$$\int_0^{2\pi} |R + e^{i\beta}|^r d\beta \geq \int_0^{2\pi} |r' + e^{i\beta}|^r d\beta. \quad (4.4)$$

For point $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $\psi'(e^{i\theta}) \neq 0$, we have

$$R = \left| \frac{\eta'(e^{i\theta})}{\psi'(e^{i\theta})} \right|,$$

and $r' = S'_0$, then from (4.2) and (4.3),

$$R \geq r' \geq 1.$$

Now, we have for $r > 0$

$$\begin{aligned} \int_0^{2\pi} \left| \eta'(e^{i\theta}) + e^{i\beta} \psi'(e^{i\theta}) \right|^r d\beta &= |\psi'(e^{i\theta})|^r \int_0^{2\pi} \left| \frac{\eta'(e^{i\theta})}{\psi'(e^{i\theta})} + e^{i\beta} \right|^r d\beta \\ &= |\psi'(e^{i\theta})|^r \int_0^{2\pi} \left| \left| \frac{\eta'(e^{i\theta})}{\psi'(e^{i\theta})} \right| + e^{i\beta} \right|^r d\beta \quad (\text{by Lemma 2.6}) \\ &\geq |\psi'(e^{i\theta})|^r \int_0^{2\pi} |S'_0 + e^{i\beta}|^r d\beta. \end{aligned} \quad (4.5)$$

For point $e^{i\theta}$, $0 \leq \theta < 2\pi$ for which $\psi'(e^{i\theta}) = 0$, inequality (4.5) trivially holds.

Now using (4.5) to Lemma 2.7, we obtain for each $r > 0$,

$$\int_0^{2\pi} |S'_0 + e^{i\beta}|^r d\beta \int_0^{2\pi} |\psi'(e^{i\theta})|^r d\theta \leq 2\pi(n-s)^r \int_0^{2\pi} |\psi(e^{i\theta})|^r d\theta. \quad (4.6)$$

By Lemma 2.3,

$$S'_0 \geq S_0 \geq 1, \quad (4.7)$$

and by (4.4), we have for each $r > 0$,

$$\int_0^{2\pi} |S_0 + e^{i\beta}|^r d\beta \leq \int_0^{2\pi} |S'_0 + e^{i\beta}|^r d\beta. \quad (4.8)$$

Using (4.8) to (4.6), we have

$$\int_0^{2\pi} |S_0 + e^{i\beta}|^r d\beta \int_0^{2\pi} |\psi'(e^{i\theta})|^r d\theta \leq 2\pi(n-s)^r \int_0^{2\pi} |\psi(e^{i\theta})|^r d\theta, \quad (4.9)$$

where

$$S_0 = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\alpha| \frac{m}{(k+|z_0|)^s}} k^{\mu+1} + 1} \right\}.$$

Using (4.1) to (4.9) and noting that $\phi(z) = \frac{p(z)}{(z-z_0)^s}$, we have

$$\|\phi'(e^{i\theta})\|_r \leq (n-s)G_r \left\{ \int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta} - z_0)^s} - \frac{\alpha m}{(k+|z_0|)^s} \right|^r d\theta \right\}^{\frac{1}{r}}, \quad (4.10)$$

where

$$G_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |e^{i\beta} + S_0|^r d\beta \right\}^{\frac{-1}{r}}.$$

Since $p'(z) = (z-z_0)^s \phi'(z) + s(z-z_0)^{s-1} \phi(z)$,

$$(z-z_0)p'(z) = (z-z_0)^{s+1} \phi'(z) + sp(z). \quad (4.11)$$

Using (4.11) to (4.10), we have

$$\left\| \frac{p'(z)}{(z-z_0)^s} - s \frac{p(z)}{(z-z_0)^{s+1}} \right\|_r \leq (n-s)G_r \left\| \frac{p(z)}{(z-z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right\|_r. \quad (4.12)$$

For $|z| = 1$, we have

$$|z - z_0| \geq |1 - |z_0|| = 1 - |z_0|,$$

from which, we get

$$\frac{1}{|z - z_0|} \leq \frac{1}{1 - |z_0|}, \quad (4.13)$$

and

$$\frac{1}{|z - z_0|} \geq \frac{1}{1 + |z_0|}. \quad (4.14)$$

Using (4.13) and (4.14) to the left hand side of (4.12), we get

$$\frac{1 - |z_0|}{(1 + |z_0|)^{s+1}} \left\| p'(z) - s \frac{p(z)}{(z-z_0)} \right\|_r \leq (n-s)G_r \left\| \frac{p(z)}{(z-z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right\|_r,$$

which is equivalent to

$$\left\| p'(z) - s \frac{p(z)}{(z-z_0)} \right\|_r \leq \frac{(n-s)(1+|z_0|)^{s+1}}{1-|z_0|} G_r \left\| \frac{p(z)}{(z-z_0)^s} - \alpha \frac{m}{(k+|z_0|)^s} \right\|_r,$$

which completes the proof of Theorem 3.11.

Proof of Theorem 3.1. Applying Lemma 2.5 to the polynomial $\psi(z)$ as given by equality (4.1) in the proof of Theorem 3.11, we have for each $r > 0$,

$$\|\phi'(z)\|_r \leq (n-s)F_r \left\| \phi(z) - \alpha \frac{m}{(k+|z_0|)^s} \right\|_r, \quad (4.15)$$

where

$$F_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |e^{i\alpha} + k^\mu|^r d\alpha \right\}^{\frac{-1}{r}}.$$

Proceeding the similar procedure adopted after inequality (4.10) in the proof of Theorem 3.11, to inequality (4.15), we have inequality (3.1) of Theorem 3.1 and the proof of Theorem 3.1 ends.

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