RING IN WHICH EVERY ELEMENT IS SUM OF \( N \) IDEMPOTENTS

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Abstract. In this paper we discuss about the ring \( R \) in which every element is sum of \( n \) commuting idempotents and discuss the properties of it.

1. Introduction

In the paper[6] the authors discuss about the ring \( R \) in which every element is sum of two commuting idempotents and their related properties and shows that the elements satisfy the identity \( x^3 = x \) and \( R \cong R_1 \times R_2 \), where \( R_1 \) is Boolean and \( R_2 \) is zero or subdirect product of \( Z_3 \)'s. In the paper[2] the authors show in a ring every element is difference of two commuting idempotents if and only if \( R \) has the identity \( x^3 = x \). In the paper [7] the author discuss about the ring in which every element is sums of three or differences of two commuting idempotents. In the paper [8] the author discuss about the rings whose elements are linear combinations of three commuting idempotents. In the paper [9] the author discuss about the rings whose elements are sums or minus sums of two commuting idempotents. In the paper [10] the author discuss about the ring whose elements are sum of four commuting idempotents. Here we find the structure of the ring in which every element is sum of \( n \) commuting idempotents and discuss the properties of the ring. Then we find the structure of the ring in which every element is sum of \( n \) commuting idempotents and difference of \( m \) commuting idempotents. Then we find the structure of the ring in which every element is sum of \( n_1 \) and difference of \( m_1 \) commuting idempotents or / sum of \( n_2 \) and difference of \( m_2 \) commuting idempotents or /...or/ sum of \( n_i \) and difference of \( m_i \) commuting idempotents.

2. Preliminaries

All ring consider here is associative with unity. The Jacobson radical is denoted by \( J(R) \) for a ring \( R \). Also the all the units of a ring \( R \) is denoted by \( U(R) \). Again the Chinese Remainder Theorem states "Let \( R \) be ring and \( I,J \) be ideals in \( R \)
such that \(I + J = R \) then there exists a ring isomorphism \( R/(I \cap J) = R/I \times R/J \)
. For our work we take the generalized version which states if \( I_i, 1 \leq i \leq n \)

are ideals of a ring \( R \) with \( \sum_{i=1}^{n} I_i = R \) and \( \cap_{i=1}^{n} I_i = 0 \) then \( R \cong \left( \frac{R}{I_1} \right) \times \left( \frac{R}{I_2} \right) \times \cdots \times \left( \frac{R}{I_n} \right) \).

If a ring \( R \) is sum of \( n \) commuting idempotents we denote it by \( SI^n \). So the ring in which every element is sum of \( 3, 4, 5 \) commuting idempotents are \( SI^3, SI^4, SI^5 \) respectively.

In the whole paper \( e_i \) represent the idempotent.

3. RINGS IN WHICH EVERY ELEMENT IS SUM OF \( n \) COMMUTING IDEMPOTENTS

Proposition 3.1. if every element of a ring \( R \) is sum of \( n \) idempotents (i.e \( R \) is \( SI^n \)) then for every \( k \in R \) we have

\[(k - n)\{k - (n - 1)\}\{k - (n - 2)\} \cdots (k - 3)(k - 2)(k - 1)k = 0\]

Proof. 1st we check for \( n = 2, 3, 4 \) then by induction we prove it.

1) Suppose \( R \) is a \( SI^2(n = 2) \) ring. So for every \( k \in R \) there exist idempotents \( e_i \in R, 1 \leq i \leq 2 \) with \( 1 \leq i, j \leq 2, e_i e_j = e_j e_i \) such that \( k = e_1 + e_2 \).

Now \( k^2 = k + 2e_1e_2 \Rightarrow k^2 - k = 2e_1e_2 \Rightarrow (k^2 - k)k = 2e_1e_2(e_1 + e_2) = 4e_1e_2 = 2(k^2 - k) \Rightarrow (k - 2)(k - 1)k = 0 \).

2) Suppose \( R \) is a \( SI^3(n = 3) \) ring. So for every \( k \in R \) there exist idempotents \( e_i \in R, 1 \leq i, j \leq 3 \) with \( e_i e_j = e_j e_i \) such that \( k = e_1 + e_2 + e_3 \).

Now \( k^3 = e_1^3 + e_2^3 + e_3^3 + 3\{e_1^2(e_2 + e_3) + e_2^2(e_3 + e_1) + e_3^2(e_1 + e_2)\} + 6e_1e_2e_3 \Rightarrow k^3 - k = 3\{2(e_1e_2 + e_2e_3 + e_3e_1)\} + 6e_1e_2e_3 \Rightarrow k^3 - k = 2(e_1e_2 + e_2e_3 + e_3e_1) + 6e_1e_2e_3 \Rightarrow (k^3 - k)k = 2(k^2 - k)k = 6e_1e_2e_3 \Rightarrow (k^3 - k)(k^3 - 3k^2 + 2k) = \cdots = 0 \Rightarrow (k^3 - (k - 3)(k - 2)(k - 1)k = 0 \).

3) Suppose \( R \) is a \( SI^4(n = 4) \) ring. So for every \( k \in R \) there exist idempotents \( e_i \in R, 1 \leq i, j \leq 4 \) with \( e_i e_j = e_j e_i \) such that \( k = e_1 + e_2 + e_3 + e_4 \).

Now \( k^3 - k = 3(k^2 - k) + 6(e_1e_2e_3 + e_2e_3e_4 + e_3e_4e_1 + e_4e_1e_2) \Rightarrow (k^3 - 3k^2 + 2k)k = 3(k^3 - 3k^2 + 2k) + 24e_1e_2e_3e_4 \Rightarrow (k - 3)(k^3 - 3k^2 + 2k) = 4.24(e_1e_2e_3e_4 = 4(k - 3)(k^3 - 3k^2 + 2k) \Rightarrow (k - 4)(k - 3)(k - 2)(k - 1)k = 0 \).

Now suppose the result is true for \( n \). We are going to prove the result for \( n + 1 \) also. Therefore when \( k = e_1 + e_2 + e_3 + \cdots + e_n \) where \( e_i, 1 \leq i \leq n \)’s commute each other. Then \( k \) satisfy

\[(k - n)\{k - (n - 1)\}\{k - (n - 2)\} \cdots (k - 3)(k - 2)(k - 1)k = 0 \quad (3.1)\]

. Now, let \( k = e_1 + e_2 + e_3 + \cdots + e_n + e_{n+1} \) where \( e_i, 1 \leq i \leq (n + 1) \). So \( k - e_{n+1} \) is sum of \( n \) idempotents which commute each other so it satisfy the equation (3.1). Therefore \( \{k - e_{n+1} - n\}\{k - (e_{n+1} - 1)\}\{k - (e_{n+1} - 2)\} \cdots \{k - e_{n+1} - (n - 1)\}\{k - (e_{n+1} - 2)\} \cdots \{k - e_{n+1} - (n - 2)\} \cdots \{k - e_{n+1} - 3\} \{k - e_{n+1} - 2\} \{k - e_{n+1} - 1\}\{k - e_{n+1} - 0\} \Rightarrow \{k - n\}\{k - (n - 1)\}\{k - (n - 2)\} \cdots \{k - e_{n+1} - 1\} \Rightarrow \{k - (n - e_{n+1})\}\{k - (n - 1)\}\{k - (n - 2)\} \cdots \{k - (3)\} \{k - (2)\} \{k - (1)\} \{k - (0)\} \Rightarrow \{k - (n - e_{n+1})\}\{k - (n - 1)\}\{k - (n - 2)\} \cdots \{k - 3\} \{k - 2\} \{k - 1\} \{k - 0\} = 0 \Rightarrow (k - n)\{k - (n - 1)\}\{k - (n - 2)\} \cdots \{k - 3\} \{k - 2\} \{k - 1\} \{k - 0\} = 0 \Rightarrow (k - n)\{k - (n - 1)\} \cdots \{k - 3\} \{k - 2\} \{k - 1\} \{k - 0\} = 0
\[\Rightarrow \{(k-n)e_{n+1} - e_{n+1}\} = \{(k-(n-1))e_{n+1} - e_{n+1}\} = \{(k-(n-2))e_{n+1} - e_{n+1}\} = \ldots = \{(k-3)e_{n+1} - e_{n+1}\} = \{(k-2)e_{n+1} - e_{n+1}\} = \{(k-1)e_{n+1} - e_{n+1}\} = 0\]

Corollary 3.1. If every element of a ring \(R\) is the sum of \(n\) idempotents and difference of \(m\) idempotents which commute each other i.e every \(k \in R\) can be express as

\[k = e_1 + e_2 + \ldots + e_n - f_1 - f_2 - \ldots - f_m\]  \(\text{(3.3)}\)

where \(e_i, f_j; 1 \leq i, j \leq n\) and \(e_i f_j = f_j e_i; 1 \leq i \leq n, 1 \leq j \leq m\) and \(f_i, f_j; 1 \leq i, j \leq m\), then for every \(k \in R\) we have

\[(k-n)(k-(n-1))(k-(n-2))(k-(n-3)) \ldots (k+1)(k+m) = 0\]

Proof. 1st we see, if \(e \in R\) with \(e^2 = e\) then \((1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e\). So 1 - e is also idempotent. Now, \((3.3) \Rightarrow k + m = e_1 + e_2 + \ldots + e_n + (1 - f_1) + (1 - f_2) + \ldots + (1 - f_m)\) so \(k + m\) is sum of \(m + n\) idempotents and they commute each other. Therefore using the Proposition 3.1 we have

\[
\{(k + m) - (m + n)\} \{(k + m) - (m + n - 1)\} \ldots \{(k + m) - 2\} \{(k + m) - 1\} (k + m) = 0
\]

\[
\Rightarrow (k-n)(k-(n-1))(k-(n-2))(k-(n-3)) \ldots (k+1)(k+m) = 0.
\]

Corollary 3.1. If every element of a ring \(R\) is sum of \(n\) idempotents and difference of \(n\) idempotents which commute each other then for every \(k \in R\) we have

\[(k-n)(k-(n-1))(k-(n-2)) \ldots (k+(n-2))(k+(n-1))(k+n) = 0\]

Corollary 3.2. If every element of a ring \(R\) is sum of \(n\) commuting idempotents \((SI^m)\) ring then every element of the ring can be express as sum of \(l\) idempotents and difference \(m\) idempotents which commute each other such that \(l + m = n\). Proof. Suppose \(k \in R\) so \(k\) can be express as sum of \(n\) idempotents (i.e \(R\) is \(SI^m\) ring) that commute each other. Now for \(k \in R, k + j, 1 \leq j \leq n\) is sum of \(n\) idempotents. Therefore \(k + j = e_1 + e_2 + \ldots + e_{n-j} + e_{n-(j-1)} + e_n\)

\[k = e_1 + e_2 + \ldots + e_{n-j} - (1 - e_{n-(j-1)}) - \ldots (1 - e_n)\]  \(\text{(3.4)}\)

As \((1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e\), so \(1 - e\) is idempotent. So from \((3.4)\) we get \(k\) is sum of \(n - j\) idempotents and difference of \(j\) idempotents that commute each other. Putting \(n - j = l\) and \(j = m\) we get the result. The result can be express as if \(R\) is \(SI^m\) then every \(k \in R\) can be express as \(\pm e_1 \pm e_2 \pm \ldots \pm e_n\). For example if \(R\) is \(SI^3\) ring then for every \(k \in R\) can be express as \(\pm e_1 \pm e_2 \pm e_3\). Therefore we have

\[(1) \quad (k-3)(k-2)(k-1)k = 0.\]
(2) \((k - 2)(k - 1)k(k + 1) = 0\).
(3) \((k - 1)k(k + 1)(k + 2) = 0\).
(4) \(k(k + 1)(k + 2)(k + 3) = 0\).

Corollary 3.3. If every element of a ring \(R\) is sum of \(m\) and difference of \(n\) commuting idempotents then every element of the ring is sum of \(m + n\) commuting idempotents i.e \(R\) is a \(SI^{m+n}\) ring.

Proof. Suppose \(k \in R\) be arbitrary so \((k - n) \in R\). So \(k - n\) can be expressed as 
\[k - n = e_1 + e_2 + \ldots + e_m - e_{m+1} - e_{m+2} - \ldots - e_{m+n} \]
where \(e_i, 1 \leq i \leq (m + n)'s\) are commuting idempotents. Now \(k = e_1 + e_2 + \ldots + e_m + (1 - e_{m+1}) + (1 - e_{m+2}) + \ldots + (1 - e_{m+n})\) which is sum of \(m + n\) commuting idempotents. So \(R\) is \(SI^{m+n}\) ring.

Corollary 3.4. If every element of a ring \(R\) of the type \(l_1e_1 + l_2e_2 + \ldots + l_m e_m - p_1f_1 - p_2f_2 - \ldots - p_n e_n\) where \(l_i, p_j \in N ; 1 \geq i \geq m, 1 \geq j \geq n\). \(e_i, f_j\) are commuting idempotents then for every \(k \in R\) we have
\[\{k-(l_1+l_2+\ldots+l_m)\}\{k-(l_1+l_2+\ldots+l_m-1)\}\ldots(k-1)k(k+1)\ldots\{k+(p_1+p_2+\ldots+p_n)\} = 0\]

Also in this ring we have \((l_1 + l_2 + \ldots + l_m + p_1 + p_2 + \ldots + p_n + 1)! = 0\).

Using Proposition 3.4 we get the result.

Till now, ring in which every element is sum of 2, 3, 4 commuting idempotents are studied. So we study the rings in which every element is sum of \(n(n \geq 5)\) commuting idempotents.

Lemma 3.1[2]. Let \(a \in R\). If \(a^2 - a\) is nilpotent, then there exists a monic polynomial \(\theta(t) \in Z[t]\) such that \(\theta(a) = \theta(a)\) and \(a - \theta(a)\) is nilpotent.

Lemma 3.2[6]. Let \(p\) be a prime. The following are equivalent for a ring \(R:\)

1. \(p \in Nil(R)\) and \(a^p - a\) is nilpotent for all \(a \in R\).
2. \(J(R)\) is nil and \(R/J(R)\) is a subdirect product of \(Z_p\)'s.

Lemma 3.3[1]. [Theorem 2.7] A ring \(R\) is strongly nil-clean iff \(R/J(R)\) is Boolean and \(J(R)\) is nil.

Now we have to find the general properties of \(SI^n\) ring \(R\) where \(n \in N\). Clearly for a \(SI^n\) ring \(R\) we have \(n! = 0\). The other properties of \(SI^n\) ring is given below the proposition.

Proposition 3.6. If \(R\) is a \(SI^n\) \((n > 2)\) ring with unity and \((n+1)! = 2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}...p_1^{a_j}...p_m^{a_m}\).

Then we have

(a) For every \(k \in R\) we have \((k-n)(k-(n-1))\ldots(k-(n-2))\ldots(k-j)...(k-2)(k-1)k = 0\).

(b) \(R \cong R_1 \times R_2 \times R_3 \times \ldots \times R_j \times \ldots \times R_m\) where

1. \(R_1\) is zero or \(SI^n\) ring with \(2^{a_1} = 0\) and for every \(k \in R_1\) we have
\[(k^2-k)^{a_1} = 0; 2^{a_1-1}(k^2-k) = 0\text{ and }k^{2^{a_1-1}}. R_1\) is periodic. For every \(n \in Nil(R_1)\) we have \(n^{a_1} = 0, 2^{a_1-1}n = 0. R_1/J(R_1)\) is Boolean. For every \(j \in J(R_1)\) we have \(j^{a_1} = 0; 2^{a_1-1}j = 0. U(R_1)\) is a group of exponent \(2^{a_1-1}\) For every \(u \in U(R_1)\) we have \(2^{a_1-1}u = 2^{a_1-1}\).

2. \(R_2\) is zero or \(SI^n\) ring with \(3^{a_2} = 0\) and for every \(k \in R_2\) we have
\[(k^3-k)^{a_2} = 0; 3^{a_2-1}(k^3-k) = 0\text{ and }k^{3^{a_2-1}}. R_2\) is periodic. For every \(n \in Nil(R_2)\) we have \(n^{a_2} = 0, 3^{a_2-1}n = 0. R_2/J(R_2)\) is subdirect product of \(Z_3\)'s. For every \(j \in J(R_2)\) we have \(j^{a_2} = 0\).
0; 3^{a_2-1}j = 0. U(R_2) is a group of exponent $2 \times 3^{a_2-1}$. For every $u \in U(R_2)$ we have $3^{a_2-1}u^2 = 3^{a_2-1}$. If $a_2 = 1$ then $R_2$ is zero or subdirect product of $Z_3$'s.

(i) $R_i$ is zero or $ST^n$ ring with $p_i^{a_i} = 0$ and for every $k \in R_i$ we have $(k^{p_i} - k)^{a_i} = 0; p_i^{a_i-1}(k^{p_i} - k) = 0$ and $k^{p_i a_i} = k^{p_i a_i-1}$. $R_i$ is periodic. For every $n \in Nil(R_i)$ we have $n^{a_i} = 0, p_i^{a_i-1}n = 0$. $R_i/J(R_i)$ is subdirect product of $Z_{p_i}$. For every $j \in J(R_i)$ we have $j^{a_i} = 0; p_i^{a_i-1}j = 0$. $U(R_i)$ is a group of exponent $(p_i - 1)p_i^{a_i-1}$. For every $u \in U(R_i)$ we have $p_i^{a_i-1}u^{a_i-1} = p_i^{a_i-1}$.

If $a_i = 1$ then $R_i$ is zero or subdirect product of $Z_{p_i}$'s.

(m) $R_m$ is zero or $ST^n$ ring with $p_m^{a_m} = 0$ and for every $k \in R_m$ we have $(k^{p_m} - k)^{a_m} = 0; p_m^{a_m-1}(k^{p_m} - k) = 0$ and $k^{p_m a_m} = k^{p_m a_m-1}$. $R_m$ is periodic. For every $n \in Nil(R_i)$ we have $n^{a_m} = 0, p_m^{a_m-1}n = 0$. $R_m/J(R_m)$ is subdirect product of $Z_{p_m}$. For every $j \in J(R_m)$ we have $j^{a_m} = 0; p_m^{a_m-1}j = 0$. $U(R_m)$ is group of exponent $(p_m - 1)p_m^{a_m-1}$. For every $u \in U(R_i)$ we have $p_m^{a_m-1}u^{a_m-1} = p_m^{a_m-1}$.

Proof. As $R$ is $ST^n$ ring then by Proposition 3.1 for every $k \in R$ we have $(k - n)\{k - (n - 1)\}\{k - (n - 2)\}...(k - j)\cdots(k - 2)\cdots(k - 1)k = 0$.

Putting $k = n + 1$ we have $(n + 1)! = 0 \Rightarrow 2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}...p_i^{a_i}...p_m^{a_m} = 0$. Using Chinese Remainder Theorem we have $R \cong R_1 \times R_2 \times R_3 \times ... \times R_i \times ... \times R_m$ where $R_1 \cong \frac{R}{a_1 R}; R_2 \cong \frac{R}{a_2 R}; R_3 \cong \frac{R}{a_3 R}; ...; R_i \cong \frac{R}{a_i R}; ...; R_m \cong \frac{R}{a_m R}$.

Suppose $R_1 \neq 0$. In $R_1$ we have $2^{a_1} = 0$. Now for every $k \in R_1$ we have $k = \sum_{i=1}^{n} e_i \Rightarrow k^2 - k = 2 \sum_{e_i} e_1 e_2 \Rightarrow (k^2 - k)^{a_1} = 0; 2^{a_1-1}(k - 2) = 0$. Again $k^{2^2} = k^{2^{a_1-1}} + 2^{a_1}F_2$, where $F_2$ is a function of $e_i$'s. Similarly we get $k^{2^3} = k^{2^{a_1-1}} + 2^{a_1}F_3$, $k^{2^4} = k^{2^{a_1-1}} + 2^{a_1}F_4$, ..., $k^{2^n} = k^{2^{a_1-1}} + 2^{a_1}F_n$, and finally $k^{2^n} = k^{2^{a_1-1}} + 2^{a_1}F_n = k^{2^{a_1-1}}$ where $F_1, F_2, ...F_n$ are functions of $e_i$'s.

As $k^{2^n} = k^{2^{a_1-1}}$ for every $k \in R_1$ so $R_1$ is periodic. Now for every $n \in Nil(R_1)$ we have $1 - n^{a_1} \in U(R_i)$ for $\alpha \in N$. Now $(n^2 - n)^{a_1} = 0 \Rightarrow n^{a_1}(n - 1)^{a_1} = 0 \Rightarrow n^{a_1} = 0$.

Again $2^{a_1-1}(n^2 - n) = 0 \Rightarrow 2^{a_1-1}n(n - 1) = 0 \Rightarrow 2^{a_1-1}n = 0$.

Again $k^{2^2} - k$ is nilpotent for every $k \in R_1$ so by Lemma 3.1 and Corollary 3.1 there exist idempotent $e = \theta(k) \in R_1$ and a nilpotent $b = k - \theta(k) = k - e \in R_1$ such that $eb = be$. So $k = e + b$ is sum of idempotent and nilpotent that commute each other. So $k$ is strongly nil clean. Hence $R_1$ is strongly nil clean, so by Lemma 3.3[1], $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil.

For $j \in J(R_1)$ we have $1 \pm j \in U(R_1)$. Now $(j^2 - j)^{a_1} = 0 \Rightarrow j^{a_1} = 0$. Also $2^{a_1-1}(j - 1) = 0 \Rightarrow 2^{a_1-1}j = 0$.

Again for $u \in U(R_1)$ we have $2^{a_1-1}u(u - 1) = 0 \Rightarrow 2^{a_1-1}u = 2^{a_1-1}$. Again $u^{2^{a_1-1}} = u^{2^{a_1-1}} \Rightarrow u^{2^{a_1-1}}(u^{2^{a_1-1}} - 1) = 0 \Rightarrow u^{2^{a_1-1}} = 1$. So $U(R)$ is group of exponent $2^{a_1-1}$.
Suppose $R_i \neq 0$. In $R_i$ we have $p_i^{a_i} = 0$. Now for every $k \in R_i$ we have $k = \sum_{i=1}^{n} e_i \Rightarrow k^{p_i} - k = p_i P\{e_i | 1 \geq i \geq n\} \Rightarrow (k^{p_i} - k)^{a_i} = 0; p_i^{a_i-1}(k^{p_i} - k) = 0$ where $P\{e_i | 1 \geq i \geq n\}$ is a polynomial in commuting $e_i, 1 \geq i \geq n$.

Now for every $n \in \text{Nil}(R_i)$ we have $1 - n^\alpha \in \text{Nil}(R_i)$ for $\alpha \in N$. Again $(n^{p_i} - n)^{a_i} = 0 \Rightarrow n^{a_i}(n^{p_i-1} - 1)^{a_i} = 0 \Rightarrow n^{a_i} = 0$. Also $p_i^{a_i-1}(n^{p_i} - n) = 0 \Rightarrow (p_i)^{a_i-1}n(n^{p_i-1} - 1) = 0 \Rightarrow p_i^{a_i-1}n = 0$. Again $k^{p_i} = k^{p_i-1} + p_i^{1}F_2, k^{p_i} = k^{p_i-1} + p_i^{1}F_3, \ldots$ and finally $k^{p_i} = k^{p_i-1} + p_i^{1}F_{a_i} \Rightarrow k^{p_i} = k^{p_i-1}$ where $F_i$'s are functions of $e_i$'s.

As $k^{p_i} = k^{p_i-1}$ for every $k \in R_i$ so $R_i$ is periodic.

Using Lemma 3.2 we have $R_i/J(R_i)$ is subdirect product of $Z_{p_i}$'s and $J(R_i)$ is nil.

For $j \in J(R_i) \Rightarrow j^{p_i-1} \in J(R_i)$ we have $1 - j^{p_i-1} \in U(R_i)$. Now $(j^{p_i} - j)^{a_i} = 0 \Rightarrow j^{a_i}(j^{p_i-1} - 1)^{a_i} = 0 \Rightarrow j^{a_i} = 0$. Also $p_i^{a_i-1}j(j^{a_i-1} - 1) = 0 \Rightarrow p_i^{a_i-1}j = 0$.

Again for $u \in U(R_i)$ we have $p_i^{a_i-1}u(u-1) = 0 \Rightarrow p_i^{a_i-1}u = p_i^{a_i-1}$. Again $u^{p_i} = u^{p_i-1} \Rightarrow u^{p_i-1}(u^{p_i-1} - 1) = 0 \Rightarrow u^{p_i} - 1 = 1$. So $U(R_i)$ is group of exponent $(p_i - 1)p_i^{a_i-1}$.

Again if $a_i = 1$ then $p_i = 0$. Now if $k^2 = 0$ where $k = \sum_{i=1}^{n} e_i \Rightarrow k^{p_i} - k = p_i P\{e_i | 1 \geq i \geq n\} \Rightarrow k^{p_i} - k = 0 \Rightarrow k = 0$ where $P\{e_i | 1 \geq i \geq n\}$ is a polynomial in commuting $e_i, 1 \geq i \geq n$. So $R_i$ is reduced ring. So $R_i$ is subdirect product of the domains $\{R_\alpha\}$. But $R_\alpha$ has only trivial idempotents 0, 1. We infer that $R_\alpha = \{0, 1, 2, 3, \ldots, p_i - 1\}$ as $p_i = 0$ in $R_\alpha$. Hence $R_i$ is subdirect product of $Z_{p_i}$'s.

Similarly we can prove the results for $R_2, \ldots, R_m$.

Examples. In $SI^5$ we have $6! = 0 \Rightarrow 2^4 \times 3^2 \times 5 = 0$. $\prod_{\alpha \in \wedge} R_\alpha$, where $R_\alpha = Z_5$ is $SI^5$ ring. Similarly $\prod_{\alpha \in \wedge} R_\alpha \times \prod_{\beta \in \wedge} R_\beta$, where $R_\alpha = Z_5$ and $R_\beta = Z_7$ is a $SI^6$ ring.

Proposition 3.7. If the ring $R$ is subdirect product of $Z_p$ where $p$ is prime then every element of $R$ is a sum of $p - 1$ commuting idempotents.

Proof. Here $R$ is subdirect product of $\{R_\alpha : \alpha \in \wedge\}$ where $R_\alpha = Z_p$ for all $\alpha \in \wedge$. So $R$ is a subring of $\prod_{\alpha \in \wedge} R_\alpha$. Let $x = (x_\alpha) \in R$. Then $\wedge$ is a disjoint union of $\wedge_0, \wedge_1, \wedge_2, \ldots, \wedge_{p-2}, \wedge_{p-1}$ such that $x_\alpha = i$ if and only if $\alpha \in \wedge_i$ for $i = 0, 1, 2, 3, \ldots, p - 1$. Without loss of generality, we can denote $x = (0_{\wedge_0}, 1_{\wedge_1}, 2_{\wedge_0}, \ldots, i_{\wedge_i}, \ldots, (p - 1)_{\wedge_{p-1}})$. Now

\[
e_1 = (0_{\wedge_0}, 1_{\wedge_1}, 1_{\wedge_2}, \ldots, 1_{\wedge_i}, \ldots, 1_{\wedge_{p-1}}) \in R, \\
e_2 = (0_{\wedge_0}, 0_{\wedge_1}, 1_{\wedge_2}, \ldots, 1_{\wedge_i}, \ldots, 1_{\wedge_{p-1}}) \in R, \\
\vdots \\
e_{i-1} = (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_2}, \ldots, 1_{\wedge_{i-1}}, 1_{\wedge_i}, 1_{\wedge_{i+1}}, \ldots, 1_{\wedge_{p-1}}) \in R, \\
e_i = (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_2}, \ldots, 1_{\wedge_i}, 1_{\wedge_{i+1}}, \ldots, 1_{\wedge_{p-1}}) \in R, \\
e_{i+1} = (0_{\wedge_0}, 0_{\wedge_1}, 1_{\wedge_2}, \ldots, 0_{\wedge_i}, 1_{\wedge_{i+1}}, \ldots, 1_{\wedge_{p-1}}) \in R, \\
\vdots \\
e_{p-1} = (0_{\wedge_0}, 0_{\wedge_1}, 0_{\wedge_2}, \ldots, 0_{\wedge_i}, \ldots, 1_{\wedge_{p-1}}) \in R.
\]
Clearly we can see that \( e_i^2 = e_i, e_ie_j = e_je_i \forall 1 \leq i, j \leq p - 1 \) and \( x = e_1 + e_2 + ... + e_i + ... + e_{p-1} \). This shows every element of \( R \) is a sum of \( p - 1 \) commuting idempotents.

4. **Rings in which every element is sum of \( n_1 \) and difference of \( m_1 \) commuting idempotents / or sum of \( n_2 \) and difference of \( m_2 \) commuting idempotents / or........../or sum of \( n_t \) and difference of \( m_t \) commuting idempotents**

From the Corollary 3.2 we get if \( R \) is a \( SI^n \) ring then every element can be express as sum of \( l \) idempotents and difference of \( m \) idempotents which commute each other such that \( l + m = n \). i.e the elements of of a \( SI^n \) ring can be express as \( \pm e_1 \pm e_2 \pm e_3 \ldots \pm e_n \). For example if we take \( SI^3 \) ring \( R \) then it’s element can be express as \( \pm e_1 \pm e_2 \pm e_3 \).

But there is a difference, i.e if we take a ring in which every element can be express as sum of three commuting idempotents or/ sum of two and difference of one commuting idempotents or/ sum of one and difference of two commuting idempotents or/ difference of three commuting idempotents then it does not mean that it is \( SI^3 \) ring. Because if we take \( k \) is sum of 3 commuting idempotents then \( k + 1 \) may not sum of three commuting idempotents. Let’s first find the structure of this ring.

**Proposition 4.1.** If every element of a ring \( R \) is of the type \( e_1 + e_2 + e_3 \) or \( e_1 + e_2 - e_3 \) or \( e_1 - e_2 - e_3 \) or \( -e_1 - e_2 - e_3 \) where \( e_i(1 \leq i \leq 3) \) are commute each other then \( R \) satisfies

\[
(k - 3)(k - 2)(k - 1)k(k + 1)(k + 2)(k + 3) = 0
\]

for every \( k \in R \) and \( R \) satisfy the same properties as \( SI^6 \) ring as in the Proposition 3.4.

**Proof.** Using Proposition 3.2 we have When if \( k \in R \) with \( k = e_1 + e_2 + e_3 \), where \( e_i(1 \leq i \leq 3) \) are commute each other then

\[
(k - 3)(k - 2)(k - 1)k = 0 \quad (4.1)
\]

When if \( k \in R \) with \( k = e_1 + e_2 - e_3 \), where \( e_i(1 \leq i \leq 3) \) are commute each other then

\[
(k - 2)(k - 1)k(k + 1) = 0 \quad (4.2)
\]

When if \( k \in R \) with \( k = e_1 - e_2 - e_3 \), where \( e_i(1 \leq i \leq 3) \) are commute each other then

\[
(k - 1)k(k + 1)(k + 2) = 0 \quad (4.3)
\]

When if \( k \in R \) with \( k = -e_1 - e_2 - e_3 \), where \( e_i(1 \leq i \leq 3) \) are commute each other then

\[
k(k + 1)(k + 2)(k + 3) = 0 \quad (4.4)
\]

Combining all the equations (4.1),(4.2),(4.3),(4.4) we have

\[
(k - 3)(k - 2)(k - 1)k(k + 1)(k + 2)(k + 3) = 0
\]

which is nothing but the equation of a ring in which every element is sum of 3 and difference of 3 commuting idempotents. Now if \( k \in R \) then \( (k - 3) \in R \) so
Then we have 

So using Chinese Remainder Theorem

So

Now we going to prove the generalized version.

Proposition 4.2. If in a ring \( R \) every element is sum of \( n_1 \) and difference of \( m_1 \) commuting idempotents / or sum of \( n_2 \) and difference of \( m_2 \) commuting idempotents / or........./or sum of \( n_t \) and difference of \( m_t \) commuting idempotents. Then \( R \) satisfies

\[
(k - n)k - (n - 1)....\{k + (m - 1)\}(k + m) = 0
\]

where \( (k - n)k - (n - 1)....\{k + (m - 1)\}(k + m) = L.C.M\{\{k - (n_1)\}{k - (n_1 - 1)}\}...\{k + (m_1 - 1)\}{k + m_1}, (k - n_2)\{k - (n_2 - 1)}\}...\{k + (m_2 - 1)\}{k + m_2}, ......., (k - n_t)\{k - (n_t - 1)}\}...\{k + (m_t - 1)\}{k + m_t}.)

\( R \) is a \( SI^{m+n} \) ring and satisfies the properties as in the Proposition 3.6.

Proof. Using Proposition 3.2 we have If \( k \in R \) is sum of \( n_1 \) and difference of \( m_1 \) commuting idempotents then

\[
(k - n_1)\{k - (n_1 - 1)}\}...\{k + (m_1 - 1)\}(k + m_1) = 0
\]

If \( k \in R \) is sum of \( n_2 \) and difference of \( m_1 \) commuting idempotents then

\[
(k - n_2)\{k - (n_2 - 1)}\}...\{k + (m_2 - 1)\}(k + m_2) = 0
\]

................................................................................

If \( k \in R \) is sum of \( n_t \) and difference of \( m_t \) commuting idempotents then

\[
(k - n_t)\{k - (n_t - 1)}\}...\{k + (m_t - 1)\}(k + m_t) = 0
\]

................................................................................

If \( k \in R \) is sum of \( n_t \) and difference of \( m_t \) commuting idempotents then

\[
(k - n_t)\{k - (n_t - 1)}\}...\{k + (m_t - 1)\}(k + m_t) = 0
\]

Now if we take L.C.M of all the right hand side equations and letting \( (k - n)k - (n - 1)....\{k + (m - 1)\}(k + m) = L.C.M\{\{k - (n_1)\}{k - (n_1 - 1)}\}...\{k + (m_1 - 1)\}{k + m_1}, (k - n_2)\{k - (n_2 - 1)}\}...\{k + (m_2 - 1)\}{k + m_2}, ......., (k - n_t)\{k - (n_t - 1)}\}...\{k + (m_t - 1)\}{k + m_t}.

Then we have

\[
(k - n)k - (n - 1)....\{k + (m - 1)\}(k + m) = 0
\]

So \( R \) is a \( SI^{m+n} \) ring. Therefore \( (m + n + 1)! = 0 \). We get get the properties of the ring \( R \) using the Proposition 3.6. For example if if every element of a ring \( R \) is sum of 200 and difference of 100 commuting idempotents /or sum of 80 and difference of 500 idempotents then \( R \) is a \( SI^{200+500} = SI^{700} \) ring and 701! = 0 and we get the result of the ring using Proposition 4.2.
References


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