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RING IN WHICH EVERY ELEMENT IS SUM OF N IDEMPOTENTS

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Abstract. In this paper we discuss about the ring R in which every element is sum of n commuting idempotents and discuss the properties of it.

1. Introduction

In the paper[6]the authors discuss about the ring R in which every element is sum of two commuting idempotents and their related properties and shows that the elements satisfy the identity $x^3 = x$ and $R \cong R_1 \times R_2$, where R_1 is Boolean and R_2 is zero or subdirect product of Z_3's. In the paper[2] the authors show in a ring every element is difference of two commuting idempotents if and only if R has the identity $x^3 = x$. In the paper [7] the author discuss about the ring in which every element is sums of three or differences of two commuting idempotents. In the paper [8] the author discuss about the rings whose elements are linear combinations of three commuting idempotents. In the paper [9] the author discuss about the rings whose elements are sums or minus sums of two commuting idempotents. In the paper [9] the author discuss about the rings whose elements are sums or minus sums of two commuting idempotents. In the paper[10] the author discuss about the ring whose elements are sum of four commuting idempotents. Here we find the structure of the ring in which every element is sum of n commuting idempotents and discuss the properties of the ring. Then we find the structure of the ring in which every element is sum of n and difference of m commuting idempotents. Then we find the structure of the ring in which every element is sum of n_1 and difference of m_1 commuting idempotents or / sum of n_2 and difference of m_2 commuting idempotents or/....or/ sum of n_i and difference of m_i commuting idempotents.

2. Preliminaries

All ring consider here is associative with unity. The Jacobson radical is denoted by $J(R)$ for a ring R. Also the all the units of a ring R is denoted by $U(R)$. Again the Chinese Remainder Theorem states "Let R be ring and $I.J$ be ideals in R

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such that $I+J=R$ then there exists a ring isomorphism $R/(I\cap J)=R/I\times R/J$ ". For our work we take the generalized version which states if $I_i, 1\leq i\leq n$ are ideals of a ring R with $\sum_{i=1}^{n} I_i=R$ and $\cap_{i=1}^{n} I_i=0$ then $R\cong(\frac{R}{I_1})\times(\frac{R}{I_2})\times...\times(\frac{R}{I_n})$. If a ring R is sum of n commuting idempotents we denote it by SI^n. So the ring in which every element is sum of 3,4,5 commuting idempotents are SI^3, SI^4, SI^5 respectively.
In the whole paper e_i represent the idempotent.

3. Rings in which every element is sum of n commuting idempotents

Proposition 3.1. if every element of a ring R is sum of n idempotents (i.e R is SI^n) then for every $k\in R$ we have

$$(k-n)\{k-(n-1)\}\{k-(n-2)\}....(k-3)(k-2)(k-1)k=0$$

Proof. 1st we check for $n=2,3,4$ then by induction we prove it.

(1) Suppose R is a $SI^2(n=2)$ ring. So for every $k\in R$ there exist idempotents $e_i\in R, 1\leq i\leq 2$ with $1\leq i,j\leq 2, e_ie_j=e_je_i$ such that $k=e_1+e_2$. Now $k^2=k+2e_1e_2\Rightarrow k^2-k=2e_1e_2\Rightarrow(k^2-k)k=2e_1e_2(e_1+e_2)=4e_1e_2=2(k^2-k)\Rightarrow(k-2)(k-1)k=0$.

(2) Suppose R is a $SI^3(n=3)$ ring. So for every $k\in R$ there exist idempotents $e_i\in R, 1\leq i,j\leq 3$ with $e_ie_j=e_je_i$ such that $k=e_1+e_2+e_3$. Now $k^3=e_1^3+e_2^3+e_3^3+3\{e_1^2(e_2+e_3)+e_2^2(e_3+e_1)+e_3^2(e_1+e_2)\}+6e_1e_2e_3\Rightarrow k^3-k=3\{2(e_1e_2+e_2e_3+e_3e_1)\}+6e_1e_2e_3$. Again $k^2-k=2(e_1e_2+e_2e_3+e_3e_1)$. Therefore $k^3-k=3(k^2-k)+6e_1e_2e_3\Rightarrow k^3-3k^2+2k=6e_1e_2e_3\Rightarrow(k^3-3k^2+2k)(e_1+e_2+e_3)=18e_1e_2e_3=3\times 6e_1e_2e_3=3(k^3-3k^2+2k)\Rightarrow(k-3)(k^3-3k^2+3k)=0\Rightarrow(k-3)(k-2)(k-1)k=0$.

(3) Suppose R is a $SI^4(n=4)$ ring. So for every $k\in R$ there exist idempotents $e_i\in R, 1\leq i,j\leq 4$ with $e_ie_j=e_je_i$ such that $k=e_1+e_2+e_3+3_4$. Now $k^3-k=3(k^2-k)+6(e_1e_2e_3+e_2e_3e_4+e_3e_4e_1+e_4e_1e_2)$. Therefore $(k^3-3k^2+2k)(e_1+e_2+e_3+e_4)=3.6(e_1e_2e_3+e_2e_3e_4+e_3e_4e_1+e_4e_1e_2)+6.4e_1e_2e_3e_4\Rightarrow(k^3-3k^2+2k)k=3(k^3-3k^2+2k)+24e_1e_2e_3e_4\Rightarrow(k-3)(k^3-3k^2+2k)=24e_1e_2e_3e_4\Rightarrow(k-3)(k^3-3k^2+2k)(e_1+e_2+e_3+e_4)=4.24e_1e_2e_3e_4=4(k-3)(k^3-3k^2+2k)\Rightarrow(k-4)(k-3)(k-2)(k-1)k=0$.

Now suppose the result is true for n. We are going to prove the result for $n+1$ also. Therefore when $k=e_1+e_2+e_3+\therefore+e_n$, where $e_i, 1\leq i\leq n$'s commute each other. Then k satisfy

$$(k-n)\{k-(n-1)\}\{k-(n-2)\}....(k-3)(k-2)(k-1)k=0 \tag{3.1}$$

. Now, let $k=e_1+e_2+e_3+....+e_n+e_{n+1}$ where $e_i, 1\leq i\leq(n+1)$. So $k-e_{n+1}=e_1+e_2+.....+e_n$. As $k-e_{n+1}$ is sum of n idempotents which commute each other so it satisfy the equation (3.1). Therefore $\{(k-e_{n+1})-n\}\{(k-e_{n+1})-(n-1)\}\{(k-e_{n+1})-(n-2)\}....\{(k-e_{n+1})-3\}\{(k-e_{n+1})-2\}\{(k-e_{n+1})-1\}(k-e_{n+1})=0$
$\Rightarrow\{(k-n)-e_{n+1}\}[\{(k-(n-1)\}-e_{n+1}][\{(k-(n-2)\}-e_{n+1}]....\{(k-3)-e_{n+1}\}\{(k-2)-e_{n+1}\}\{(k-1)-e_{n+1}\}(k-e_{n+1})=0$
$\Rightarrow\{(k-n)e_{n+1}-e_{n+1}^2\}[\{(k-(n-1)\}e_{n+1}-e_{n+1}^2][\{(k-(n-2)\}e_{n+1}-e_{n+1}^2]....\{(k-3)e_{n+1}-e_{n+1}^2\}\{(k-2)e_{n+1}-e_{n+1}^2\}\{(k-1)e_{n+1}-e_{n+1}^2\}(ke_{n+1}-e_{n+1}^2)=0$

$\Rightarrow \{(k-n)e_{n+1}-e_{n+1}\}[\{(k-(n-1)\}e_{n+1}-e_{n+1}][\{(k-(n-2)\}e_{n+1}-e_{n+1}]....\{(k-3)e_{n+1}-e_{n+1}\}\{(k-2)e_{n+1}-e_{n+1}\}\{(k-1)e_{n+1}-e_{n+1}\}(ke_{n+1}-e_{n+1}) = 0$
$\Rightarrow \{k-(n+1)\}(k-n)\{k-(n-1)\}\{k-(n-2)\}....(k-3)(k-2)(k-1)e_{n+1} = 0.$
Now e_{n+1} is arbitrary. So for every e_i in $k = e_1 + e_2 + ... + e_{n+1}$ we have

$$\{k-(n+1)\}(k-n)\{k-(n-1)\}\{k-(n-2)\}....(k-3)(k-2)(k-1)e_i = 0; 1 \leq i \leq (n+1) \tag{3.2}$$

Adding all these we get $\{k-(n+1)\}(k-n)\{k-(n-1)\}\{k-(n-2)\}....(k-3)(k-2)(k-1)(e_1+e_2+...+e_{n+1}) = 0$
$\Rightarrow \{k-(n+1)\}(k-n)\{k-(n-1)\}\{k-(n-2)\}....(k-3)(k-2)(k-1)k = 0.$
Therefore by induction the result is true.
Proposition 3.2. If every element of a ring R is sum of n idempotents and difference of m idempotents which commute each other i.e every $k \in R$ can be express as

$$k = e_1 + e_2 + ... + e_n - f_1 - f_2 - ... - f_m \tag{3.3}$$

where $e_ie_j = e_je_i; 1 \leq i,j \leq n$ and $e_if_j = f_je_i; 1 \leq i \leq n, 1 \leq j \leq m$ and $f_if_j = f_jf_i; 1 \leq i,j \leq m$, then for every $k \in R$ we have

$$(k-n)\{k-(n-1)\}\{k-(n-2)\}\{k-(n-3)\}....k(k+1)...\{k+(m-1)\}(k+m) = 0$$

.
Proof. 1st we see, if $e \in R$ with $e^2 = e$ then $(1-e)^2 = 1-2e+e^2 = 1-2e+e = 1-e$. So $1-e$ is also idempotent. Now, (3.3) $\Rightarrow k+m = e_1+e_2+...+e_n+(1-f_1)+(1-f_2)+...+(1-f_m)$ so $k+m$ is sum of $m+n$ idempotents and they commute each other. Therefore using the Proposition 3.1 we have $\{(k+m)-(m+n)\}\{(k+m)-(m+n-1)\}\{(k+m)-(m+n-2)\}...\{(k+m)-m\}...\{(k+m)-2\}\{(k+m)-1\}(k+m) = 0$
$\Rightarrow (k-n)\{k-(n-1)\}\{k-(n-2)\}\{k-(n-3)\}....k(k+1)...\{k+(m-1)\}(k+m) = 0.$
Corollary 3.1. If every element of a ring R is sum of n idempotents and difference of n idempotents which commute each other then for every $k \in R$ we have

$$(k-n)\{k-(n-1)\}\{k-(n-2)\}...\{k+(n-2)\}\{k+(n-1)\}(k+n) = 0$$

.
Corollary 3.2. If every element of a ring R is sum of n commuting idempotents(SI^n ring) then every element of the ring can be express as sum of l idempotents and difference m idempotents which commute each other such that $l+m = n$.
Proof. Suppose $k \in R$ so k can be express as sum of n idempotents (i.e R is SI^n ring)that commute each other. Now for $k \in R$,$k+j, 1 \leq j \leq n$ is sum of n idempotents. Therefore $k+j = e_1+e_2+...+e_{n-j}+e_{n-(j-1)}.+e_n$

$$k = e_1+e_2+...+e_{n-j}-(1-e_{n-(j-1)})-...(1-e_n) \tag{3.4}$$

As $(1-e)^2 = 1-2e+e^2 = 1-2e+e = 1-e$, so $1-e$ is idempotent. So from (3.4) we get k is sum of $n-j$ idempotents and difference of j idempotents that commute each other. Putting $n-j = l$ and $j = m$ we get the result.
The result can be express as if R is SI^n then every $k \in R$ can be express as $\pm e_1 \pm e_2 \pm ... \pm e_n$. For example if R is SI^3 ring then for every $k \in R$ can be express as $\pm e_1 \pm e_2 \pm e_3$. Therefore we have

(1) $(k-3)(k-2)(k-1)k = 0.$

(2) $(k-2)(k-1)k(k+1) = 0$.
(3) $(k-1)k(k+1)(k+2) = 0$.
(4) $k(k+1)(k+2)(k+3) = 0$.

Corollary 3.3. If every element of a ring R is sum of m and difference of n commuting idempotents then every element of the ring is sum of $m+n$ commuting idempotents i.e R is a SI^{m+n} ring.
Proof. Suppose $k \in R$ be arbitrary so $(k-n) \in R$. So $k-n$ can be expresed as $k-n = e_1 + e_2 + ... + e_m - e_{m+1} - e_{m+2} - .... - e_{m+n}$ where $e_i, 1 \leqslant i \leqslant (m+n)$'s are commuting idempotents. Now $k = e_1 + e_2 + ... + e_m + (1 - e_{m+1}) + (1 - e_{m+2}) + ... + (1 - e_{m+n})$ which is sum of $m+n$ commuting idempotents (as $1 - e_i$ is idempotent $m+1 \leqslant i \leqslant m+n$). So R is SI^{m+n} ring.
Corollary 3.4. If every element of a ring R of the type $l_1e_1 + l_2e_2 + ... + l_me_m - p_1f_1 - p_2f_2 - ..... - p_ne_n$ where $l_i, p_j \in N$;$1 \geq i \geq m, 1 \geq j \geq n$. e_i, f_j are commuting idempotents then for every $k \in R$ we have

$$\{k-(l_1+l_2+...+l_m)\}\{k-(l_1+l_2+...+l_m-1)\}....(k-1)k(k+1)....\{k+(p_1+p_2+...+p_n)\} = 0$$

Also in this ring we have $(l_1 + l_2 + ... + l_m + p_1 + p_2 + ... + p_n + 1)! = 0$.
Using Proposition 3.4 we get the result.

Till now, ring in which every element is sum of $2, 3, 4$ commuting idempotents are studied. So we study the rings in which every element is sum of $n (n \geq 5)$ commuting idempotents.
Lemma 3.1[2]. Let $a \in R$. If $a^2 - a$ is nilpotent,then there exists a monic polynomial $\theta(t) \in Z[t]$ such that $\theta(a)^2 = \theta(a)$ and $a - \theta(a)$ is nilpotent.
Lemma 3.2[6]. Let p be a prime. The following are equivalent for a ring R:

(1) $p \in Nil(R)$ and $a^p - a$ is nilpotent for all $a \in R$.
(2) $J(R)$ is nil and $R/J(R)$ is a subdirect product of Z_p's.

Lemma 3.3[1]. [Theorem 2.7] A ring R is strongly nil-clean iff $R/J(R)$ is Boolean and $J(R)$ is nil.

Now we have to find the general properties of SI^n ring R where $n \in \mathbb{N}$. Clearly for a SI^n ring R we have $n! = 0$. The other properties of SI^n ring is given below the proposition.
Proposition 3.6. If R is a SI^n $(n > 2)$ ring with unity and $(n+1)! = 2^{a_1}3^{a_2}5^{a_3}7^{a_3}11^{a_4}...{p_i}^{a_j}...{p_m}^{a_m}$. Then we have

(a) For every $k \in R$ we have $(k-n)\{k-(n-1)\}\{k-(n-2)\}...(k-j)...(k-2)(k-1)k = 0$.
(b) $R \cong R_1 \times R_2 \times R_3 \times ... \times R_j \times ... \times R_m$ where
 (1) R_1 is zero or SI^n ring with $2^{a_1} = 0$ and for every $k \in R_1$ we have $(k^2-k)^{a_1} = 0; 2^{a_1-1}(k^2-k) = 0$ and $k^{2^{a_1}} = k^{2^{a_1-1}}$. R_1 is periodic. For every $n \in Nil(R_1)$ we have $n^{a_1} = 0, 2^{a_1-1}n = 0$. $R_1/J(R_1)$ is Boolean. For every $j \in J(R_1)$ we have $j^{a_1} = 0; 2^{a_1-1}j = 0$. $U(R_1)$ is a group of exponent 2^{a_1-1} For every $u \in U(R_1)$ we have $2^{a_1-1}u = 2^{a_1-1}$.
 (2) R_2 is zero or SI^n ring with $3^{a_2} = 0$ and for every $k \in R_2$ we have $(k^3 - k)^{a_2} = 0; 3^{a_2-1}(k^3 - k) = 0$ and $k^{3^{a_2}} = k^{3^{a_2-1}}$. R_2 is periodic. For every $n \in Nil(R_2)$ we have $n^{a_2} = 0; 3^{a_2-1}n = 0$. $R_2/J(R_2)$ is subdirect product of Z_3's. For every $j \in J(R_2)$ we have $j^{a_2} =$

$0; 3^{a_2-1}j = 0$. $U(R_2)$ is a group of exponent $2 \times 3^{a_2-1}$ For every $u \in U(R_2)$ we have $3^{a_2-1}u^2 = 3^{a_2-1}$.
If $a_2 = 1$ then R_2 is zero or subdirect product of Z_3's.

... .

(i) R_i is zero or SI^n ring with ${p_i}^{a_i} = 0$ and for every $k \in R_i$ we have $(k^{p_i} - k)^{a_i} = 0; {p_i}^{a_i-1}(k^{p_i} - k) = 0$ and $k^{{p_i}^{a_i}} = k^{{p_i}^{a_i-1}}$. R_i is periodic. For every $n \in Nil(R_i)$ we have $n^{a_i} = 0, {p_i}^{a_i-1}n = 0$. $R_i/J(R_i)$ is subdirect product of $Z'_{p_i}s$. For every $j \in J(R_i)$ we have $j^{a_i} = 0; {p_i}^{a_i-1}j = 0$. $U(R_i)$ is a group of exponent $(p_i - 1)p_i^{a_i-1}$. For every $u \in U(R_i)$ we have ${p_i}^{a_i-1}u^{a_i-1} = {p_i}^{a_i-1}$.
If $a_i = 1$ then R_i is zero or subdirect product of Z_{p_i}'s.

... .

(m) R_m is zero or SI^n ring with ${p_m}^{a_m} = 0$ and for every $k \in R_m$ we have $(k^{p_m} - k)^{a_m} = 0; {p_m}^{a_m-1}(k^{p_m} - k) = 0$ and $k^{{p_m}^{a_m}} = k^{{p_m}^{a_m-1}}$. R_m is periodic. For every $n \in Nil(R_i)$ we have $n^{a_m} = 0, {p_m}^{a_m-1}n = 0$. $R_m/J(R_m)$ is subdirect product of $Z'_{p_m}s$. For every $j \in J(R_m)$ we have $j^{a_m} = 0; {p_m}^{a_m-1}j = 0$. $U(R_m)$ is group of exponent $(p_m - 1)p_m^{a_m-1}$. For every $u \in U(R_i)$ we have ${p_m}^{a_m-1}u^{a_m-1} = {p_m}^{a_m-1}$.

Proof. As R is SI^n ring then by **Proposition 3.1** for every $k \in R$ we have $(k - n)\{k - (n - 1)\}\{k - (n - 2)\}...(k - j)...(k - 2)(k - 1)k = 0$.
Putting $k = n + 1$ we have $(n + 1)! = 0 \Rightarrow 2^{a_1}3^{a_2}5^{a_3}7^{a_3}11^{a_4}...{p_i}^{a_j}...{p_m}^{a_m} = 0$. Using Chinese Remainder Theorem we have $R \cong R_1 \times R_2 \times R_3 \times ... \times R_i \times ... \times R_m$ where $R_1 \cong \frac{R}{2^{a_1}R}$; $R_2 \cong \frac{R}{3^{a_2}R}$; $R_3 \cong \frac{R}{5^{a_3}R}$;... ;$R_i \cong \frac{R}{{p_i}^{a_i}R}$;...;$R_m \cong \frac{R}{{p_m}^{a_m}R}$.
Suppose $R_1 \neq 0$. In R_1 we have $2^{a_1} = 0$. Now for every $k \in R_1$ we have $k = \sum_{i=1}^{n} e_i \Rightarrow k^2 - k = 2\sum_{cyc} e_1e_2 \Rightarrow (k^2 - k)^{a_1} = 0; 2^{a_1-1}(k^2 - k) = 0$.Again $k^{2^2} = k^2 + 2^2F_2$, where F_2 is a function of e_i's. Similarly we get $k^{2^3} = k^{2^2} + 2^3F_3$, $k^{2^4} = k^{2^3} + 2^4F_4$,...,$k^{2^i} = k^{2^{i-1}} + 2^iF_i$,...and finally $k^{2^a_1} = k^{2^{a_1-1}} + 2^a_1F_{a_1} = k^{2^{a_1-1}}$ where $F_1, F_2, ...F_i, .., F_{a_i}$ are functions of e_i's.
As $k^{2^a_1} = k^{2^{a_1-1}}$ for every $k \in R_1$ so R_1 is periodic. Now for every $n \in Nil(R_1)$ we have $1-n^{\alpha} \in U(R_1)$ for $\alpha \in N$. Now $(n^2-n)^{a_1} = 0 \Rightarrow n^{a_1}(n-1)^{a_1} = 0 \Rightarrow n^{a_1} = 0$. Again $2^{a_1-1}(n^2 - n) = 0 \Rightarrow 2^{a_1-1}n(n - 1) = 0 \Rightarrow 2^{a_1-1}n = 0$.
Again $k^2 - k$ is nilpotent for every $k \in R_1$ so by Lemma 3.1 and Corollary 3.1 there exist idempotent $e = \theta(k) \in R_1$ and a nilpotent $b = k - \theta(k) = k - e \in R_1$ such that $eb = be$. So $k = e+b$ is sum of idempotent and nilpotent that commute each other. So k is strongly nil clean. Hence R_1 is strongly nil clean, so by Lemma 3.3[1] , $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil.

For $j \in J(R_1)$ we have $1 \pm j \in U(R_1)$. Now $(j^2 - j)^{a_1} = 0 \Rightarrow j^{a_1} = 0$. Also $2^{a_1-1}j(j - 1) = 0 \Rightarrow 2^{a_1-1}j = 0$.
Again for $u \in U(R_1)$ we have $2^{a_1-1}u(u - 1) = 0 \Rightarrow 2^{a_1-1}u = 2^{a_1-1}$. Again $u^{2^{a_1}} = u^{2^{a_1-1}} \Rightarrow u^{2^{a_1-1}}(u^{2^{a_1-1}} - 1) = 0 \Rightarrow u^{2^{a_1-1}} = 1$. So $U(R)$ is group of exponent2^{a_1-1}.

Suppose $R_i \neq 0$. In R_i we have ${p_i}^{a_i} = 0$. Now for every $k \in R_i$ we have $k = \sum_{i=1}^{n} e_i \Rightarrow k^{p_i} - k = p_i P(\{e_i | 1 \geq i \geq n\}) \Rightarrow (k^{p_i} - k)^{a_i} = 0; {p_i}^{a_i - 1}(k^{p_i} - k) = 0$ where $P(\{e_i | 1 \geq i \geq n\})$ is a polynomial in commuting $e_i, 1 \geq i \geq n$.
Now for every $n \in Nil(R_i)$ we have $1 - n^{\alpha} \in Nil(R_i)$ for $\alpha \in N$. Again $(n^{p_i} - n)^{a_i} = 0 \Rightarrow n^{a_i}(n^{p_i - 1} - 1)^{a_i} = 0 \Rightarrow n^{a_i} = 0$. Also ${p_i}^{a_i - 1}(n^{p_i} - n) = 0 \Rightarrow (p_i)^{a_i - 1} n(n^{p_i - 1} - 1) = 0 \Rightarrow {p_i}^{a_i - 1} n = 0$. Again $k^{p_i^2} = k^{p_i} + p_i^2 F_2$, $k^{p_i^3} = k^{p_i^2} + p_i^3 F_3$,..., $k^{p_i^j} = k^{p_i^{j-1}} + p_i^j F_j$,..and finally $k^{p_i^{a_i}} = k^{p_i^{a_i - 1}} + p_i^{a_i} F_{a_i} \Rightarrow k^{p_i^{a_i}} = k^{p_i^{a_i - 1}}$ where F_i's are functions of e_i's.
As $k^{p_i^{a_i}} = k^{p_i^{a_i - 1}}$ for every $k \in R_i$ so R_i is periodic.
Using Lemma 3.2 we have $R_i / J(R_i)$ is subdirect product of Z_{p_i}'s and $J(R_i)$ is nil.

For $j \in J(R_i) \Rightarrow j^{p_i - 1} \in J(R_i)$ we have $1 - j^{p_i - 1} \in U(R_i)$. Now $(j^{p_i} - j)^{a_i} = 0 \Rightarrow j^{a_i}(j^{p_i - 1} - 1)^{a_i} = 0 \Rightarrow j^{a_i} = 0$. Also ${p_i}^{a_i - 1} j(j^{a_i - 1} - 1) = 0 \Rightarrow {p_i}^{a_i - 1} j = 0$.
Again for $u \in U(R_i)$ we have ${p_i}^{a_i - 1} u(u - 1) = 0 \Rightarrow {p_i}^{a_i - 1} u = {p_i}^{a_i - 1}$. Again $u^{p_i^{a_i}} = u^{p_i^{a_i - 1}} \Rightarrow u^{p_i^{a_i - 1}}(u^{(p_i - 1)p_i^{a_i - 1}} - 1) = 0 \Rightarrow u^{(p_i - 1)p_i^{a_i - 1}} = 1$. So $U(R_i)$ is group of exponent $(p_i - 1)p_i^{a_i - 1}$.

Again if $a_i = 1$ then $p_i = 0$. Now if $k^2 = 0$ where $k = \sum_{i=1}^{n} e_i \Rightarrow k^{p_i} - k = p_i P(\{e_i | 1 \geq i \geq n\}) \Rightarrow k^{p_i} - k = 0 \Rightarrow k = 0$ where $P(\{e_i | 1 \geq i \geq n\})$ is a polynomial in commuting $e_i, 1 \geq i \geq n$. So R_i is reduced ring.So R_i is subdirect product of the domains $\{R_\alpha\}$. But R_α has only trivial idempotents $0, 1$. We infer that $R_\alpha = \{0, 1, 2, 3, ..., p_i - 1\}$ as $p_i = 0$ in R_α. Hence R_i is subdirect product of Z_{p_i}'s.

Similarly we can prove the results for $R_2, ..., R_m$.
Examples. In SI^5 we have $6! = 0 \Rightarrow 2^4 \times 3^2 \times 5 = 0$. $\prod_{\alpha \in \wedge} R_\alpha$,where $R_\alpha = Z_5$ is SI^5 ring. Similarly $\prod_{\alpha \in \wedge} R_\alpha \times \prod_{\beta \in \wedge} R_\beta$, where $R_\alpha = Z_5$ and $R_\beta = Z_7$ is a SI^6 ring.
Proposition 3.7. If the ring R is subdirect product of Z_p where p is prime then every element of R is a sum of $p - 1$ commuting idempotents.
Proof. Here R is subdirect product of $\{R_\alpha : \alpha \in \wedge\}$ where $R_\alpha = Z_p$ for all $\alpha \in \wedge$. So R is a subring of $\prod_{\alpha \in \wedge} R_\alpha$. Let $x = (x_\alpha) \in R$. Then $\wedge$ is a disjoint union of $\wedge_0, \wedge_1, \wedge_2, ..., \wedge_{p-2}, \wedge_{p-1}$ such that $x_\alpha = i$ if and only if $\alpha \in \wedge_i$ for $i = 0, 1, 2, 3, ..., p - 1$. Without loss of generality , we can denote $x = (0_{\wedge_0}, 1_{\wedge_1}, 2_{\wedge_0}, ..., i_{\wedge_i}, ..., (p-1)_{\wedge_{p-1}})$. Now

$$\begin{aligned}
e_1 &= (0_{\wedge_0}, 1_{\wedge_1}, 1_{\wedge_2}, ..., 1_{\wedge_i}, ..., 1_{\wedge_{p-1}}) \in R, \\
e_2 &= (0_{\wedge_0}, 0_{\wedge_1}, 1_{\wedge_0}, ..., 1_{\wedge_i}, ..., 1_{\wedge_{p-1}}) \in R, \\
& \quad .... \\
e_{i-1} &= (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_2}, ...1_{\wedge_{i-1}}, 1_{\wedge_i}, 1_{\wedge_{i+1}} ..., 1_{\wedge_{p-1}}) \in R, \\
e_i &= (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_2}, ..., 1_{\wedge_i}, 1_{\wedge_{i+1}} ..., 1_{\wedge_{p-1}}) \in R, \\
e_{i+1} &= (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_0}, ..., 0_{\wedge_i}, 1_{\wedge_{i+1}} ..., 1_{\wedge_{p-1}}) \in R, \\
& \quad .... \\
e_{p-1} &= (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_2}, ..., 0_{\wedge_i}, ..., 1_{\wedge_{p-1}}) \in R.
\end{aligned}$$

Clearly we can see that $e_i^2 = e_i, e_ie_j = e_je_i \forall 1 \leq i, j \leq p-1$ and $x = e_1 + e_2 + ... + e_i + ... + e_{p-1}$. This shows every element of R is a sum of $p-1$ commuting idempotents.

4. Rings in which every element is sum of n_1 and difference of m_1 commuting idempotents / or sum of n_2 and difference of m_2 commuting idempotents / or........./or sum of n_i and difference of m_i commuting idempotents / or......../or sum of n_t and difference of m_t commuting idempotents

From the Corollary 3.2 we get if R is a SI^n ring then every element can be express as sum of l idempotents and difference of m idempotents which commute each other such that $l + m = n$. i.e the elements of of a SI^n ring can be express as $\pm e_1 \pm e_2 \pm e_3 ... \pm e_n$. For example if we take SI^3 ring R then it's element can be express as $\pm e_1 \pm e_2 \pm e_3$.
But there is a difference, i.e if we take a ring in which every element can be express as sum of three commuting idempotents or/ sum of two and difference of one commuting idempotents or/ sum of one and difference of two commuting idempotents or/ difference of three commuting idempotents then it does not mean that it is SI^3 ring. Because if we take k is sum of 3 commuting idempotents then $k+1$ may not sum of three commuting idempotents. Let's first find the structure of this ring.
Proposition 4.1. If every element of a ring R is of the type $e_1+e_2+e_3$ or $e_1+e_2-e_3$ or $e_1 - e_2 - e_3$ or $-e_1 - e_2 - e_3$ where $e_i (1 \leqslant i \leqslant 3)$ are commute each other then R satisfies

$$(k-3)(k-2)(k-1)k(k+1)(k+2)(k+3) = 0$$

for every $k \in R$ and R satisfy the same properties as SI^6 ring as in the Proposition 3.4.
Proof. Using Proposition 3.2 we have When if $k \in R$ with $k = e_1 + e_2 + e_3$,where $e_i (1 \leqslant i \leqslant 3)$ are commute each other then

$$(k-3)(k-2)(k-1)k = 0 \tag{4.1}$$

When if $k \in R$ with $k = e_1 + e_2 - e_3$,where $e_i (1 \leqslant i \leqslant 3)$ are commute each other then

$$(k-2)(k-1)k(k+1) = 0 \tag{4.2}$$

When if $k \in R$ with $k = e_1 - e_2 - e_3$,where $e_i (1 \leqslant i \leqslant 3)$ are commute each other then

$$(k-1)k(k+1)(k+2) = 0 \tag{4.3}$$

When if $k \in R$ with $k = -e_1 - e_2 - e_3$,where $e_i (1 \leqslant i \leqslant 3)$ are commute each other then

$$k(k+1)(k+2)(k+3) = 0 \tag{4.4}$$

Combining all the equations (4.1),(4.2),(4.3),(4.4) we have

$$(k-3)(k-2)(k-1)k(k+1)(k+2)(k+3) = 0$$

which is nothing but the equation of a ring in which every element is sum of 3 and difference of 3 commuting idempotents. Now if $k \in R$ then $(k-3) \in R$ so

$k-3 = e_1+e_2+e_3-e_4-e_5-e_6 \Rightarrow k = e_1+e_2+e_3+(1-e_4)+(1-e_5)+(1-e_6)$ which is the sum of 6 commuting idempotents (as $1-e_i$ is also idempotent $1 \leqslant i \leqslant 3$). As $k \in R$ is arbitrary, so every element of the ring R is sum of 6 commuting idempotents. So R is SI^6 ring. Putting $k = 4$ we get $7! = 0 \Rightarrow 2^4 \times 3^2 \times 5 \times 7 = 0$. So using Chinese Remainder Theorem $R \cong R_1 \times R_2 \times R_3 \times R_4$ where R_1, R_2, R_3, R_4 have same properties as in Proposition 3.4.
Now we going to prove the generalized version.
Proposition 4.2. If in a ring R every element is sum of n_1 and difference of m_1 commuting idempotents / or sum of n_2 and difference of m_2 commuting idempotents / or........./or sum of n_i and difference of m_i commuting idempotents / or......../or sum of n_t and difference of m_t commuting idempotents. Then R satisfies

$$(k-n)k-(n-1).....\{k+(m-1)\}(k+m) = 0$$

where $(k-n)k-(n-1).....\{k+(m-1)\}(k+m) = L.C.M\{(k-n_1)\{k-(n_1-1)\}...\{k+(m_1-1)\}(k+m_1), (k-n_2)\{k-(n_2-1)\}...\{k+(m_2-1)\}(k+m_2), ..., (k-n_i)\{k-(n_i-1)\}...\{k+(m_i-1)\}(k+m_i), ..., (k-n_t)\{k-(n_t-1)\}...\{k+(m_t-1)\}(k+m_t)\}$.
R is a SI^{m+n} ring and satisfies the properties as in the Proposition 3.6.
Proof. Using Proposition 3.2 we have If $k \in R$ is sum of n_1 and difference of m_1 commuting idempotents then

$$(k-n_1)\{k-(n_1-1)\}...\{k+(m_1-1)\}(k+m_1) = 0$$

If $k \in R$ is sum of n_2 and difference of m_1 commuting idempotents then

$$(k-n_2)\{k-(n_2-1)\}...\{k+(m_2-1)\}(k+m_2) = 0$$

....
If $k \in R$ is sum of n_i and difference of m_i commuting idempotents then

$$(k-n_i)\{k-(n_i-1)\}...\{k+(m_i-1)\}(k+m_i) = 0$$

....
If $k \in R$ is sum of n_t and difference of m_t commuting idempotents then

$$(k-n_t)\{k-(n_t-1)\}...\{k+(m_t-1)\}(k+m_t) = 0$$

Now if we take L.C.M of all the right hand side equations and letting $(k-n)k-(n-1).....\{k+(m-1)\}(k+m) = L.C.M\{(k-n_1)\{k-(n_1-1)\}...\{k+(m_1-1)\}(k+m_1), (k-n_2)\{k-(n_2-1)\}...\{k+(m_2-1)\}(k+m_2), ..., (k-n_i)\{k-(n_i-1)\}...\{k+(m_i-1)\}(k+m_i), ..., (k-n_t)\{k-(n_t-1)\}...\{k+(m_t-1)\}(k+m_t)\}$.
Then we have

$$(k-n)k-(n-1).....\{k+(m-1)\}(k+m) = 0$$

So R is a SI^{m+n} ring. Therefore $(m+n+1)! = 0$. We get get the properties of the ring R using the Proposition 3.6. For example if if every element of a ring R is sum of 200 and difference of 100 commuting idempotents /or sum of 80 and difference of 500 idempotents then R is a $SI^{200+500} = SI^{700}$ ring and $701! = 0$ and we get the result of the ring using Proposition 4.2.

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