RESULTS OF SEMIGROUP OF LINEAR OPERATORS IN EXTRAPOLATION SPACES

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ABSTRACT. Results of an omega-order preserving partial contraction mapping in generalized spaces are presented in this study. Assumed to be a closed linear operator on a Banach space $X$ with a non-empty resolvent set in a semigroup of linear operators. If $A$ is densely defined, the extrapolation spaces $X^{-1}$ and $X^{-1}$ will be associated with $A$ in agreement. However, $X^{-1}$ is a proper closed subspace of $X^{-1}$ if $A$ is not densely defined. Then, we demonstrated that the reason these spaces exist is because $(X^*)^{-1}$ and $(D(A_0))$ are naturally isomorphic to $(X^*)_{-1}$ and $(X^*)^{-1}$, respectively.

1. INTRODUCTION

According to Nagel et al. (see [7]), extrapolation spaces for strongly continuous semigroups of linear operators on Banach space have been created using a variety of techniques. They further stated that they typically show up as tools employed in a preliminary process to address the Cauchy problem on the original space. Assume $X$ is a Banach space, $X_n \subseteq X$ is a finite set, $T(t)$ the $C_0$-semigroup, $\omega - OCP_n$ the $\omega$-order preserving partial contraction mapping, $M_m$ be a matrix, $L(X)$ be a bounded linear operator on $X$, $P_n$ a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of $A$ and $A \in \omega - OCP_n$ is a generator of $C_0$-semigroup. This paper consist of results of $\omega$-order preserving partial contraction mapping generating some results of semigroup of linear operators in extrapolation space.

In the field of spectrum theory, Akinyele et al. [1] discovered some semigroup of linear operator findings. A semigroup of operators had some asymptotic behavior, according to Batty et al. [2]. Balakrishnan [3] was able to create an operator calculus for semigroup’s infinitesimal generators. Chill and Tomilov [4] came to some conclusions on a resolvent method for the stability operator semigroup. The spectra of linear operators were first presented by Davies [5]. Engel and Nagel [6], obtained one-parameter semigroup for linear evolution equations. Nagel et al. [7], identified extrapolation spaces for unbounded operators. Neerven [8], presented some results on adjoint of semigroup of linear operators. Both in [9] and [10], Omosowon et al. built a regular weak*-continuous semigroup of
linear operators and presented quasilinear equations of evolution on the semigroup of linear operator, a differential operator is produced by partially contraction mapping in reverse.

Some spectral and asymptotic characteristics of the dominated operator were established by Räbiger and Wolf [11]. Additionally, in [12], Rauf et al. reported some results of stability and spectrum features on semigroup of linear operator. Akinyele et al. [13] further Vrabie [14], proved some results of $C_0$-semigroup and its applications. Yosida [15], obtained some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

**Definition 2.1 (C₀-Semigroup) [14]**
A $C_0$-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

**Definition 2.2 (ω-OCPₙ) [13]**
A transformation $\alpha \in P_n$ is called $\omega$-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t + s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

**Definition 2.3 (Extrapolation ) [16]**
Extrapolation is a type of estimation, beyond the original observation range, of the value of a variable on the basis of its relationship with another variable.

**Definition 2.4 (closed linear operator) [15]**
Let $X, Y$ be two Banach spaces. A linear operator $A : D(A) \subseteq X \to Y$ is closed if for every sequence $x_n$ in $D(A)$ converging to $x$ in $X$ such that $Ax_n \to y \in Y$ as $n \to \infty$, one has $x \in D(A)$ and $Ax = y$. Equivalently, $A$ is closed if its graph is closed in the direct sum $X \oplus Y$.

**Example 1**
$2 \times 2$ matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^t \\ e^t & e^{2t} \end{pmatrix}.$$ 

**Example 2**
$3 \times 3$ matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$
and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^t & e^{2t} & e^{2t} \end{pmatrix}.$$

**Example 3**

$3 \times 3$ matrix $[M_m(C)]$, we have

for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on $X$.

Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA \lambda}$, then

$$e^{tA \lambda} = \begin{pmatrix} e^{2t \lambda} & e^{2t \lambda} & e^{3t \lambda} \\ e^{2t \lambda} & e^{2t \lambda} & e^{2t \lambda} \\ e^{t \lambda} & e^{2t \lambda} & e^{2t \lambda} \end{pmatrix}.$$

**Theorem 2.1** Hille-Yoshida [13]

A linear operator $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator for a $C_0$-semigroup of contraction if and only if

i. $A$ is densely defined and closed,

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}.$$  \hspace{1cm} (2.1)

3. **Main Results**

This section present results of semigroup of linear operators in extrapolation space generated by $\omega-OCP_n$:

**Theorem 3.1.** Suppose $A \in \omega - OCP_n$ is a generator of a semigroup of linear operator. Then,

(i) For each $X \in \rho(A), \lambda - A^{-1}$ is an isomorphism of $X$ onto $X^{-1}$.

(ii) If $\lambda \in \rho(A)$, then $\lambda \in \rho(A^{-1})$ and $R(\lambda, A^{-1})(x, y) = (R(\lambda, A)x R(\lambda, A)y) = -AR(\lambda, A)x + R(\lambda, A)y$.

Specifically, $R(\lambda, A) + R(\lambda, A^{-1})|_x$ and $A$ is the part of $A^{-1}$ in $X$. Moreover, there is a constant $C > 0$ independent of $\lambda$ such that

$$\|R(\lambda, A^{-1})\|_{X^{-1}} \leq C\|R(\lambda, A)\|.$$  \hspace{1cm} (3.1)

**Proof.** First we prove that $A^{-1}$ maps $X$ one-one onto $X^{-1}$. If

$$(\lambda - A^{-1})x = (\lambda - A^{-1})y,$$  \hspace{1cm} (3.1)

then

$$(x, \lambda x) = (y, \lambda y).$$  \hspace{1cm} (3.2)
which means that \( x - y \in D(A) \) and
\[
A(x - y) = \lambda(x - y).
\]
(3.3)

But \( \lambda - A \) is one-one and therefore
\[
x = y.
\]
(3.4)

For surjectivity, let \( (x, y) \in X \times X \), we can check that
\[
(\lambda - A^{-1})(-AR(\lambda, A)x + R(\lambda, A)y) = (x, y).
\]
(3.5)

It is clear that \( \lambda - A^{-1} \) is bounded as map \( X \to X^{-1} \). Therefore it is an isomorphism by the open mapping theorem. It then follows that each \( \mu \in \rho(A) \) and \( A \in \omega - OCP_n \), we have that
\[
\|(x, y)\|_\mu := \|AR(\mu, A)x - R(\mu, A)y\|
\]
defines and equivalent norm \( X^{-1} \). In fact with respect to \( \cdot \|_\mu, \mu - A^{-1} : X \to X^{-1} \) is isoperimetrical isomorphism.

To prove (ii), we have that the inverse \( R(\lambda, A^{-1}) : X^{-1} \to X \) of \( \lambda - A^{-1} \) exists by (i) of Theorem 3.1 and is given by
\[
R(\lambda, A^{-1})(x, y) = -AR(\lambda, A)x + R(\lambda, A)y
= (0, -AR(\lambda, A)x + R(\lambda, A)y)
= (R(\lambda, -A)x, R(\lambda, A)y)
\]
(3.6)

particularly for \( y \in X \) and \( A \in \omega - OCP_n \), we have
\[
R(\lambda, A)y = R(\lambda, A^{-1})(0, y) = R(\lambda, A)y.
\]
(3.7)

To prove that \( A \) is the part of \( A^{-1} \) in \( X \), let \( x \in D(A^{-1}) = X \) with \( A^{-1}x \in X \). Then also \( (\lambda - A^{-1})x \in X \) and consequently
\[
x = R(\lambda, A^{-1})(\lambda - A^{-1})x = R(\lambda, A)(\lambda - A^{-1})x
\]
(3.8)

This shows that \( x \in D(A) \). Moreover, applying \( \lambda - A \) to this identity gives
\[
(\lambda - A)x = (\lambda - A)R(\lambda, A)(\lambda - A^{-1})x = (\lambda - A^{-1})x,
\]
so
\[
Ax = A^{-1}x.
\]
(3.9)

Therefore, \( A \) is the part of \( A^{-1} \) in \( X \). Fix any \( \mu \in \rho(A) \) and \( A \in \omega - OCP_n \). Then
\[
|R(\lambda, A^{-1})(x, y)|_\mu = \|AR(\mu, A)R(\lambda, A)x - R(\mu, A)R(\lambda, A)y\|
\leq \|R(\lambda, A)|||AR(\mu, A)x - R(\mu, A)y||
= \|R(\lambda, A)|||(x, y)|_\mu.
\]
(3.10)

Since \( \cdot \|_\mu \) is an equivalent norm on \( X^{-1} \), then the result follows. Hence the proof is complete.

\[\square\]

**Theorem 3.2.** Let \( A \) be closed linear operator and densely defined on \( X \) with \( \lambda \in \rho(A) \). Then

(i) \( |x_1| = \|(\lambda - A)x_1\| \) defines an equivalent norm on \( X_1 \).
Proof. We have
\[ |x_1| = \|(\lambda - A)x_1\| \leq \max(\lambda, 1)(\|x_1\| + \|Ax_1\|). \]
Also,
\[ \|x_1\| + \|Ax_1\| = \|R(\lambda, A)(\lambda - A)x_1\| + \|\lambda R(\lambda, A) - I\| (\lambda - A)x_1\| \leq ((1 + |\lambda|)\|R(\lambda, A)\| + 1)|x_1|, \]
and this complete the proof of (i).

To prove (ii), we have that the independence of $\lambda$ follows easily from the resolvent identity. We need to prove that $\phi$ is an isomorphism. Clearly, $\phi$ is one-one. By the open mapping theorem, it suffices to prove continuity and surjectivity of $\phi$. For a continuity, we let $C$ be the norm of $R(\lambda, A^{*-1})$ when regarded as an isomorphism $(X^*)^{-1} \rightarrow X^*$ and $C^*$ the constant for the equivalent norms of the in (i) of Theorem 3.2, we then have
\[ |\phi(x^*, y^*)(x_1)| \leq C\|(x^*, y^*)\||(\lambda - A)x_1\| \leq CC^*\|(x^*, y^*)\|\|x_1\|_{D(A)}. \]
For surjectivity, we have that if $x_1^* \in (X_1)$; then by (i) of Theorem 3.2, have
\[ \langle z^*, (\lambda - A)x_1 \rangle := \langle x_1^*, x_1 \rangle \]
defines a boundary linear functional $z^* \in X^*$. In view of the identity
\[ (\lambda - A^{*-1})(-A^*R(\lambda, A^*)x^* + R(\lambda, A^*)y^*) = (x^*, y^*), \]
for all $x_1 \in X_1$ we have
\[ \phi(z^*, \lambda z^*)(x_1) = \langle R(\lambda, A^{*-1})(z^*, \lambda^*), (\lambda - A)x_1 \rangle \]
\[ = \langle -A^*R(\lambda, A^*)z^* + R(\lambda, A^*)\lambda z^*, (\lambda - A)x_1 \rangle \]
\[ = \langle z^*, (\lambda - A)x_1 \rangle = \langle x_1^*, x_1 \rangle. \]
Therefore, $x_1^* = \phi(z^*, \lambda z^*)$. This proves the first part of (ii). Next we show that $\phi$ maps $X^*$ onto $D((A_1)^*)$.

**First into:** If $x^* \in X^*$, then for $x_1 \in D(A)$ and $A \in \omega - OCP_n$, we have
\[ \langle \phi(0, x^*), A_1 x_1 \rangle = \langle R(\lambda, A^{*-1})(0, x^*), (\lambda - A)Ax_1 \rangle \]
\[ = \langle R(\lambda, A^*)x^*, (\lambda - A)Ax_1 \rangle \]
\[ = \langle A^*R(\lambda, A^*)x^*, (\lambda - A)x_1 \rangle \]
\[ = \langle R(\lambda, A^{*-1})(-x^*, 0), (\lambda - A)x_1 \rangle \]
\[ = \langle \phi(-x^*, 0), x_1 \rangle. \]
This proves $(A_1)^*\phi x^* = \phi A^{*-1}x^*$. At the same time we have
\[ |\langle \phi(0, x^*), A_1 x_1 \rangle| \leq \|\phi(-x^*, 0)\|\|x_1\|, \]
which by definition of $(A_1)^*$ implies that $\phi(0, x^*) \in D((A_1)^*)$. 

Next onto: If $x^*_1 \in D((A)^*)$ and $A \in \omega - OCP_n$, then letting $z^*$ as above, in view of (i) in Theorem 3.2,
\[
|\langle z^*, Ax_1 \rangle| = |\langle x^*_1, R(\lambda, A)x_1 \rangle| \\
= |\langle x^*_1, A_1 R(\lambda, A)x_1 \rangle| \\
\leq \|(A_1)^* x^*_1 \|_{(X_1), C''} \|x_1\|. \quad (3.14)
\]
Therefore $z^* \in D(A^*)$. But then
\[
x^*_1 = \phi(z^*, \lambda z^*) = \phi(0, \lambda z^* - A^* z^*) \in \phi X^*.
\]
Hence, the proof is completed.
\[\square\]

**Theorem 3.3.** Let $A \in \omega - OCP_n$ be linear closed operator and the generator of a $C_0$-semigroup on $X$ with $\lambda \in \rho(A)$. Then:

(i) the isomorphism $\phi : (X^*)^{-1} \simeq (X_1)^\circ$ induces an isomorphism $(X^\circ)^{-1} \simeq (X_1)^\circ$.

(ii) $X$ is dense in $X^{-1}$ if and only if $A$ is densely defined.

**Proof.** We have
\[
(X_1)^\circ = \overline{D((A_1)^*)}^{(X_1)^*}.
\]
Under the identification, $\phi$, the space $D((A_1)^*)$ is just
\[
D(A^{*-1}) = X^*.
\]
Therefore, $(X_1)^\circ$ can be identifies with the closure of $X^*$ in $(X^*)^{-1}$. But this closure is easily seen to be $(X^\circ)^{-1}$ such that inclusion $\supset$ follows from
\[
(0, x^*) = (0, (\lambda - A^*) R(\lambda, A^*) x^*) \\
= (R(\lambda, A^*) x^*, \lambda R(\lambda, A^*) x^*) \in (X^\circ)^{-1}. \quad (3.15)
\]
The inclusion $\supset$ follows from
\[
(x^\circ, y^\circ) = \lim_{\lambda} (\lambda R(\lambda, A^*) x^\circ, \lambda R(\lambda, A^*) y^\circ) \\
= \lim_{\lambda} (0, \lambda R(\lambda, A^*) y^\circ - A^* \lambda R(\lambda, A^*) x^\circ) \in \overline{X^*}^{(X^*)^{-1}},
\]
which proves (i).

To prove (ii), we assume $A$ is densely defined and let $(x, y) \in X^{-1}$ be arbitrary. Choose $z \in X$ such that
\[
(\lambda - A^{-1}) z = (x, y).
\]
Suppose there exists $z_n \subset D(A)$ such that $z_n \to z$ in $X$. Then by (i) of Theorem 3.1,
\[
(\lambda - A) z_n = (\lambda - A^{-1}) z_n \to (\lambda - A^{-1}) z = (x, y), \quad (3.16)
\]
the convergence being in $X^{-1}$. Therefore $X$ is dense $X^{-1}$.

Conversely, let $X$ be dense in $X^{-1}$. Let $x \in X$, assume a sequence $(x_n) \subset X$ such that $x_n \to (\lambda - A^{-1}) x$ in $X^{-1}$. Then, again by (i) of Theorem 3.1, we have
\[
R(\lambda, A) x_n = R(\lambda, A^{-1}) x_n \to R(\lambda, A^{-1}) (\lambda - A^{-1}) x = x,
\]
and the convergence being in \( X \). Therefore \( D(A) \) is dense in \( X \), and this achieved the proof.

\[ \square \]

**Theorem 3.4.** Suppose \( A \in \omega - OCP_n \) is a Hille-Yosida operator on \( X \). Then:

i) \( A_0 \) generates a \( C_0 \)-semigroup \( T_0(t) \) on \( X_0 \) and \( R(\lambda, A_0) = R(\lambda, A)|_{X_0} \);

ii) \( X_0 \) is \( X_{-1} \)-dense in \( X \) and the map \( i_0 \) extends to an isomorphism \( (X_0)_{-1} \cong X_{-1} \);

iii) under the identification \( (X_0)_{-1} = X_{-1} \), we have \( (A_0)_{-1} = A_{-1} \);

iv) \( T_0(t) \) extends to \( C_0 \)-semigroup \( T_{-1}(t) \) on \( X_{-1} \), whose generator is \( A_{-1} \).

**Proof.** It is understandable that \( A_0 \) is a Hille-Yosida operator (of the same type) on \( X_0 \) again. We claim that \( A_0 \) is densely defined. This follows from

\[ D(A_0) \ni R(\lambda, A)R(\mu, A)x \to R(\mu, A)x \text{ as } \lambda \to \infty \]

(applying the resolvent identity), showing that \( D(A_0) \) is dense in the dense subspace \( R(\lambda, A)X \) of \( X_0 \). The assertion concerning the resolvent is trivial, and this proves (i).

To prove (ii), let fix \( x \in X \) arbitrary. Since \( R(\mu, A)x \in X, A \in \omega - OCP_n \) and \( D(A_0) \) is dense in \( X_0 \), then there is a sequence \( (x_n) \subset X_0 \) such that \( R(\mu, A)x_n \to R(\mu, A)x \). We assert that \( x_n \to x \) in \( X_{-1} \). In fact,

\[ \|x_n - x\|_{-1} = \|R(\mu, A)(x - x_n)\| \to 0. \] (3.17)

This proves the denseness assertion. For any \( x_0 \in X_0 \) and \( A \in \omega - OCP_n \), we have

\[ \|x_0\|_{(x_0)_{-1}} = \|R(\mu, A)x_0\| = \|x_0\|_{X_{-1}}. \] (3.18)

Thus, \( i_0 \) extends to an isomorphism \( (X_0)_{-1} \to X_{-1} \).

To prove (iii), assume \( A \) has type \((M, w)\). Let \( A \in \omega - OCP_n \) be a closed operator with \( \lambda \in \rho(A) \). Then \( D(A_{-1}) = X_0 \) and \( \lambda - A_{-1} : X_0 \to X_{-1} \) is an isomorphism and \( A \) is the part of \( A_{-1} \) in \( X \). Since \( \lambda \in \rho(A) \), then \( \lambda \in \rho(A_{-1}) \) and

\[ R(\lambda, A) = R(\lambda, A_{-1})|_X. \] (3.19)

Then we have that \((w, \infty) \subset \rho(A_{-1})\) and

\[ R(\lambda, A_{-1})|_X = R(\lambda, A) \text{ for } \lambda > w. \]

Applying this to \( A_0 \) shows that

\[ R(\lambda, (A_0)_{-1})|_{X_0} = R(\lambda, A_0) = R(\lambda, A)|_{X_0}. \] (3.20)

Since \( X_0 \) is dense in \( X_{-1} \) it follows that

\[ R(\lambda, (A_0)_{-1}) = R(\lambda, A_{-1}). \] (3.21)

Thus,

\[ (A_0)_{-1} = A_{-1}. \] (3.22)

and this proves (iii).
To prove (iv), first we show that $A_{-1}$ is Hille-Yosida on $X_{-1}$. In fact, for any $x \in X$, $A \in \omega - \text{OCP}_n$ and $n \in \mathbb{N}$, we have

$$
\|R(\lambda, A_{-1})^n x\|_{-1} = \|R(\lambda, A)^n R(\mu, A)x\|
\leq \frac{M}{(\lambda - w)^n} \|R(\mu, A)x\|
= \frac{M}{(\lambda - w)^n} \|x\|_{-1}.
$$

(3.23)

Since $D(A_{-1}) = X_0$ is dense in $X_{-1}$ it follows that $A_{-1}$ is the generator for $C_0$-semigroup $T_{-1}(t)$ on $X_{-1}$. For $x_0 \in X_0$ and $A \in \omega - \text{OCP}_n$, we have by applying exponential formula to $A_{-1}$, we have

$$
T_{-1}x_0 = \lim_{n \to \infty} \left( \frac{n}{t} R \left( \frac{n}{t}, A_{-1} \right) \right)^n x_0 = \lim_{n \to \infty} \left( \frac{n}{t} R \left( \frac{n}{t}, A_0 \right) \right)^n x_0 = T_0(t).
$$

(3.24)

We note that the convergence in both limits is with respect to the norm of $X_{-1}$. Bt since $A_0$ is also a generator, by applying the exponential formula to $A_0$, the latter limit is actually in the sense of $X_0$. Hence the prove is completed.

□

**Theorem 3.5.** Assume $A \in \omega - \text{OCP}_n$ is the generator of a $C_0$-semigroup. Then:

(i) $X^\circ = \left\{ x^* \in X^* : \lim_{\lambda \to \infty} \|\lambda R(\lambda, A)^* x^* - x^*\| = 0 \right\}$;

(ii) the restriction map $i_0$ induces an isomorphism $X^\circ \simeq (X_0)^\circ$, under which we have $i_0^* T(\circ)(t) = T_0^0(t) i_0^*$.

Furthermore, for all $x \in X$, and $x^\circ \in X^\circ$, we have

$$
\langle x^\circ, x \rangle = \lim_{n \to \infty} \langle i_0^* x^\circ, x_n \rangle,
$$

(3.25)

where $(x_n)$ is any bounded sequence in $X_0$ such that $R(\mu, A)x_n \to R(\mu, A)x$ in $X$, that is $x_n \to x$ in $X_{-1}$.

**Proof.** By the resolvent identity, it is clear that

$$
\lim_{\lambda} R(\lambda, A)^* R(\mu, A)^* x^* = R(\mu, A)^* x^*
$$

for all $x^* \in X^*$ and $A \in \omega - \text{OCP}_n$. By the uniform boundedness of $\|\lambda R(\lambda, A)^*\|$ near $\lambda = \infty$, the inclusion $\subset$ follows. The converse inclusion is trivial, and that complete the prove of (i).

To prove (ii), first let $x^* \in R(\lambda, A)^* X^*$ and $A \in \omega - \text{OCP}_n$, say $x^* = R(\lambda, A)^* y^*$. We assert that the restriction $i_0^* x^*$ belongs to $R(\lambda, A_0)^* X_0^*$. For arbitrary $x_0 \in X$ we have

$$
\langle i_0^* x^*, x_0 \rangle = \langle y^*, R(\lambda, A)x_0 \rangle = \langle i_0^* y, R(\lambda, A_0)x_0 \rangle = \langle R(\lambda, A_0)^* i_0^* y, x_0 \rangle
$$

(3.27)

and therefore,

$$
i_0^* x^* = R(\lambda, A_0)^* i_0^* y^* \in R(\lambda, A_0)^* X_0^*.
$$
providing the claim. Next since the closure of $M \cdot R(\lambda, A)Bx_0$ contains $R(\lambda, A)Bx,
\|i_0^*x\|_{X_0^*} \leq \|x\|_{X^*} = \sup_{\|x\|} |\langle y^*, R(\lambda, A)x \rangle|
\leq M \sup |\langle y^*, R(\lambda, A)x_0 \rangle|
= M \|R(\lambda, A_0)\|_{X_0^*}
= M \|i_0^*x\|_{X_0^*}.
(3.28)
Therefore $i_0^*$ maps the space $(R(\lambda, A)X^*, \| \cdot \|_{X^*})$ isomorphically into the space
$(R(\lambda, A_0)X^*, \| \cdot \|_{X_0^*})$. It is an easy consequence of the Hahn-Banach theorem
that this map is actually onto. Thus the first assertion of the theorem follows by
taking closures.

In order to prove the formula for $\langle x^\odot, x \rangle$, we choose $(x_n^*) \subset X^*$ such that
$R(\mu, A)\mu^x_n \rightarrow x^\odot$. Since
$$\lim_n \lim_m |\langle i_0^*R(\mu, A)x_n^*, x_n \rangle - \langle i_0^*x_n^*, x_n \rangle| = 0 \quad (3.29)$$
we have
$$\langle x^\odot, x \rangle = \lim_n \langle R(\mu, A)x_n^*, x_n \rangle
= \lim_n \lim_m \langle x_n^*, R(\mu, A)x_m \rangle
= \lim_n \lim_m \langle i_0^*R(\mu, A)^x_n^*, x_m \rangle
= \lim_m \langle i_0^*x^\odot, x_m \rangle. \quad (3.30)$$
The relation between $T^\odot(t)$ and $T_0^\odot(t)$ is proved as follows. First, by definition
of $A^\odot$ and by what we have proved so far,
$$\langle R(\lambda, A^\odot)x^\odot, x \rangle = \langle R(\lambda, A)^x^\odot, x \rangle = \langle x^\odot, R(\lambda, A)x \rangle
= \langle i_0^*x^\odot, R(\lambda, A)x \rangle = \lim_{\mu} \langle i_0^*x^\odot, R(\lambda, A)\mu R(\mu, A)x \rangle
= \lim_{\mu} \langle R(\lambda, A_0)^x_i^*x^\odot, \mu R(\mu, A)x \rangle
= \langle kR(\lambda, A_0)^x_i^*x^\odot, x \rangle. \quad (3.31)$$
where $k : X^\odot_0 \rightarrow X^\odot$ is the inverse of $i_0^{|X^\odot}$. This shows that
$$R(\lambda, A^\odot) = kR(\lambda, A_0)^x_i^*.$$ 
Thus,
$$T^\odot(t) = kT_0^*(t)i_0^*$$
by the exponential formula. Hence the proof is completed.

**Conclusion**
In this paper, it has been established that $\omega$-order preserving partial contraction
mapping generate some results of semigroup of linear operators in extrapolation
space.

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References


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