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RESULTS OF SEMIGROUP OF LINEAR OPERATORS IN EXTRAPOLATION SPACES

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Abstract. Results of an omega-order preserving partial contraction mapping in generalized spaces are presented in this study. Assumed to be a closed linear operator on a Banach space X with a non-empty resolvent set in a semigroup of linear operators. If A is densely defined, the extrapolation spaces X^{-1} and X_{-1} will be associated with A in agreement. However, X_{-1} is a proper closed subspace of X^{-1} if A is not densely defined. Then, we demonstrated that the reason these spaces exist is because $(X^*)^{-1}$ and $(D(A_0))$ are naturally isomorphic to $(X^*)_{-1}$ and $(X^*)^{-1}$, respectively.

1. Introduction

According to Nagel *et al.* (see [7]), extrapolation spaces for strongly continuous semigroups of linear operators on Banach space have been created using a variety of techniques. They further stated that they typically show up as tools employed in a preliminary process to address the Cauchy problem on the original space. Assume X is a Banach space, $X_n \subseteq X$ is a finite set, $T(t)$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of ω-order preserving partial contraction mapping generating some results of semigroup of linear operators in extrapolation space.

In the field of spectrum theory, Akinyele *et al.* [1] discovered some semigroup of linear operator findings. A semigroup of operators had some asymptotic behavior, according to Batty *et al.* [2]. Balakrishnan [3] was able to create an operator calculus for semigroup's infinitesimal generators. Chill and Tomilov [4] came to some conclusions on a resolvent method for the stability operator semigroup. The spectra of linear operators were first presented by Davies [5]. Engel and Nagel [6], obtained one-parameter semigroup for linear evolution equations. Nagel *et al.* [7], identified extrapolation spaces for unbounded operators. Neerven [8], presented some results on adjoint of semigroup of linear operators. Both in [9] and [10], Omosowon *et al.* built a regular weak*-continuous semigroup of

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linear operators and presented quasilinear equations of evolution on the semigroup of linear operator, a differential operator is produced by partially contraction mapping in reverse.

Some spectral and asymptotic characteristics of the dominated operator were established by Räbiger and Wolf [11]. Additionally, in [12], Rauf *et al.* reported some results of stability and spectrum features on semigroup of linear operator. Akinyele *et al.* [13] Further Vrabie [14], proved some results of C_0-semigroup and its applications. Yosida [15], obtained some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [14]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (ω-OCP_n) [13]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (Extrapolation) [16]
Extrapolation is a type of estimation, beyond the original observation range, of the value of a variable on the basis of its relationship with another variable.
Definition 2.4 (closed linear operator) [15]
Let X, Y be two Banach spaces. A linear operator $A : D(A) \subseteq X \to Y$ is closed if for every sequence x_n in $D(A)$ converging to x in X such that $Ax_n \to y \in Y$ as $n \to \infty$, one has $x \in D(A)$ and $Ax = y$. Equivalently, A is closed if its graph is closed in the direct sum $X \oplus Y$.

Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3

3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Theorem 2.1 Hille-Yoshida [13]

A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed,
ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

3. Main Results

This section present results of semigroup of linear operators in extrapolation space generated by ω-OCP_n:

Theorem 3.1. *Suppose $A \in \omega - OCP_n$ is a generator of a semigroup of linear operator. Then,*

(i) *For each $X \in \rho(A), \lambda - A^{-1}$ is an isomorphism of X onto X^{-1}.*
(ii) *If $\lambda \in \rho(A)$, then $\lambda \in \rho(A^{-1})$ and $R(\lambda, A^{-1})(x, y) = (R(\lambda, A)xR(\lambda, A)y) = -AR(\lambda, A)x + R(\lambda, A)y$.*

Specifically, $R(\lambda, A) + R(\lambda, A^{-1})|_x$ and A is the part of A^{-1} in X. Moreover, there is a constant $C > 0$ independent of λ such that

$$\|R(\lambda, A^{-1})\|_{X^{-1}} \leq C\|R(\lambda, A)\|.$$

Proof. First we prove that A^{-1} maps X one-one onto X^{-1}. If

$$(\lambda - A^{-1})x = (\lambda - A^{-1})y, \tag{3.1}$$

then

$$(x, \lambda x) = (y, \lambda y) \tag{3.2}$$

which means that $x - y \in D(A)$ and

$$A(x-y) = \lambda(x-y). \tag{3.3}$$

But $\lambda - A$ is one-one and therefore

$$x = y. \tag{3.4}$$

For surjectieity, ley $(x, y) \in X \times X$, we can check that

$$(\lambda - A^{-1})(-AR(\lambda, A)x + R(\lambda, A)y) = (x, y). \tag{3.5}$$

It is clear that $\lambda - A^{-1}$ is bounded as map $X \to X^{-1}$. Therefore it is an isomorphism by the open mapping theorem. It then follows that each $\mu \in \rho(A)$ and $A \in \omega - OCP_n$, we have that

$$|(x, y)|_\mu := \|AR(\mu, A)x - R(\mu, A)y\|$$

defines and equivalent norm X^{-1}. In fact with respect to $|\cdot|_\mu$, $\mu - A^{-1} : X \to X^{-1}$ is isoperimetrical isomorphism.

To prove (ii), we have that the inverse $R(\lambda, A^{-1}) : X^{-} \to X$ of $\lambda - A^{-1}$ exists by (i) of Theorem 3.1 and is given by

$$\begin{aligned} R(\lambda, A^{-1})(x, y) &= -AR(\lambda, A)x + R(\lambda, A)y \\ &= (0, -AR(\lambda, A)x + R(\lambda, A)y) \\ &= (R(\lambda, -A)x, R(\lambda, A)y) \end{aligned} \tag{3.6}$$

particularly for $y \in X$ and $A \in \omega - OCP_n$, we have

$$R(\lambda, A)y = R(\lambda, A^{-1})(0, y) = R(\lambda, A)y. \tag{3.7}$$

To prove that A is the part of A^{-1} in X, let $x \in D(A^{-1}) = X$ with $A^{-1}x \in X$. Then also $(\lambda - A^{-1})_x \in X$ and consequently

$$x = R(\lambda, A^{-1})(\lambda - A^{-1})x = R(\lambda, A)(\lambda - A^{-1})x \tag{3.8}$$

This shows that $x \in D(A)$. Moreover, applying $\lambda - A$ to this identity gives

$$(\lambda - A)x = (\lambda - A)R(\lambda, A)(\lambda - A^{-1})x = (\lambda - A^{-1})x,$$

so

$$Ax = A^{-1}x. \tag{3.9}$$

Therefore, A is the part of A^{-1} in X. Fix any $\mu \in \rho(A)$ and $A \in \omega - OCP_n$. Then

$$\begin{aligned} |R(\lambda, A^{-1})(x, y)|_\mu &= \|AR(\mu, A)R(\lambda, A)x - R(\mu, A)R(\lambda, A)y\| \\ &\leq \|R(\lambda, A)\| \|AR(\mu, A)x - R(\mu, A)y\| \\ &= \|R(\lambda, A)\| |(x, y)|_\mu. \end{aligned} \tag{3.10}$$

Since $|\cdot|_\mu$ is an equivalent norm on X^{-1}, then the result follows. Hence the proof is complete.

□

Theorem 3.2. *Let A be closed linear operator and densely defined on X with $\lambda \in \rho(A)$. Then*

(i) $|x_1| = \|(\lambda - A)x_1\|$ *defines an equivalent norm on* X_1.

(ii) *ϕ defines an isomorphism from $(X^*)^{-1}$ onto $(X_1)^*$, which is independent of λ. Moreover, ϕ maps X^* onto $D((A_1)^*)$ and for all $x^* \in X^*$ and $A \in \omega - OCP_n$, we have*

$$\phi A^{*-1}x^* = (A_1)^* \phi x^*.$$

Proof. We have

$$|x_1| = \|(\lambda - A)x_1\| \leq \max(\lambda, 1)(\|x_1\| + \|Ax_1\|).$$

Also,

$$\begin{aligned}\|x_1\| + \|Ax_1\| &= \|R(\lambda, A)(\lambda - A)x_1\| + \|(\lambda R(\lambda, A) - I)(\lambda - A)x_1\| \\ &\leq ((1 + |\lambda|)\|R(\lambda, A)\| + 1)|x_1|,\end{aligned}$$

and this complete the proof of (i).

To prove (ii), we have that the independence of λ follows easily from the resolvent identity. We need to prove that ϕ is an isomorphism. Clearly, ϕ is one-one. By the open mapping theorem, it suffices to prove continuity and surjectivity of ϕ. For a continuity, we let C be the norm of $R(\lambda, A^{*-1})$ when regarded as an isomorphism $(X^*)^{-1} \to X^*$ and C' the constant for the equivalent norms of the in (i) of Theorem 3.2, we then have

$$|\phi(x^*, y^*)(x_1)| \leq C\|(x^*, y^*)\|\|(\lambda - A)x_1\| \leq CC'\|(x^*, y^*)\|\|x_1\|_{D(A)}.$$

For surjectivity, we have that if $x_1^* \in (X_1)$; then by (i) of Theorem 3.2, have

$$\langle z^*, (\lambda - A)x_1\rangle := \langle x_1^*, x_1\rangle$$

defines a boundary linear functional $z^* \in X^*$. In view of the identity

$$(\lambda - A^{*-1})(-A^*R(\lambda, A^*)x^* + R(\lambda, A^*)y^*) = (x^*, y^*), \tag{3.11}$$

for all $x_1 \in X_1$ we have

$$\begin{aligned}\phi(z^*, \lambda z^*)(x_1) &= \langle R(\lambda, A^{*-1})(z^*, \lambda^*), (\lambda - A)x_1\rangle \\ &= \langle -A^*R(\lambda, A^*)z^* + R(\lambda, A^*)\lambda z^*, (\lambda - A)x_1\rangle \\ &= \langle z^*, (\lambda - A)x_1\rangle = \langle x_1^*, x_1\rangle.\end{aligned} \tag{3.12}$$

Therefore, $x_1^* = \phi(z^*, \lambda z^*)$. This proves the first part of (ii). Next we show that ϕ maps X^* onto $D((A_1)^*)$.

First into: If $x^* \in X^*$, then for $x_1 \in D(A)$ and $A \in \omega - OCP_n$, we have

$$\begin{aligned}\langle \phi(0, x^*), A_1x_1\rangle &= \langle R(\lambda, A^{*-1})(0, x^*), (\lambda - A)Ax_1\rangle \\ &= \langle R(\lambda, A^*)x^*, (\lambda - A)Ax_1\rangle \\ &= \langle A^*R(\lambda, A^*)x^*, (\lambda - A)x_1\rangle \\ &= \langle R(\lambda, A^{*-1})(-x^*, 0), (\lambda - A)x_1\rangle \\ &= \langle \phi(-x^*, 0), x_1\rangle.\end{aligned} \tag{3.13}$$

This proves $(A_1)^*\phi x^* = \phi A^{*-1}x^*$. At the same time we have

$$|\langle \phi(0, x^*), A_1x_1\rangle| \leq \|\phi(-x^*, 0)\|\|x_1\|,$$

which by definition of $(A_1)^*$ implies that $\phi(0, x^*) \in D((A_1)^*)$.

Next onto: If $x_1^* \in D((A)^*)$ and $A \in \omega - OCP_n$, then letting z^* as above, in view of (i) in Theorem 3.2,

$$\begin{aligned} |\langle z^*, Ax_1\rangle| &= |\langle x_1^*, R(\lambda, A)Ax_1\rangle| \\ &= |\langle x_1^*, A_1 R(\lambda, A)x_1\rangle| \\ &\leq \|(A_1)^* x_1^*\|_{(X_1)^*} C' \|x_1\|. \end{aligned} \tag{3.14}$$

Therefore $z^* \in D(A^*)$. But then

$$x_1^* = \phi(z^*, \lambda z^*) = \phi(0, \lambda z^* - A^* z^*) \in \phi X^*.$$

Hence, the proof is completed.

□

Theorem 3.3. *Let $A \in \omega - OCP_n$ be linear closed operator and the generator of a C_0-semigroup on X with $\lambda \in \rho(A)$. Then:*

(i) *the isomorphism $\phi : (X^*)^{-1} \simeq (X_1)^*$ induces an isomorphism $(X^{\odot})^{-1} \simeq (X_1)^{\odot}$.*

(ii) *X is dense in X^{-1} if and only if A is densely defined.*

Proof. We have

$$(X_1)^{\odot} = \overline{D((A_1)^*)}^{(X_1)^*}.$$

Under the identification, ϕ, the space $D((A_1)^*)$ is just

$$D(A^{*-1}) = X^*.$$

Therefore, $(X_1)^{\odot}$ can be identifies with the closure of X^* in $(X^*)^{-1}$. But this closure is easily seen to be $(X^{\odot})^{-1}$ such that inclusion $\subset$ follows from

$$\begin{aligned} (0, x^*) &= (0, (\lambda - A^*)R(\lambda, A^*)x^*) \\ &= (R(\lambda, A^*)x^*, \lambda R(\lambda, A^*)x^*) \in (X^{\odot})^{-1}. \end{aligned} \tag{3.15}$$

The inclusion $\supset$ follows from

$$\begin{aligned} (x^{\odot}, y^{\odot}) &= \lim_{\lambda} (\lambda R(\lambda, A^*)x^{\odot}, \lambda R(\lambda, A^*)y^{\odot}) \\ &= \lim_{\lambda} (0, \lambda R(\lambda, A^*)y^{\odot} - A^* \lambda R(\lambda, A^*)x^{\odot}) \in \overline{X^*}^{(X^*)^{-1}}, \end{aligned}$$

which proves (i).

To prove (ii), we assume A is densely defined and let $(x, y) \in X^{-1}$ be arbitrary. Choose $z \in X$ such that

$$(\lambda - A^{-1})_z = (x, y).$$

Suppose there exists $z_n \subset D(A)$ such that $z_n \to z$ in X. Then by (i) of Theorem 3.1,

$$(\lambda - A)_{z_n} = (\lambda - A^{-1})_{z_n} \to (\lambda - A^{-1})_z = (x, y), \tag{3.16}$$

the convergence being in X^{-1}. Therefore X is dense X^{-1}.

Conversely, let X be dense in X^{-1}. Let $x \in X$, assume a sequence $(x_n) \subset X$ such that $x_n \to (\lambda - A^{-1})x$ in X^{-1}. Then, again by (i) of Theorem 3.1, we have

$$R(\lambda, A)x_n = R(\lambda, A^{-1})x_n \to R(\lambda, A^{-1})(\lambda - A^{-1})x = x,$$

and the convergence being in X. Therefore $D(A)$ is dense in X, and this achieved the proof.

□

Theorem 3.4. *Suppose $A \in \omega - OCP_n$ is a Hille-Yosida operator on X. Then:*

(i) *A_0 generates a C_0-semigroup $T_0(t)$ on X_0 and $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$;*
(ii) *X_0 is X_{-1}-dense in X and the map i_0 extends to an isomorphism $(X_0)_{-1} \simeq X_{-1}$;*
(iii) *under the identification $(X_0)_{-1} = X_{-1}$, we have $(A_0)_{-1} = A_{-1}$;*
(iv) *$T_0(t)$ extends to C_0-semigroup $T_{-1}(t)$ on X_{-1}, whose generator is A_{-1}.*

Proof. It is understandable that A_0 is a Hille-Yosida operator (of the same type) on X_0 again. We claim that A_0 is densely defined. This follows from

$$D(A_0) \ni R(\lambda, A)R(\mu, A)x \to R(\mu, A)x \text{ as } \lambda \to \infty$$

(applying the resolvent identity), showing that $D(A_0)$ is dense in the dense subspace $R(\lambda, A)X$ of X_0. The assertion concerning the resolvent is trivial, and this proves (i).

To prove (ii), let fix $x \in X$ arbitrary. Since $R(\mu, A)x \in X$, $A \in \omega - OCP_n$ and $D(A_0)$ is dense in X_0, then there is a sequence $(x_n) \subset X_0$ such that $R(\mu, A)x_n \to R(\mu, A)x$. We assert that $x_n \to x$ in X_{-1}. In fact,

$$\|x_n - x\|_{-1} = \|R(\mu, A)(x - x_n)\| \to 0. \tag{3.17}$$

This proves the denseness assertion. For any $x_0 \in X_0$ and $A \in \omega - OCP_n$, we have

$$\|x_0\|_{(x_0)_{-1}} = \|R(\mu, A)x_0\| = \|x_0\|_{X_{-1}}. \tag{3.18}$$

Thus, i_0 extends to an isomorphism $(X_0)_{-1} \to X_{-1}$.

To prove (iii), assume A has type (M, w). Let $A \in \omega - OCP_n$ be a closed operator with $\lambda \in \rho(A)$. Then $D(A_{-1}) = X_0$ and $\lambda - A_{-1} : X_0 \to X_{-1}$ is an isomorphism and A is the part of A_{-1} in X. Since $\lambda \in \rho(A)$, then $\lambda \in \rho(A_{-1})$ and

$$R(\lambda, A) = R(\lambda, A_{-1})|_X. \tag{3.19}$$

Then we have that $(w, \infty) \subset \rho(A_{-1})$ and

$$R(\lambda, A_{-1})|_X = R(\lambda, A) \text{ for } \lambda > w.$$

Applying this to A_0 shows that

$$R(\lambda, (A_0)_{-1})|_{X_0} = R(\lambda, A_0) = R(\lambda, A)|_{X_0}. \tag{3.20}$$

Since X_0 is dense in X_{-1} it follows that

$$R(\lambda, (A_0)_{-1}) = R(\lambda, A_{-1}). \tag{3.21}$$

Thus,

$$(A_0)_{-1} = A_{-1}, \tag{3.22}$$

and this proves (iii).

To prove (iv), first we show that A_{-1} is Hille-Yosida on X_{-1}. In fact, for any $x \in X$, $A \in \omega - OCP_n$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \|R(\lambda, A_{-1})^n x\|_{-1} &= \|R(\lambda, A)^n R(\mu, A)x\| \\ &\leq \frac{M}{(\lambda - w)^n} \|R(\mu, A)x\| \\ &= \frac{M}{(\lambda - w)^n} \|x\|_{-1}. \end{aligned} \tag{3.23}$$

Since $D(A_{-1}) = X_0$ is dense in X_{-1} it follows that A_{-1} is the generator for C_0-semigroup $T_{-1}(t)$ on X_{-1}. For $x_0 \in X_0$ and $A \in \omega - OCP_n$, we have by applying exponential formula to A_{-1}, we have

$$T_{-1}x_0 = \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, A_{-1}\right)\right)^n_{x_0} = \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, A_0\right)\right)^n_{x_0} = T_0(t). \tag{3.24}$$

We note that the convergence in both limits is with respect to the norm of X_{-1}. Bt since A_0 is also a generator, by applying the exponential formula to A_0, the latter limit is actually in the sense of X_0. Hence the prove is completed.

□

Theorem 3.5. *Assume $A \in \omega - OCP_n$ is the generator of a C_0-semigroup. Then:*

(i) $X^{\odot} = \left\{x^* \in X^* : \lim_{\lambda\to\infty} \|\lambda R(\lambda, A)^* x^* - x^*\| = 0\right\}$;

(ii) *the restriction map i_0 induces an isomorphism $X^{\odot} \simeq (X_0)^{\odot}$, under which we have $i_0^* T^{\odot}(t) = T_0^{\odot}(t) i_0^*$.*

Furthermore, for all $x \in X$, and $x^{\odot} \in X^{\odot}$, we have

$$\langle x^{\odot}, x\rangle = \lim_{n\to\infty} \langle i_0^* x^{\odot}, x_n\rangle, \tag{3.25}$$

where (x_n) is any bounded sequence in X_0 such that $R(\mu, A)x_n \to R(\mu, A)x$ in X, that is $x_n \to x$ in X_{-1}.

Proof. By the resolvent identity, it is clear that

$$\lim_{\lambda} R(\lambda, A)^* R(\mu, A)^* x^* = R(\mu, A)^* x^* \tag{3.26}$$

for all $x^* \in X^*$ and $A \in \omega - OCP_n$. By the uniform boundedness of $\|\lambda R(\lambda, A)^*\|$ near $\lambda = \infty$, the inclusion $\subset$ follows. The converse inclusion is trivial, and that complete the prove of (i).

To prove (ii), first let $x^* \in R(\lambda, A)^* X^*$ and $A \in \omega - OCP_n$, say $x^* = R(\lambda, A)^* y^*$. We assert that the restriction $i_0^* x^*$ belongs to $R(\lambda, A_0)^* X_0^*$. For arbitrary $x_0 \in X$ we have

$$\langle i_0^* x^*, x_0\rangle = \langle y^*, R(\lambda, A)x_0\rangle = \langle i_0^* y, R(\lambda, A_0)x_0\rangle = \langle R(\lambda, A_0)^* i_0^* y, x_0\rangle \tag{3.27}$$

and therefore,

$$i_0^* x^* = R(\lambda, A_0)^* i^* i_0^* y^* \in R(\lambda, A_0)^* X_0^*,$$

providing the claim. Next since the closure of $M \cdot R(\lambda, A)Bx_0$ contains $R(\lambda, A)Bx$,

$$\begin{aligned} \|i_0^* x^*\|_{X_0^*} \leq \|x^*\|_{X^*} &= \sup_{\|x\|_{-1}} |\langle y^*, R(\lambda, A)x\rangle| \\ &\leq M \sup |\langle y^*, R(\lambda, A)x_0| \\ &= M\|R(\lambda, A_0)^* i_0^* y^*\|_{X_0^*} \\ &= M\|i_0^* x_0^*\|_{X_0^*}. \end{aligned} \tag{3.28}$$

Therefore i_0^* maps the space $(R(\lambda, A)^* X^*, \|\cdot\|_{X^*})$ isomorphically into the space $(R(\lambda, A_0)^* X^*, \|\cdot\|_{X_0^*})$. It is an easy consequence of the Hahn-Banach theorem that this map is actually onto. Thus the first assertion of the theorem follows by taking closures.

In order to prove the formula for $\langle x^{\odot}, x\rangle$, we choose $(x_n^*) \subset X^*$ such that $R(\mu, A)^* x_n^* \to x^{\odot}$. Since

$$\lim_n \lim_m \sup |\langle i_0^* R(\mu, A)^* x_n^* - i_0^* x, x_n\rangle| = 0 \tag{3.29}$$

we have

$$\begin{aligned} \langle x^{\odot}, x\rangle &= \lim_n \langle R(\mu, A)^* x_n^*, x\rangle \\ &= \lim_n \lim_m \langle x_n^*, R(\mu, A)x_m\rangle \\ &= \lim_n \lim_m \langle i_0^* R(\mu, A)^* x_n^*, x_m\rangle \\ &= \lim_m \langle i_0^* x^{\odot}, x_m\rangle. \end{aligned} \tag{3.30}$$

The relation between $T^{\odot}(t)$ and $T_0^{\odot}(t)$ is proved as follows. First, by definition of $A^{\odot}$ and by what we have proved so far,

$$\begin{aligned} \langle R(\lambda, A^{\odot})x^{\odot}, x\rangle &= \langle R(\lambda, A)^* x^{\odot}, x\rangle = \langle x^{\odot}, R(\lambda, A)x\rangle \\ &= \langle i_0^* x^{\odot}, R(\lambda, A)x\rangle = \lim_{\mu} \langle i_0^* x^{\odot}, R(\lambda, A)\mu R(\mu, A)x\rangle \\ &= \lim_{\mu} \langle R(\lambda, A_0)^* i_0^* x^{\odot}, \mu R(\mu, A)x\rangle \\ &= \langle kR(\lambda, A_0)^* i_0^* x^{\odot}, x\rangle. \end{aligned} \tag{3.31}$$

where $k : X_0^{\odot} \to X^{\odot}$ is the inverse of $i_0^*|_{X^{\odot}}$. This shows that

$$R(\lambda, A^{\odot}) = kR(\lambda, A_0)^* i_0^*.$$

Thus,

$$T^{\odot}(t) = kT_0^*(t) i_0^*$$

by the exponential formula. Hence the proof is completed.

□

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generate some results of semigroup of linear operators in extrapolation space.

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References

1. A. Y. Akinyele, O. Y. Saka-Balogun, and M. A. Ganiyu, *Results of semigroup of linear operator in spectra theory,* Asia Pac. J. Math. **8** (2021). 1-8. https://doi.org/10.28924/APJM/8-2
2. C. J. K. Batty, R. Chill, and Y. Tomilov, *Strong stability of bounded evolution families and semigroup,* Journal of Functional Analysis. **193** (2002), 116 - 139. https://doi.org/10.1006/jfan.2001.3917
3. A. V. Balakrishnan, *An Operator calculus for infinitesimal generators of semigroup,* Trans Amer. Math. Soc. **91** (1959), 330 - 353. https://doi.org/10.1090/S0002-9947-1959-0107179-0
4. R. Chill, and Y. Tomilov, *Stability operator semigroup* Banach Center Publication 75, Polish Academy of Sciences, Warsaw, (2007), 71-73. https://doi.org/10.4064/bc75-0-6.
5. E. B. Davies, *Linear operator and their spectra,* Cambridge Studies in Advanced Mathematics, Cambridge University Press, New York. (2007). https://doi.org/10.1017/CBO9780511618864
6. K. Engel, and R. Nagel, *One-parameter semigroups for linear equations,* Graduate Texts in Mathematics, 194, Springer, New York, (2000). https://doi.org/10.1007/s002330010042
7. R. Nagel, G. Nickel, and S. Romanelli, *Identification of extrapolation spaces for unbounded operators*, Quaestiones Mathematicae (1996), 1 - 14. https://doi.org/10.1080/16073606.1996.9631827
8. J. Neerven, *The adjoint of semigroup of linear operators,* Graduate Texts in Mathematics, 194, Springer-Verlag, Berlin Heidelberg, (1992). https://doi.org/10.1007/BFb0085008
9. J. B. Omosowon, A. Y. Akinyele, K. A. Bello, and B. M. Ahmed, *Results of semigroup of linear operators generating a regular weak*-continuous semigroup*, Earthline Journal of Mathematical Sciences **10(2)** (2022), 289 - 304. https://doi.org/10.34198/ejms.10222.289304
10. J. B. Omosowon, A. Y. Akinyele, K. A. Bello, and B. M. Ahmed, *Results of semigroup of linear operator generating a quasilinear equations of evolution*, Earthline Journal of Mathematical Sciences **10(2)** (2022), 409 - 421. https://doi.org/10.34198/ejms.10222.409421
11. F. Rabiger, and M. P. H. Wolf, *Spectral and asymptotic properties of resolvent dominated-operators*, Journal of the Australian Mathematical Society Series A **68**(2000), 18 - 201. https://doi.org/10.1017/S144678870000029X
12. K. Rauf, A. Y. Akinyele, M. O. Etuk, R. O. Zubair, and M. A. Aasa *Some result of stability and spectra properties on semigroup of linear operator*, Advances in Pure Mathematics **9** (2019), 43 - 51. https://doi.org/10.4236/apm.2019.91003
13. Akinyele, A. Y., Abubakar, J. U., Bello, K. A., Alhassan, L. K., & Aasa, M. A. (2021). Results of ω-order reversing partial contraction mapping generating a differential operator. Malaya Journal of Matematik, 9(03), 91-98.
14. I. I. Vrabie, *C_0-semigroup and application*, Mathematics Studies, 191, Elsevier, North-Holland, (2003). https://doi.org/10.4064/-19-1-351-361
15. K. Yosida,, *On the differentiability and representation of one-parameter semigroups of linear operators*, J. Math. Soc. **1** (1948), 15 - 21. https://doi.org/10.2969/JMSJ/00110015
16. https://en.wikipedia.org/wiki/Extrapolation.

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