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SOME COMMON FIXED POINT THEOREMS USING G-CONE METRIC SPACES WITH BANACH ALGEBRA

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Abstract. In this paper, we have proved some common fixed point results using generalized contraction mapping in G-cone metric spaces over Banach algebras. Our results are a generalization and extension of several well-known results related to fixed point theory.

1. Introduction

Fixed point theory plays a major role in the analysis field. The Banach contraction principle [4] is widely used in existence and uniqueness theories. In the study of fixed points of contractive type mappings, some geometric assumptions are needed on the structure of mappings or spaces. Determining the fixed point of contractive mappings is a frequent problem in Mathematics.

The concept of D-metric space was introduced by Dhage [6]. It has been shown that certain theorems involving Dhage's D-metric space are flawed and most of the results claimed by Dhage and others are invalid. These errors were pointed out by Mustafa and Sims [12]. They introduced G-metric spaces as a generalization of metric spaces.

Huang and Zhang [7] introduced the notion of cone metric spaces by replacing real numbers with an ordering Banach space and obtaining a fixed point theorem. Abbas and Jungck [1] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence point and proved some common fixed point theorems in cone metric spaces. Recently, Rezapour and Hamlbarani [15] omitted the assumption of normality in cone metric spaces, a milestone in developing fixed point theory in cone metric spaces.

More recently, Olaleru [14] proved some common fixed point theorems in cone metric spaces for weakly compatible mappings. Adewale [2] introduced the concept of G-cone metric spaces by replacing the set of real numbers with an ordered Banach space, proved convergence properties of sequences, and proved some fixed point theorems in this space. The concept of a G-cone metric space is more general than that of G-metric space and cone metric space.

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The purpose of this paper is to obtain some common fixed point theorems of maps satisfying the contractive conditions in G-cone metric spaces over Banach algebra. Our results are generalizations of some theorems in literature.

2. Preliminaries

In this paper, we assume that $\mathcal{P}$ is a cone in $\mathcal{A}$ with int $\mathcal{P} \neq \phi(\theta$, the additive identity element of $\mathcal{A})$ and $\preceq$ is the partial ordering with respect to $\mathcal{P}$ where $\mathcal{A}$ is a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, satisfying the following properties [10] (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$):

(i) $(xy)z = x(yz)$;
(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(iv) there exists $e \in E$ such that $xe = ex = x$;
(v) $||e|| = 1$;
(vi) $||xy|| \leq ||x||.||y||$.

An element $x \in \mathcal{A}$ is called invertible if there exists $x^{-1} \in \mathcal{A}$ such that

$$xx^{-1} = x^{-1}x = e.$$

Proposition 2.1. [10] *Let $x \in \mathcal{A}$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in \mathcal{A}$ holds*

$$\rho(u) = \lim_{n\to\infty} ||u_n||^{\frac{1}{n}} = inf||u^n||^{\frac{1}{n}} < 1$$

Further, $(e-u)$ is invertible and $(e-u)^{-1} = \sum_{i=0}^{\infty} u^i$.

Consider a Banach algebra $\mathcal{A}, \theta$ be the null vector, e be the identity element of $\mathcal{A}$ and a subset $\mathcal{P}$ of $\mathcal{A}$ is called a cone if it satisfies the following:

(i) $\{\theta, e\} \subset \mathcal{P}$ and $\mathcal{P}$ is closed;
(ii) $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;
(iii) $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$, for all non-negative real numbers α and β;
(iv) $\mathcal{P} \cap (-C\mathcal{P}) = \{\theta\}$.

With respect to cone $\mathcal{P}$, a partial ordering $\preceq$ is defined as $u \preceq w$ if and only if $(w-u) \in \mathcal{P}$ and $u \prec w$ if $u \preceq w$ and $u \neq w$ whereas $u \ll w$ means $(w-u) \in int\mathcal{P}$.

If $\mathcal{A}$ is a Banach space and $\mathcal{P} \subset \mathcal{A}$, satisfies the conditions 1, 3 and 4 then $\mathcal{P}$ is called a cone of $\mathcal{A}$.

Remark 2.2. [10] If $\rho(x) < 1$, then $\|x_n\| \to 0$ as $n \to \infty$.

Definition 2.3. [8] Consider X is a non-empty set, $\mathcal{A}$ be a Banach algebra and $\mathcal{P} \subseteq \mathcal{A}$ be a cone. Suppose the mapping $d : X \times X \to \mathcal{A}$ satisfies the following for all $x, y, z \in X$:

(i) $d(x, z) = \theta$ if and only if $x = z$, and $\theta \preceq d(x, z)$;
(ii) $d(x, z) = d(z, x)$;
(iii) $d(x, z) \preceq d(x, y) + d(y, z)$ for every $x, y, z \in X$.

Here d is called a cone metric and (X, d) is called Cone metric space over a Banach algebra $\mathcal{A}$ (In Short CMSBA). Note that $d(x, z) \in \mathcal{P}$ for all $x, y \in X$.

Definition 2.4. [2] Let X be a non-empty set, $\mathcal{A}$ be a Banach algebra and $G : X^3 \to \mathcal{A}$ be a function satisfying the following properties:

(i) $G(x, y, z) = \theta$ if and only if $x = y = z$;
(ii) $\theta \prec G(x, y, z)$, for all $x, y \in X$, with $x \neq y$;
(iii) $G(x, x, y) \preceq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$;
(iv) $G(x, y, z) = G(y, z, x) = G(x, z, y) = ...$ (symmetry);
(v) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z)$ for all $a, x, y, z \in X$ (rectangle inequality).

Then G is called a G-cone metric over Banach algebra $\mathcal{A}$ and the pair (X, G) denotes a G-cone metric space over Banach algebra.

Definition 2.5. [5] Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and $\{x_n\}$ a sequence in X. We say that

(i) $\{x_n\}$ is a convergent sequence if, for every $c \in B$ with $\theta \ll c$, there is an N such that $d(x_n, x) \ll c$ for all $n \geq N$. Ones write it by $x_n \to x (n \to \infty)$.
(ii) $\{x_n\}$ is a Cauchy sequence if, for every $c \in B$ with $\theta \ll c$, there is an N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 2.6. [2] *Let $\mathcal{A}$ be a Banach algebra and k, a vector in $\mathcal{A}$. If $0 \leq r(k) < 1$, then we have*

$$r((e - k)^{-1}) < (1 - r(k))^{-1}.$$

Lemma 2.7. [3] *Let $\mathcal{A}$ be a Banach algebra and x, y be vectors in $\mathcal{A}$. If x and y commute, then the following holds:*

(i) $r(xy) \leq r(x)r(y)$;
(ii) $r(x + y) \leq r(x) + r(y)$;
(iii) $|r(x) - r(y)| \leq r(x - y)$.

Lemma 2.8. [3] *If $\mathcal{A}$ is real Banach algebra with a solid cone $\mathcal{P}$ and $\{x_n\}$ is a sequence in $\mathcal{A}$. Suppose $||x_n|| \to 0 (n \to \infty)$ for any $\theta \ll c$. Then $x_n \ll c$ for all $n > N^1, N^1 \in N$.*

Lemma 2.9. [2] *If E is a real Banach space with a solid cone $\mathcal{P}$ and if $||x_n|| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

Lemma 2.10. [11] *Let $\mathcal{A}$ be a Banach algebra and $k \in A$. If $\rho(k) < 1$, then*

$$\lim_{n \to \infty} ||k^n|| = 0$$

.

Lemma 2.11. [1] *If X is a symmetric G-cone metric space, then*

$$d_G(x, y) = 2G(x, y, y).$$

Example 2.12. [13] Let $\mathcal{A}$ be the Banach space of all continuous real-valued functions $C(K)$ on a compact Hausdorff topological space K, with multiplication defined pointwise. Then $\mathcal{A}$ is a Banach algebra, and the constant function $f(t) = 1$ is the unit of $\mathcal{A}$.

Let $P = \{f \in \mathcal{A} : f(t) \geq 0 \text{ for all } t \in K\}$. Then $\mathcal{P} \subset \mathcal{A}$ is a normal cone with normal constant $M = 1$. Let $X = C(K)$ with the metric $d : X \times X \to \mathcal{A}$ defined by

$$d(f, g) = |f(t) - g(t)|$$

where $t \in K$. Then (X, d) is a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.13. [2] Let (X, d) be a G-cone metric space over Banach algebra. G is said to be symmetric if

$$G(x, y, y) = G(x, x, y)$$

for all $x, y, z \in X$.

Definition 2.14. [2] A G-cone metric space over Banach algebra $\mathcal{A}$ is said to be G-bounded if for any $x, y, z \in X$, there exists $K \succ \theta$ such that

$$||G(x, y, z)|| \preceq K$$

Definition 2.15. [2] Let (X, G) be a G-cone metric space over Banach algebra and $\{x_n\}$ a sequence in X, $c \gg \theta$ with $c \in \mathcal{A}$. Then

(i) $\{x_n\}$ converges to $x \in X$ if and only if $G(x_m, x_n, x) \ll c$ for all $n, m > N^1, N^1 \in N$.
(ii) $\{x_n\}$ is Cauchy sequence if and only if $G(x_n, x_m, x_p) \ll c$ for all $n, m > p > N^1, N^1 \in N$.
(iii) (X, G) is complete G-cone metric space over Banach algebra if every Cauchy sequence converges.

Definition 2.16. [9] Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 2.17. [1] *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique coincidence point $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. Main Result

Theorem 3.1. *Let (X, G) be a complete symmetric G-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a cone in $\mathcal{A}$. Suppose that the mapping $f, g : X \to X$ satisfies one of the following contractive condition*

$$\begin{aligned} G(fx, fy, fz) \preceq\ & aG(gx, gy, gz) + bG(gx, fx, fx) \\ & + cG(gy, fy, fy) + dG(gz, fz, fz) \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} G(fx, fy, fz) \preceq\ & aG(gx, gy, gz) + bG(gx, gx, fx) \\ & + cG(gy, gy, fy) + dG(gz, gz, fz) \end{aligned} \tag{3.2}$$

For all $x, y, z \in X$ and $a, b, c, d \in [0, 1), r(a) + r(b) + r(c) + r(d) < 1$. Suppose f and g are weakly compatible and $f(X) \subset g(X)$ such that $f(X)$ or $g(X)$ complete *subspace of X. Then the mappings f and g have a unique common fixed point. Moreover, for any $x_0 \in X$ the sequence, $x_n \in X$ defined by $gx = fx_{n-1}$ for all n, converges to the fixed point.*

Proof. Suppose that f and g satisfy condition (3.1). Then for all $x, y \in X$ we have

$$G(fx, fy, fy) \preceq aG(gx, gy, gy) + bG(gx, fx, fx) + (c+d)G(gy, fy, fy) \quad (3.3)$$

Also

$$G(fy, fx, fx) \preceq aG(gy, gx, gx) + bG(gy, fy, fy) + (c+d)G(gx, fx, fx) \quad (3.4)$$

Since X is a symmetric G-cone metric space by adding (3.3) and (3.4) we have

$$d_G(fx, fy) \preceq ad_G(gx, gy) + \frac{(b+c+d)}{2}d_G(fx, gx) + \frac{(b+c+d)}{2}d_G(gy, fy)$$

Let

$$a = \alpha_1, \frac{(b+c+d)}{2} = \alpha_2, \frac{(b+c+d)}{2} = \alpha_3$$

$$r(\alpha_1) + r(\alpha_2) + r(\alpha_3) < 1$$

$$d_G(fx, fy) \preceq \alpha_1 d_G(gx, gy) + \alpha_2 d_G(fx, gx) + \alpha_3 d_G(gy, fy) \quad (3.5)$$

If $fx_n = fx_{n-1}$ for all $n \in N$ then fx_n is a Cauchy sequence. If $fx_n \neq fx_{n-1}$ for all $n \in N$ then put $x = x_{n+1}$ and $y = x_n$ in (3.5), we get

$$d_G(fx_{n+1}, fx_n) \preceq \alpha_1 d_G(gx_{n+1}, gx_n) + \alpha_2 d_G(fx_{n+1}, gx_{n+1}) + \alpha_3 d_G(gx_n, fx_n)$$

Using the fact that $gx_n = fx_{n-1}$ for all n, we have

$$\begin{aligned} d_G(fx_{n+1}, fx_n) \preceq\ & \alpha_1 d_G(fx_n, fx_{n-1}) + \alpha_2 d_G(fx_{n+1}, fx_n) \\ & + \alpha_3 d_G(fx_n, fx_{n-1}) \end{aligned}$$

$$\begin{aligned} d_G(fx_{n+1}, fx_n) \preceq\ & \alpha_1 d_G(fx_n, fx_{n-1}) + \alpha_2 d_G(fx_{n+1}, fx_n) \\ & + \alpha_3 d_G(fx_n, fx_{n-1}) \end{aligned}$$

$$\{1 - \alpha_2\} d_G(fx_{n+1}, fx_n) \preceq \{\alpha_1 + \alpha_3\} d_G(fx_n, fx_{n-1})$$

It further implies that

$$d_G(fx_{n+1}, fx_n) \preceq \mu d_G(fx_n, fx_{n-1})$$

Where $\mu = \frac{\alpha_1+\alpha_3}{1-\alpha_2} < 1$,
Consequently

$$d_G(fx_{n+1}, fx_n) \preceq \mu^n d_G(fx_n, fx_{n-1})$$

Now for all $m, n \in N$ with $m > n$, we have

$$\begin{aligned}
d_G(fx_m, fx_n) &\preceq d_G(fx_m, fx_{m-1}) + d_G(fx_{m-1}, fx_{m-2}) + ... + d_G(fx_{n+1}, fx_n) \\
&= (\mu^{m-1} + \mu^{m-2} + ... + \mu^n) d_G(fx_1, fx_0) \\
&\preceq (\sum_{n=0}^{\infty} \mu^n) \mu^m d_G(fx_1, fx_0) \\
&\preceq \frac{\mu^n}{e-\mu} d_G(fx_1, fx_0) \\
&\preceq (e-\mu)^{-1} \mu^m d_G(fx_1, fx_0)
\end{aligned}$$

Then by Lemma 2.8, one has

$$\|(e-\mu)^{-1}\mu^m d_G(fx_1, fx_0)\| \leq \|(e-\mu)^{-1}\|.\|\mu^m\|.\|d_G(fx_1, fx_0)\| \to 0(m \to \infty)$$

Which implies that

$$(e-\mu)^{-1}\mu^m d_G(fx_1, fx_0) \to 0(m \to \infty)$$

So by Lemma 2.9 for each $\theta \ll c$ there exists N_1 such that

$$d_G(fx_m, fx_n) \ll c$$

for all $n > N_1$.

Therefore fx_n is a Cauchy sequence. Since $f(X)$ or $g(X)$ is a complete subspace of X, then there exists $w \in f(X)$ such that $fx_n \to w$ and $gx_n \to w$ as $n \to \infty$. Hence we can find $v \in X$ such that $gv = w$. Note that

$$fx_{n-1} \to w(n \to \infty)$$

It is easy to see for every real $\epsilon > 0$ choose c_0 with $\theta \ll c_0$ and $\|\mu\|\|c_0\| < \epsilon$. Then there is N_2 such that $d_G(fx_{n-1}, w) \ll c_0$ for all $n > N_2$. Consequently,

$$d_G(fx_n, gv) = d_G(gx_{n-1}, gv) \preceq \mu d_G(fx_{n-1}, fv) = \mu d_G(fx_{n-1}, w) \preceq \mu c_0$$

That is to say

$$fx_n \to gv(n \to \infty)$$

Indeed, owing to

$$\|\mu c_0\| \leq \|\mu\|\|c_0\| < \epsilon \Rightarrow \mu c_0 \to \theta(n \to \infty)$$

Then by Lemma 2.9 for each $\theta \ll$ c there exists N_3 such that $d_G(fx_n, gv) \ll$ for all $n > N_3$. As a result

$$fx_n \to gv(n \to \infty)$$

By

$$fx_n \to w = fv(n \to \infty)$$

Then

$$fv = gv$$

In the following, we shall prove that f and g have a unique point of coincidence. Let's suppose, for the sake of absurdity, that there exists something.$v^* \neq v$ such that $fv^* = gv^*$. Then

$$d_G(fv^*, fv) = d_G(gv^*, gv) \preceq \mu d_G(fv^*, fv) \preceq .... \preceq \mu^m d_G(fv^*, fv)$$

Making use of Lemma 2.9 and the following fact

$$\|\mu^m d_G(fv^*, fv)\| \leq \|\mu^m\|.\|d_G(fx_1, fx_0)\| \to 0(n \to \infty),$$

We speculate

$$\mu^n d_G(fv^*, fv) \to \theta(n \to \infty)$$

That is

$$fv^* = fv.$$

Finally, if (f, g) is weakly compatible, then by using Lemma 2.17, we claim that f and g have a unique common fixed point. □

Theorem 3.2. *Let (X, G) be a complete symmetric G-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a cone in $\mathcal{A}$. Suppose that the mapping $f, g : X \to X$ satisfies one of the following contractive condition*

$$G(fx, fy, fy) \preceq a\{G(gx, fy, fy) + G(gy, fx, fx)\} \tag{3.6}$$

or

$$G(fx, fy, fy) \preceq a\{G(gx, gx, fy) + G(gy, gy, fx)\} \tag{3.7}$$

For all $x, y \in X$ and $a \in [0, \frac{1}{2})$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. If X is symmetric, then from (3.6), we have

$$G(fy, fx, fx) \preceq a\{G(gy, fx, fx) + G(gx, fy, fy)\} \tag{3.8}$$

Adding (3.6) and (3.8), we have

$$d_G(fx, fy) \preceq a\{d_G(gx, fy) + d_G(fx, gy)\}$$

Since $a < \frac{1}{2}$, the existence and uniqueness of a common fixed point follow. However, if X is not symmetric, then (3.6) gives no information about the maps, as in this case, the contractive constant need not be less than 1. In this case, let x_0 be an arbitrary point in X. Choose $\{x_n\}$ as in Theorem 3.1. Then

$$\begin{aligned}
G(gx_n, gx_{n+1}, gx_{n+1}) &= aG(fx_{n-1}, fx_n, fx_n) \\
&\preceq a\{G(gx_{n-1}, fx_n, fx_n) + G(fx_n, fx_{n-1}, fx_{n-1}) \\
&= aG(gx_{n-1}, gx_{n+1}, gx_{n+1}) \\
&\preceq aG(gx_{n-1}, gx_n, gx_n) + aG(gx_n, gx_{n+1}, gx_{n+1})
\end{aligned}$$

It gives,

$$G(gx_n, gx_{n+1}, gx_{n+1}) \preceq \mu G(gx_{n-1}, gx_n, gx_n)$$

Where $\mu = \frac{a}{1-a}$. Continuing the above process we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \preceq \mu^n G(gx_0, gx_1, gx_1)$$

Then, for $m > n$,

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\preceq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &+ G(gx_{n+2}, gx_{n+3}, gx_{n+3}) + ... + G(gx_{m-1}, gx_m, gx_m) \\ &\preceq (\mu^n + \mu^{n+1} + ... + \mu^{m-1}) G(gx_0, gx_1, gx_1) \\ &= (e-\mu)^{-1}\mu^m G(gx_0, gx_1, gx_1) \end{aligned}$$

Then by Lemma 2.9, one has

$$\|(e-\mu)^{-1}\mu^m G(gx_0, gx_1, gx_1)\| \leq \|(e-\mu)^{-1}\|.\|\mu^m\|.\|G(gx_0, gx_1, gx_1)\| \to 0 (m \to \infty)$$

Which implies that

$$(e-\mu)^{-1}\mu^m G(gx_0, gx_1, gx_1) \to 0 (m \to \infty)$$

So by Lemma 2.8 for each $\theta \ll c$ there exists N_1 such that

$$G(gx_n, gx_m, gx_m) \ll c$$

for all $n > N_1$.

Therefore gx_n is a Cauchy sequence. Since $f(X)$ or $g(X)$ is a complete subspace of X, then there exists $w \in f(X)$ such that $fx_n \to w$ and $gx_n \to w$ as $n \to \infty$. Hence we can find $v \in X$ such that $gv = w$. Note that

$$gx_{n-1} \to w (n \to \infty)$$

It is easy to see for every real $\epsilon > 0$ choose c_0 with $\theta \ll c_0$ and $\|\mu\|\|c_0\| < \epsilon$. Then there is N_2 such that $G(gx_{n-1}, gw, gw) \ll c_0$ for all $n > N_2$. Consequently,

$$G(fx_n, gv, gv) = G(gx_{n-1}, gv, gv) \preceq \mu G(fx_{n-1}, fv, fv) = \mu G(fx_{n-1}, w, w) \preceq \mu c_0$$

That is to say

$$fx_n \to gv (n \to \infty)$$

Indeed, owing to

$$\|\mu c_0\| \leq \|\mu\|\|c_0\| < \epsilon \Rightarrow \mu c_0 \to \theta (n \to \infty)$$

Then by Lemma 2.8 for each $\theta \ll$ c there exists N_3 such that $G(fx_n, gv, gv) \ll$ for all $n > N_3$. As a result

$$fx_n \to gv (n \to \infty)$$

By

$$fx_n \to w = fv (n \to \infty)$$

Then

$$fv = gv$$

In the following, we shall prove that f and g have a unique point of coincidence.

Let's suppose, for the sake of absurdity, that there exists something.$v^* \neq v$ such that $fv^* = gv^*$. Then

$$G(gv^*, gv, gv) = G(fv^*, fv, fv) \preceq \mu G(gv^*, gv, gv) \preceq .... \preceq \mu^m G(gv^*, gv, gv)$$

Making use of Lemma 2.9 and the following fact

$$\|\mu^m G(gv^*, gv, gv)\| \leq \|\mu^m\|.\|G(gx_0, gx_1, gx_1)\| \to 0(n \to \infty),$$

We speculate

$$\mu^n G(gv^*, gv, gv) \to \theta(n \to \infty)$$

That is

$$gv^* = gv.$$

Finally, if (f, g) is weakly compatible, then using Lemma 2.17, we claim that f and g have a unique common fixed point. □

Example 3.3. Let $X = [0, 1)$ and

$$G(u, v, w) = (|u - v| + |v - w| + |w - u|, a(|u - v| + |v - w| + |w - u|)),$$

for $a \geq 0$, be a G-cone metric on X. Define $f, g : X \to X$ by $f(u) = \frac{u}{8}$, and $g(u) = \frac{u}{2}$. Note that

$$G(gu, fv, fv) = \frac{1}{2}(|u - \frac{v}{4}|, a|u - \frac{v}{4}|)$$

And

$$G(gv, fu, fu) = \frac{1}{2}(|v - \frac{u}{4}|, a|v - \frac{u}{4}|)$$

Now

$$\begin{aligned} G(fx, fy, fy) &= \frac{1}{8}(|u - v|, a|u - v|) \\ &\preceq \frac{1}{8}(|u - \frac{v}{4}| + |v - \frac{u}{4}|, a(|u - \frac{v}{4}| + |v - \frac{u}{4}|)) \\ &= \frac{1}{4}(\frac{1}{2}|u - \frac{v}{4}| + \frac{1}{2}|v - \frac{u}{4}|, a(\frac{1}{2}|u - \frac{v}{4}| + \frac{1}{2}|v - \frac{u}{4}| \\ &= \frac{1}{4}\{G(gu, fv, fv) + G(gv, fu, fu)\} \end{aligned}$$

Therefore

$$G(fu, fv, fv) \preceq a\{G(gu, fv, fv) + G(gv, fu, fu)\},$$

$$f0 = g0 = 0$$

is satisfied for all $u, v \in X$, where $a = \frac{1}{4} < \frac{1}{2}$. Also, the range of g contains the range of f, $g(X)$ is a complete subset of X, and f and g are weakly compatible.

Therefore, f and g satisfy all conditions of Theorem 3.2. Here, 0 is a unique common fixed point of f and g.

4. CONCLUSION

We have proved common fixed point theorems for generalized contraction mappings in G-cone metric spaces with Banach algebra $\mathcal{A}$ which generalize several existing results in the literature.

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