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EXISTENCE OF ENTIRE SOLUTIONS FOR k^{th} DERIVATIVE OF CERTAIN TYPE OF NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. The paper discusses the existence of entire solutions for k^{th} derivative of certain type of non-linear differential-difference equations. In addition to, the paper evaluates some previous results for existence of finite order transcendental entire solutions and extends to the higher order. Particularly, we obtain the conditions for occurrence and attainable forms of entire solutions.

1. Introduction and preliminaries

So far, we have seen the existence of entire solution of some non-linear differential-difference equations, difference polynomials, differential-difference polynomials and linear equations. But, the present paper reviews some research on non-linear differential-difference equations and work on the higher order derivative of these equations. We assume that the reader is well known the fundamentals of Nevanlinna theory([16], [2], [8]), such as the first and second fundamental theorems. We indicate the characteristic, proximity and counting function of f as $T(r,f), m(r,f), N(r,f)$ respectively. Also we have $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of any set E of finite logarithmic measure. We say that $b(z)$ is a small function of f if and only if $T(r,b) = S(r,b)$ and $\rho(f)$ is order of growth of f.

A theorem proposed by Tumura[6] was proved by Clunie[4] on non-linear differential equation is known as Tumura-Clunie theory has been studied by many researchers over the years. In 1964, Hayman[16] extended this theory by considering a non-linear differential equation

$$f^n + G_d(f) = H(z) \tag{1.1}$$

where d is a degree of differential polynomial $G_d(f)$. Sequentially, numerous studies have been done on (1.1). One can get from [12, 11, 10] in detail.

In 2006, Li and Yang[14] further investigated on (1.1), when $H(z)$ is of the form $r_1(z)e^{h_1(z)} + r_2(z)e^{h_2(z)}$ and proved the following.

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Theorem 1.1. [14] *Let $d \leq n-3 \quad (n \geq 4)$ be the degree of differential polynomial $G_d(f)$ and r_1, r_2 be non-zero polynomials and h_1, h_2 be non-zero constants with h_1/h_2 (not rational). Then,*

$$f^n + G_d(f) = r_1(z)e^{h_1(z)} + r_2(z)e^{h_2(z)} \tag{1.2}$$

has no transcendental entire solution.

Later on, Yang and Li[21], Li[13], and Liao et al.[11] worked on the solution of (1.2) for $n = 3, n \geq 2, n \geq 3, d \leq n-2$ respectively.

In 2011, Qi[17] improved the results of Yang and Laine[9] and obtained the existence of finite order entire solution of the following equation

$$f^n + h(z)(\Delta f(z+c))^m = R(z), \quad n > m \geq 0. \tag{1.3}$$

In 2012, Wen et al.[23] examined the results on finite order entire solution of the equation

$$f^n + h(z)e^{P(z)}f(z+c) = Q(z). \tag{1.4}$$

In 2014, Qi et al.[18] improved (1.3), when $R(z)$ is replaced by $p(z)e^{q(z)}$ and obtained the following.

Theorem 1.2. [18] *Let $m, n \in \mathbb{Z}^+, h(\neq 0), p, q$ are polynomials. Then a non-linear difference equation*

$$f^n + h(z)(\Delta f(z+c))^m = p(z)e^{q(z)}, n > m \tag{1.5}$$

has no entire solution.

Definition 1.3. [15] Let $b_1, b_2, \ldots b_n(\neq 0)$ be small functions of f with $n \geq 1$. Then $L(z,f)$ is defined as

$$L(z,f) = \sum_{i=0}^{n} b_i(z)f(z+c_i)(\neq 0)$$

where $c_0(\neq 0), c_i \in \mathbb{C} - \{0\}, i = 0, 1, 2, \ldots, n$.

In 2016, Dyavanal and Mathai[5] examined the results for (1.5), by considering $m = 1$ and $\Delta f(z+c)$ is replaced by $L(z,f)$.

In 2019, X. Qi and L. Yang[20] considered the existence of entire solutions of some differential-difference equations and proved the below theorem.

Theorem 1.4. [20] *Let p_1, p_2 be non-zero polynomials and q_1, q_2 be non-constant polynomials. If*

$$lim_{z\to\infty}\left|\frac{q_1}{q_2}\right| \neq \left(\frac{m_1}{m_2}\right)^{\pm 1}, \quad m_2 > m_1 + 2$$

then

$$f^{m_1} + (f'(z+c))^{m_2} = p_1e^{q_1} + p_2e^{q_2} \tag{1.6}$$

with finite order has no transcendental entire solution.

In 2021, T. Biswas and A. Banerjee[15] studied the existence of entire solution non-linear shift equation instead of non-linear difference equation in (1.5) to get the following.

Theorem 1.5. [15] *Consider L(z,f) as in definition(1.3) and let* $m, n \in \mathbb{Z}^+, h(\neq 0), \phi_1, \phi_2$ *be polynomials. Then, the non-linear shift equation*

$$f^n(z) + h(z)(L(z, f))^m = \phi_1(z)e^{\phi_2(z)}, n > m \tag{1.7}$$

has no transcendental entire solution of finite order.

In the above results one can observe that, left side of the equation along with f^n as a dominating term has a difference polynomial, linear combination of shifts, differential-difference polynomial, and a function with its derivative or shift. Hence it is natural to ask the question, what happens when the function is of k^{th} derivative? In this paper we tried to answer this question by proving the following results.

2. Lemmas

Lemma 2.1. [15] *Let* $c \in \mathbb{C}$ *and* f *be a finite order meromorphic function. Then,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.2. [15] *Consider a meromorphic function* $f_i(z), i = 1, 2, 3$ *satisfying* $\sum_{i=1}^{3} f_i(z) \equiv 1$. *If* f_1 *in not a constant and* $T(r) = max_{1\leq i\leq 3}T(r, f_i)$ *and*

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) + 2\sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f_i}\right) < (\alpha + o(1))T(r)$$

where $\alpha < 1$, *then* $f_2(z) \equiv 1$ *or* $f_3(z) \equiv 1$.

Lemma 2.3. [9, 7] *Let* f *be a finite order non-constant meromorphic solution of*

$$f^n P(z, f) = Q(z, f)$$

where $P(z, f), Q(z, f)$ *are difference polynomials in* f *with small meromorphic function as coefficients and let* $c \in \mathbb{C}, \delta < 1$. *If the total degree of* $Q(z, f)$ *as a polynomial in* f *and its shift are at most* n, *then*

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + oT(r, f))$$

for all r *outside a possible exceptional set with finite logarithmic measure.*

Lemma 2.4. *[[19], Corollary 1.6] Let* f *be a finite order transcendental entire function,* $m_1, m_2 \in \mathbb{Z}^+$ *so that* $m_1 \geq m_2 + 2$ *and* $G(z)$ *be a meromorphic function which satisfies* $\overline{N}(r, 1/G) = S(r, f)$. *Then* f *is not a solution of*

$$f(z + c)^{m_1} + (f'(z))^{m_2} = G(z).$$

Lemma 2.5. *[[1], Theorem 1] Let $m, n \in \mathbb{Z}^+$ satisfying $\frac{1}{m} + \frac{1}{n} < 1$, then $f(z)$ and $g(z)$ are not transcendental entire solutions which satisfy*

$$a_1(z)f(z)^n + a_2(z)g(z)^m = 1$$

where $a_1(z), a_2(z)$ are small functions of f and g.

Lemma 2.6. [22] *Let f be a non-constant meromorphic function and λ_1, λ_2 be two complex numbers such that $\lambda_1 \neq \lambda_2$. Let f be a meromorphic function with order σ, then for each $\epsilon > 0$*

$$m\left(r, \frac{f(z+\lambda_1)}{f(z+\lambda_2)}\right) = O(r^{\sigma-1+\epsilon}).$$

Lemma 2.7. [3] *If $f_k(z), 1 \leq k \leq m$ and $g_k(z), 1 \leq k \leq m, m \geq 2$ are entire functions that meet the below conditions:*
(i) $\sum_{k=1}^{m} f_k(z)e^{g_k(z)} \equiv 0$;
(ii) The orders of f_k are not more than that of $e^{g_i(z)-g_j(z)}$ for $1 \leq k \leq m, 1 \leq i < j \leq m$ then $f_k(z) \equiv 0$ for $1 \leq k \leq m$.

Lemma 2.8. [3] *Consider a non-zero meromorphic function f. Then*

$$m\left(r, \frac{f'}{f}\right) = O(logr)asr \to \infty$$

when f is of finite order and

$$m\left(r, \frac{f'}{f}\right) = O(logr(T(r,f)))asr \to \infty$$

possibly outside a set E of finite logarithmic measure of r, when f is of infinite order.

3. Main results

Theorem 3.1. *Let n, m be positive integers and α, β be polynomials. Then for $n > m$ a non-linear differential-difference equation*

$$f^n(z) + \left(f^{(k)}(z+c)\right)^m = \alpha(z)e^{\beta(z)} \tag{3.1}$$

has no transcendental finite order entire function $f(z)$ as a solution.

Proof. Let us consider the following two cases.
Case 1: Suppose $\beta(z)$ is a constant or $\alpha(z) = 0$ then (3.1) becomes,

$$f^n(z) + \left(f^{(k)}(z+c)\right)^m = R(z) \tag{3.2}$$

where $R(z)$ is a polynomial.
If f is a finite order transcendental entire solution of (3.2). For our reference we take $f(z+c) = f_c$

$$\begin{aligned} nT(r,f) = T(r,f^n) &= T\left(r, R(z) - \left(f_c^{(k)}\right)^m\right) \\ &\leq T(r,R(z)) + T\left(r, \left(f_c^{(k)}\right)^m\right) + S(r,f) \\ &\leq T(r,R(z)) + T\left(r, \left(\frac{f_c^{(k)}}{f_c}\right)^m\right) + T(r,f_c^m) + S(r,f) \\ &\leq mT(r,f) + S(r,f). \end{aligned}$$

Which becomes contradiction for $n > m$.

Case 2: Suppose $\beta(z)$ (non-constant) is a polynomial or $\alpha(z) \not\equiv 0$. Let $f(z)$ is a finite order transcendental entire solution of (3.1).
Differentiating (3.1)

$$nf^{n-1}f' + m\left(f_c^{(k)}\right)^{m-1} f_c^{(k+1)} = (\alpha' + \alpha\beta')e^{\beta}. \tag{3.3}$$

From (3.1) and (3.3) eliminate $e^{\beta(z)}$ then

$$f^{n-1}\left[nf' - \left(\frac{\alpha' + \alpha\beta'}{\alpha}\right) f\right] = \left(f_c^{(k)}\right)^m \left(\frac{\alpha' + \alpha\beta'}{\alpha}\right) - m\left(f_c^{(k)}\right)^{m-1} f_c^{(k+1)}. \tag{3.4}$$

If $nf' - \left(\frac{\alpha'+\alpha\beta'}{\alpha}\right) f \equiv 0$ then

$$f^n = \kappa\alpha e^{\beta} \tag{3.5}$$

where κ is a non-zero constant.
Let $H^n(z) = \kappa\alpha(z)$, then (3.5) will be

$$f(z) = H(z)e^{\frac{\beta}{n}}. \tag{3.6}$$

Using (3.5) in (3.1)

$$(\kappa - 1)\alpha e^{\beta} + \left(f_c^{(k)}\right)^m \equiv 0. \tag{3.7}$$

If $\kappa = 1$, then clearly $\left(f_c^{(k)}\right)^m \equiv 0$, a contradiction. Therefore $\kappa \not\equiv 1$.
Let $h(z) = e^{\frac{\beta}{n}}$, then (3.6) becomes, $f(z) = H(z)h(z)$.
Now, we can explicit $\left(f_c^{(k)}\right)^m$ as

$$\begin{aligned} \left(f_c^{(k)}\right)^m &= \sum_{r=0}^{N} \binom{N}{r} \prod_{t=0}^{m} \left(f_c^{(k)}\right)^t \\ &= \sum_{r=0}^{N} \binom{N}{r} \prod_{t=0}^{m} \left((H_c h_c)^{(k)}\right)^t. \end{aligned}$$

In addition, from Lemma 2.1 we have

$$\begin{aligned}
T\left(r,\left(f_c^{(k)}\right)^m\right) &= T\left(r,\quad \sum_{r=0}^{N}\binom{N}{r}\prod_{t=0}^{m}\left((H_ch_c)^{(k)}\right)^t\right)\\
&\leq m\left(r,\frac{\sum_{r=0}^{N}\binom{N}{r}\prod_{t=0}^{m}\left((H_ch_c)^{(k)}\right)^t}{(H_ch_c)^m}\right)+m(r,(H_ch_c)^m)+S(r,f)\\
&\leq m(r,h_c^m)+S(r,f)\\
&\leq T(r,h_c^m)+S(r,f)\\
&\leq mT(r,h)+S(r,f).
\end{aligned}$$

Hence

$$\begin{aligned}
nT(r,h) &= T(r,h^n)\\
&= T(r,e^{\beta})\\
&= T(r,(\kappa-1)\alpha e^{\beta})+S(r,f)\\
&\leq T\left(r,\left(f_c^{(k)}\right)^m\right)+S(r,f)\\
&\leq mT(r,h)+S(r,f).
\end{aligned}$$

Which contradicts $n>m$. Hence we can conclude that

$$nf'(z)-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f\not\equiv 0.$$

Next, we will have the following cases.

Case I: Let $n>m+1$, then from (3.4) we have

$$f^{n-m-1}\left[nf'-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f\right]=\frac{\left(f_c^{(k)}\right)^{m-1}}{f_c^m}\left[\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f_c^{(k)}-mf_c^{(k+1)}\right]$$

and

$$f^{n-m-2}\left[nff'-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f^2\right]=\frac{\left(f_c^{(k)}\right)^{m-1}}{f_c^m}\left[\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f_c^{(k)}-mf_c^{(k+1)}\right].$$

Whereas f is an entire function, from Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
T\left(r,\left(nf'-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f\right)\right) &= m\left(r,\left(nf'-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f\right)\right)+S(r,f)\\
&= S(r,f)
\end{aligned}$$

and

$$\begin{aligned}
T\left(r,f\left(nf'-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f\right)\right) &= m\left(r,f\left(nf'-\left(\frac{\alpha'+\alpha\beta'}{\alpha}\right)f\right)\right)+S(r,f)\\
&= S(r,f).
\end{aligned}$$

Hence $T(r,f)=S(r,f)$, which is contradiction.

Case II: Let $n=m+1$, then (3.1) and (3.4) respectively will become

$$f^{m+1}+\left(f_c^{(k)}\right)^m=\alpha e^{\beta}. \tag{3.8}$$

And

$$f^m\left[(m+1)f'-\left(\frac{\alpha'}{\alpha}+\beta'\right)f\right]=\left(f_c^{(k)}\right)^{m-1}\left[\left(\frac{\alpha'}{\alpha}+\beta'\right)f_c^{(k)}-mf_c^{(k+1)}\right]. \tag{3.9}$$

Denote

$$U(z)=(m+1)f'-\left(\frac{\alpha'}{\alpha}+\beta'\right)f. \tag{3.10}$$

As $f(z)$ is entire, by Lemmas 2.1 and 2.3 we can conclude that,

$$T(r,U)=m(r,U)+S(r,U)=S(r,U). \tag{3.11}$$

From (3.10) we have

$$U'(z)=(m+1)f''-\left(\frac{\alpha'}{\alpha}+\beta'\right)f'-\left(\frac{\alpha'}{\alpha}+\beta'\right)'f. \tag{3.12}$$

Using (3.10) and (3.12)

$$\begin{aligned}(m+1)f''-K_1f'-K_2f&=0,\\(m+1)\left[\left(\frac{f'}{f}\right)'+\left(\frac{f'}{f}\right)^2\right]-K_1\frac{f'}{f}-K_2f&=0.\end{aligned} \tag{3.13}$$

where $K_1=\left[\left(\frac{\alpha'}{\alpha}+\beta'\right)+(m+1)\frac{U'}{U}\right],\quad K_2=\left[\left(\frac{\alpha'}{\alpha}+\beta'\right)'-\left(\frac{\alpha'}{\alpha}+\beta'\right)\frac{U'}{U}\right].$
Suppose z_0 is a zero of f with multiplicity $\mathfrak{p}$ and also a zero of $\alpha(z)$, then $N\left(r,\frac{1}{f}\right)=S(r,f)$. Now will discuss the two subcases by assuming z_0 is not a zero of $\alpha(z)$.

Subcase 1: Suppose z_0 is a zero of $U(z)$ with multiplicity $\mathfrak{q}$.
Let $f(z)=(z-z_0)^{\mathfrak{p}}\psi(z),\psi(z_0)\neq 0$ and $U(z)=(z-z_0)^{\mathfrak{q}}\xi(z),\xi(z_0)\neq 0$.
From (3.10) we have

$$(z-z_0)^{\mathfrak{q}}\xi(z)=(z-z_0)^{\mathfrak{p}-1}\left[(m+1)\left(\mathfrak{p}\psi(z)+(z-z_0)\psi'(z)\right)-\left(\frac{\alpha'}{\alpha}+\beta'\right)(z-z_0)\psi(z)\right].$$

That is $\mathfrak{p}=1+\mathfrak{q}\leq 2\mathfrak{q}$, then by (3.11) implies that z_0 contribution to $N\left(r,\frac{1}{f}\right)$ is $S(r,f)$.

Subcase 2: Suppose z_0 is not a zero of U(z).

Then from (3.13) we have

$$\begin{aligned}&(m+1)\left[(\mathfrak{p}^2-\mathfrak{p})\psi+2\mathfrak{p}(z-z_0)\psi'+\psi''(z-z_0)^2\right]\\ -&\left[\frac{\alpha'}{\alpha}+\beta'+(m+1)\frac{U'}{U}\right]\left[\mathfrak{p}(z-z_0)\psi+(z-z_0)^2\psi'\right]\\ &-\left[\left(\frac{\alpha'}{\alpha}+\beta'\right)'-\left(\frac{\alpha'}{\alpha}+\beta'\right)\frac{U'}{U}\right](z-z_0)^2\psi=0. \end{aligned} \tag{3.14}$$

That is $\mathfrak{p}^2=\mathfrak{p}$, then it must be a simple zero of $f(z)$. Hence $f(z)$ will take the form $f(z)=(z-z_0)\psi(z)$. By lemma of logarithmic derivative we get

$$\frac{f'}{f}=\frac{1}{z-z_0}+\frac{\psi'}{\psi}.$$

And

$$N\left(r,\frac{\psi'}{\psi}\right)=S(r,\psi)=S(r,f),$$

as left side has simple pole at $z=z_0$.
From (3.13) we have

$$\begin{aligned}\left[(m+1)\left(\frac{\psi''}{\psi}-\frac{U'}{U}\frac{\psi'}{\psi}\right)-\left(\frac{\alpha'}{\alpha}+\beta'\right)\frac{\psi'}{\psi}\right](z-z_0)-\left[\frac{\alpha'}{\alpha}+\beta'+(m+1)\frac{U'}{U}\right]\\ -\left[\left(\frac{\alpha'}{\alpha}+\beta'\right)'-\left(\frac{\alpha'}{\alpha}+\beta'\right)\frac{U'}{U}\right](z-z_0)+2(m+1)\frac{\psi'}{\psi}=0.\end{aligned}$$

From above we have

$$\begin{aligned}N(r,0:f)&\leq N\left(r,0:2(m+1)\frac{\psi'}{\psi}-\left(\frac{\alpha'}{\alpha}+\beta'+(m+1)\frac{U'}{U}\right)\right),\\ &\leq N\left(r,0:\frac{\psi'}{\psi}\right)+N\left(r,0:\left(\frac{\alpha'}{\alpha}+\beta'+(m+1)\frac{U'}{U}\right)\right),\\ &\leq T\left(r,\frac{\psi'}{\psi}\right)+S(r,f),\\ &=S(r,\psi)+S(r,f)=S(r,f).\end{aligned}$$

From the above two cases we get $N(r,\frac{1}{f})=S(r,f)$. Hence by Hadamard factorization theorem, let $f(z)=G(z)e^{\nu(z)}$, where $\nu(z)$ is a non-constant polynomial and $G(z)$ being an entire function satisfies $N\left(r,\frac{1}{G}\right)=T(r,G)=S(r,f)$. Substitute this in (3.8)

$$G^{m+1}(z)e^{(m+1)\nu(z)}+\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^m=\alpha e^{\beta}$$

$$G^{m+1}(z)e^{(m+1)\nu(z)}+\sum_{r=0}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t=\alpha e^{\beta}$$

$$G^{m+1}(z)e^{(m+1)\nu(z)}+\sum_{r=0,r\neq N}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t+[G(z)e^{\nu(z)}]^m=\alpha e^{\beta}.$$

Divide completely by $-[G(z)e^{\nu(z)}]^m$,

$$-G(z)e^{\nu(z)}+\frac{\sum_{r=0,r\neq N}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t}{-[G(z)e^{\nu(z)}]^m}+\frac{\alpha e^{\beta}}{[G(z)e^{\nu(z)}]^m}=1.$$

In view of Lemma 2.2, we have

$$f_2(z)=\frac{\alpha e^{\beta}}{[G(z)e^{\nu(z)}]^m}=1$$

and

$$f_3(z)=\frac{\sum_{r=0,r\neq N}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t}{-[G(z)e^{\nu(z)}]^m}=1.$$

From $f_2(z)=1$ we have $\alpha e^{\beta}=[G(z)e^{\nu(z)}]^m$.

From (3.8) and using Lemma 2.1 we get

$$\begin{aligned}(m+1)T(r,f)&=T\left(r,f^{(m+1)}\right)\\&=T\left(r,\alpha e^{\beta}-\left(f_c^{(k)}\right)^m\right)\\&=T\left(r,(Ge^{\nu})^m-\left(f_c^{(k)}\right)^m\right)\\&=T\left(r,f^m\right)+T\left(r,\frac{\left(f_c^{(k)}\right)^m}{f^m}\right)+S(r,f)\\&\leq mT(r,f)+S(r,f)\end{aligned}$$

a contradiction.

From $f_3(z)=1$, we have $\frac{\sum_{r=0,r\neq N}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t}{-[G(z)e^{\nu(z)}]^m}=1$

$$\Rightarrow \frac{1}{-[G(z)e^{\nu(z)}]^m}\left[\sum_{r=0}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t-\left(G(z)e^{\nu(z)}\right)^m\right]=1$$

$$\Rightarrow \frac{1}{-[G(z)e^{\nu(z)}]^m}\left[\sum_{r=0}^{N}\binom{N}{r}\prod_{t=0}^{m}\left[\left(G(z+c)e^{\nu(z+c)}\right)^{(k)}\right]^t\right]+1=1$$

$$\Rightarrow \frac{1}{-[G(z)e^{\nu(z)}]^m}\left(f_c^{(k)}\right)^m=0$$

$$\Rightarrow \left(f_c^{(k)}\right)^m=0.$$

Which is contradiction. Hence, any finite order transcendental entire function is not a solution of (3.1). □

Theorem 3.2. *Let α_1,α_2 be non-zero polynomials and β_1,β_2 are two non-constant polynomials. Let $m,n\in\mathbb{Z}^+$, with $n>m+2$. Then*

$$f^n(z)+\left(f^{(k)}(z+c)\right)^m=\alpha_1(z)e^{\beta_1(z)}+\alpha_2(z)e^{\beta_2(z)} \tag{3.15}$$

has no transcendental finite order entire function.

Proof. Let the given equation (3.15) can be written as,

$$f^n + \left(f_c^{(k)}\right)^m = \alpha_1 e^{\beta_1} + \alpha_2 e^{\beta_2}. \tag{3.16}$$

Let $f(z)$ be a finite order transcendental entire solution of (3.16). We consider the following two case.

Case 1: If $\beta_1(z) - \beta_2(z)$ is a constant, then (3.15) becomes

$$f^n + \left(f_c^{(k)}\right)^m = R_1 e^{\beta_1} \tag{3.17}$$

where $R_1(z)$ is a polynomial. From Lemma 2.4, we get a contradiction when $R_1(z) \not\equiv 0$. And for $R_1(z) \equiv 0$, (3.17) takes the form

$$f^n + \left(f_c^{(k)}\right)^m = 0. \tag{3.18}$$

Since $f(z)$ is entire, then from (3.18)

$$\begin{aligned} nT(r, f) = T(r, f^n) &= T\left(r, \left(f_c^{(k)}\right)^m\right) \\ &\leq m\left(r, \left(\frac{f_c^{(k)}}{f}\right)^m\right) + m(r, f^m) + S(r, f) \\ &\leq mm(r, f) + S(r, f) \\ &\leq mT(r, f) + S(r, f). \end{aligned}$$

Which is a contradiction to our assumption $n > m + 2$.

Case 2: If $\beta_1(z) - \beta_2(z)$ is not constant.

Differentiating (3.16) on both sides.

$$nf^{n-1}f' + m\left(f_c^{(k)}\right)^{m-1} f_c^{(k+1)} = (\alpha_1' + \alpha_1\beta_1')e^{\beta_1} + (\alpha_2' + \alpha_2\beta_2')e^{\beta_2}. \tag{3.19}$$

Eliminating e^{β_2} from (3.16) and (3.19).

$$(\alpha_2' + \alpha_2\beta_2')f^n - n\alpha_2 f^{n-1}f' + D_1(f) = Ae^{\beta_1} \tag{3.20}$$

where

$$D_1(f) = (\alpha_2' + \alpha_2\beta_2')\left(f_c^{(k)}\right)^m - \alpha_2 m\left(f_c^{(k)}\right)^{m-1} f_c^{(k+1)}. \tag{3.21}$$

And

$$A = \alpha_1(\alpha_2' + \alpha_2\beta_2') - \alpha_2(\alpha_1' + \alpha_1\beta_1'). \tag{3.22}$$

Differentiating (3.20)

$$\begin{aligned} (\alpha_2' + \alpha_2\beta_2')'f^n + (\alpha_2' + \alpha_2\beta_2')nf^{n-1}f' - n(n-1)\alpha_2 f^{n-2}(f')^2 - n\alpha_2 f^{n-1}f'' \\ + (D_1(f))' = (A' + A\beta_1')e^{\beta_1}. \end{aligned} \tag{3.23}$$

Eliminating e^{β_1} from (3.20) and (3.23)

$$\begin{aligned} \left[(A' + A\beta_1')(\alpha_2' + \alpha_2\beta_2') - A(\alpha_2' + \alpha_2\beta_2')'\right] f^n - \left[n(\alpha_2 A)' + n\alpha_2 A(\beta_1' + \beta_2')\right] f^{n-1}f' \\ + A\alpha_2 n\left[(n-1)(f')^2 + ff''\right] f^{n-2} = R_2(f). \end{aligned} \tag{3.24}$$

Where

$$R_2(f) = A(D_1(f))' - (A' + A\beta_1')D_1(f) \tag{3.25}$$

is a differential-difference polynomials in $f(z)$.
Let

$$D=\left[(A'+A\beta_1')(\alpha_2'+\alpha_2\beta_2')-A(\alpha_2'+\alpha_2\beta_2')'\right]f^2-\left[n(\alpha_2 A)'+n\alpha_2 A(\beta_1'+\beta_2')\right]ff' + A\alpha_2 n\left[(n-1)(f')^2+ff''\right]. \tag{3.26}$$

Then, from (3.24), (3.26) and Lemma 2.3, we get

$$T(r,D)=m(r,D)=S(r,f),\quad m(r,fD)=S(r,f).$$

If $D\not\equiv 0$ then

$$T(r,f)=m(r,f)\le m(r,fD)+m(r,\frac{1}{D})=S(r,f)$$

which is a contradiction. And if $D\equiv 0$ then $T(f)\equiv 0$.
Now, we will have the following two cases.

Subcase 2.1: If $D_1(f)\not\equiv 0$, we claim $A\not\equiv 0$ or else, from (3.22) we get

$$\varpi\alpha_2 e^{\beta_2}=\alpha_1 e^{\beta_1} \tag{3.27}$$

where $\varpi\not\equiv 0$ is a constant. Using (3.27) in (3.16), we get

$$f^n+\left(f_c^{(k)}\right)^m=(1+\varpi)\alpha_2 e^{\beta_2}. \tag{3.28}$$

If $\varpi\neq -1$, then from Lemma 2.4, we get a contradiction.
If $\varpi=-1$, then from (3.28)

$$f^n+\left(f_c^{(k)}\right)^m=0. \tag{3.29}$$

As in the above argument, we get a contradiction as well.
Hence, $A\not\equiv 0$. In addition, from (3.25)

$$D_1(f)=\Gamma A e^{\beta_1} \tag{3.30}$$

where $\Gamma\neq 0$ is a constant. Using (3.30) in (3.20), we get

$$f^{n-1}\left[(\alpha_2'+\alpha_2\beta_2')f-n\alpha_2 f'\right]=\left(\frac{1-\Gamma}{\Gamma}\right)D_1(f). \tag{3.31}$$

Denote

$$D_2=(\alpha_2'+\alpha_2\beta_2')f-n\alpha_2 f'. \tag{3.32}$$

Then, from (3.31), (3.32) and Lemma 2.3, we get

$$T(r,D_2)=m(r,D_2)=S(r,f),\quad m(r,fD_2)=S(r,f).$$

If $D_2\not\equiv 0$, then

$$T(r,f)=m(r,f)\le m(r,fD_2)+m(r,\frac{1}{D_2})=S(r,f)$$

which is a contradiction. Hence, $D_2\equiv 0$ and $\Gamma=1$.
Therefore

$$f^n=\varsigma\alpha_2 e^{\beta_2},\quad D_1(f)=Ae^{\beta_1} \tag{3.33}$$

where ς is a non-zero constant. Bringing together (3.16) and (3.33) one can get

$$\left(\frac{\varsigma-1}{\varsigma}\right)f^n=\frac{\alpha_1}{A}D_1(f)-\left(f_c^{(k)}\right)^m.$$

From this and Lemma 2.3, together gives that $\varsigma = 1$.
Hence

$$f^n = \alpha_2 e^{\beta_2} \tag{3.34}$$

$$\left(f_c^{(k)}\right)^m = \alpha_1 e^{\beta_1} \tag{3.35}$$

From (3.34), we get

$$f_c^{(k)} = h(z) e^{\frac{\beta_2(z+c)}{n}} \tag{3.36}$$

where $h(z) = \left[\alpha_2(z+c)^{\frac{1}{n}} \left(e^{\frac{\beta_2(z+c)}{n}}\right)^{(k)} + \left(\alpha_2(z+c)^{\frac{1}{n}}\right)^{(k)} e^{\frac{\beta_2(z+c)}{n}}\right]$ is the k^{th} derivative of differential-difference polynomial.
Hence

$$\left(f_c^{(k)}\right)^n = h^n(z) e^{\beta_2(z+c)}$$

which implies that

$$nT(r, f_c^{(k)}) = \frac{|\sigma_{\beta_2}|}{\pi} |z|^{deg(\beta_2)} (1 + o(1)), |z| \to \infty \tag{3.37}$$

where σ_{β_2} is the coefficient of the highest term of $\beta_2(z)$. On other side, from (3.35), it implies that

$$mT(r, f_c^{(k)}) = \frac{|\Upsilon_{\beta_1}|}{\pi} |z|^{deg(\beta_1)} (1 + o(1)), |z| \to \infty \tag{3.38}$$

where Υ_{β_1} is the coefficient of the highest term of $\beta_1(z)$. From (3.37) and (3.38), we have

$$\frac{m}{n} = \frac{|\Upsilon_{\beta_1}|}{|\sigma_{\beta_2}|} |z|^{deg(\beta_1) - deg(\beta_2)}. \tag{3.39}$$

If $deg(\beta_1) \neq deg(\beta_2)$, then by (3.39), we get a contradiction as $|z| \to \infty$.
If $deg(\beta_1) = deg(\beta_2)$, then from (3.39), we get

$$\frac{m}{n} = lim_{z \to \infty} \left| \frac{\beta_1(z)}{\beta_2(z)} \right|$$

which is a contradiction.

Subcase 2.2: If $D_1(f) \equiv 0$, then from (3.21), we have

$$\frac{\alpha_2' + \alpha_2 \beta_2'}{\alpha_2} = \frac{\left[\left(f_c^{(k)}\right)^m\right]'}{\left(f_c^{(k)}\right)^m}.$$

Hence

$$\left(f_c^{(k)}\right)^m = c\alpha_2 e^{\beta_2} \tag{3.40}$$

where $c \neq 0$ is a constant. Using (3.40) in (3.16), we get

$$f^n + (c-1)\alpha_2 e^{\beta_2} = \alpha_1 e^{\beta_1}$$

that is

$$\frac{1}{\alpha_1} \left(f e^{\frac{-\alpha_1}{n}}\right)^n + \frac{(c-1)\alpha_2}{\alpha_1} e^{\left(\frac{\beta_2 - \beta_1}{t}\right)^t} = 1$$

where $t \geq 3$. $\beta_2 - \beta_1$ is not a constant, then from Lemma 2.5,$c = 1$. Hence, one can have,

$$f^n = \alpha_1 e^{\beta_1}$$

and

$$\left(f_c^{(k)}\right)^m = \alpha_2 e^{\beta_2}.$$

Similar to above case, we get a contradiction. □

Theorem 3.3. *Let $n \geq 2$ be an integer and $\mathfrak{L}(z, f) = \sum_{i=1}^{l} a_i(z) f(z + \lambda_i)$ with $a_1, a_2, \ldots, a_n (\neq 0)$ be small functions of f and $\lambda_i \in \mathbb{C}, i = 0, 1, 2, \ldots, n, \lambda_0 \neq 0$. Let P_1, P_2, Q_1, Q_2, q be polynomials such that $deg(Q_1) \neq deg(Q_2)$. If f is a transcendental finite order entire solution of the equation:*

$$f^n(z) + e^{q(z)} \mathfrak{L}^{(k)}(z, f) = P_1(z) e^{Q_1(z)} + P_2(z) e^{Q_2(z)} \tag{3.41}$$

then the following holds:
(i) If $deg(Q_1) < deg(Q_2)$ and $\rho(f) = deg(Q_1)$ then every solution of f satisfies $\rho(f) < max\{deg(Q_1), deg(Q_2)\} = deg(q)$.
(ii) If $deg(Q_1) < deg(Q_2)$ and $\rho(f) \geq deg(Q_2)$ then every solution of f satisfies $\rho(f) = deg(q) \geq max\{deg(Q_1), deg(Q_2)\}$.
Also, for $deg(Q_2) < deg(Q_1), \rho(f) \geq deg(Q_1)$.

Proof. Suppose $f(z)$ be a transcendental entire solution of finite order to equation (3.41). In order to prove the theorem, we consider the following two cases.

Case 1: If $\rho(f) < max\{deg(Q_1), deg(Q_2)\}$ then from (3.41) and Lemma 2.6 it follows that

$$\begin{aligned}
T(r, e^q) = m(r, e^q) &= m\left(r, \frac{P_1 e^{Q_1} + P_2 e^{Q_2} - f^n}{\mathfrak{L}^{(k)}}\right) \\
&\leq m(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + m(r, f^n) + m\left(r, \frac{1}{\mathfrak{L}^{(k)}}\right) + O(1) \\
&\leq T(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + nT(r, f) + T(r, \mathfrak{L}^{(k)}) - N\left(r, \frac{1}{\mathfrak{L}^{(k)}}\right) + S(r, f) \\
&\leq T(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + nT(r, f) + T\left(r, \frac{\mathfrak{L}^{(k)}}{\mathfrak{L}}\right) \\
&+ T(r, \mathfrak{L}) - N\left(r, \frac{1}{\mathfrak{L}}\right) - k\overline{N}(r, \mathfrak{L}) + S(r, f) \\
&\leq T(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + nT(r, f) + m\left(r, \frac{\mathfrak{L}^{(k)}}{\mathfrak{L}}\right) + N\left(r, \frac{\mathfrak{L}^{(k)}}{\mathfrak{L}}\right) \\
&+ T(r, f) - N\left(r, \frac{1}{\mathfrak{L}}\right) + S(r, f) \\
&\leq T(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + (n+1)T(r, f) + N(r, \mathfrak{L}^{(k)}) + S(r, f) \\
&\leq T(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + (n+1)T(r, f) + S(r, f).
\end{aligned}$$

Therefore $T(r,e^q) \le T(r, P_1e^{Q_1}+P_2e^{Q_2})+S(r,f)$.
This implies

$$deg(q) \le max\{deg(Q_1), deg(Q_2)\} \tag{3.42}$$

Simultaneously, from Lemma 2.6 and (3.41) we have

$$\begin{aligned} T(r, P_1e^{Q_1}+P_2e^{Q_2}) &= m(r, P_1e^{Q_1}+P_2e^{Q_2}) \\ &= m(r, f^n + e^q\mathfrak{L}^{(k)}) \\ &\le m(r, f^n) + m(r, e^q) + m(r, \mathfrak{L}^{(k)}) + O(1) \\ &\le nT(r,f) + T(r,e^q) + T(r,f) + S(r,f) \\ &\le (n+1)T(r,f) + T(r,e^q) + S(r,f) \\ &\le T(r,e^q) + S(r,f). \end{aligned}$$

This implies

$$max\{deg(Q_1), deg(Q_2)\} \le deg(q). \tag{3.43}$$

From (3.42) and (3.43)

$$max\{deg(Q_1), deg(Q_2)\} = deg(q) \quad \text{and} \quad \rho(f) < deg(q).$$

Let us rewrite the (3.41) as

$$f^n + e^q\mathfrak{L}^{(k)} = P_1e^{Q_1} + P_2e^{Q_2}. \tag{3.44}$$

Differentiating (3.44)

$$nf^{n-1}f' + e^q[q'\mathfrak{L}^{(k)} + \mathfrak{L}^{(k+1)}] = [P_1Q_1' + P_1']e^{Q_1} + [P_2Q_2' + P_2']e^{Q_2}. \tag{3.45}$$

Eliminating e^{Q_1} and e^{Q_2} from (3.44) and (3.45)

$$A_1f^n - nf^{n-1}f' + (A_1 - A_0)\mathfrak{L}^{(k)}e^q = P_2(A_2 - A_1)e^{Q_2}. \tag{3.46}$$

$$A_2f^n - nf^{n-1}f' + (A_2 - A_0)\mathfrak{L}^{(k)}e^q = P_1(A_2 - A_1)e^{Q_1}. \tag{3.47}$$

Where, $A_0 = q' + \frac{\mathfrak{L}(k+1)}{\mathfrak{L}^{(k)}}$, $\quad A_1 = \frac{P_1'}{P_1} + Q_1'$ and $\quad A_2 = \frac{P_2'}{P_2} + Q_2'$.

Since $deg(Q_1) \ne deg(Q_2)$. Clearly $A_2 - A_1 \ne 0$. We have $deg(Q_1) = deg(Q_2)$ and $deg(Q_1) = \rho(f)$.
Differentiating (3.46) and eliminating e^{Q_2}

$$B_1 + B_2e^q = 0. \tag{3.48}$$

Where
$B_1 = f^{n-2}\left[(A_1' - A_1A_3)f^2 + (A_1 + A_3)nff' - n(n-1)(f')^2 - nff''\right]$

$B_2 = \left[\frac{(A_1-A_0)'}{A_1-A_0} + q' + \frac{\mathfrak{L}^{(k+1)}}{\mathfrak{L}^{(k)}} - A_3\right](A_1 - A_0)\mathfrak{L}^{(k)}$,

$A_3 = \frac{P_2'}{P_2} + \frac{(A_2-A_1)'}{A_2-A_1} + Q_2'$.

Since $\rho(f) < deg(q)$, then by Lemma 2.7 and (3.48) we get $B_1 \equiv B_2 \equiv 0$. Therefore $B_2 \equiv 0$. We must have neither $A_1 - A_0 = 0$ nor $\frac{(A_1-A_0)'}{A_1-A_0} + q' + \frac{\mathfrak{L}^{(k+1)}}{\mathfrak{L}^{(k)}} - A_3 = 0$.

Subcase 1.1: Suppose $A_1 - A_0 = 0$, then on integrating,

$$P_1e^{Q_1} = C_1\mathfrak{L}^{(k)}e^q, \quad C_1 \neq 0 \quad \text{a constant.} \tag{3.49}$$

If $C_1 \equiv 1$, then using (3.49) in (3.44),

$$f^n = P_2e^{Q_2}.$$

Since $\rho(f) < deg(Q_2)$. Which is a contradiction.

If $C_1 \not\equiv 1$, then from (3.49) and (3.44)

$$f^n + (1 - C_1)\mathfrak{L}^{(k)}e^q = P_2e^{Q_2}. \tag{3.50}$$

Differentiate (3.50) to eliminate e^{Q_2},

$$nf^{n-1}f' + (1 - C_1)\mathfrak{L}^{(k+1)}e^q + (1 - C_1)\mathfrak{L}^{(k)}e^q q' = (P_2' + P_2Q_2')e^{Q_2}. \tag{3.51}$$

$$A_2f^n - nf^{n-1}f' + (A_2 - A_0)(1 - C_1)\mathfrak{L}^{(k)}e^q = 0. \tag{3.52}$$

This can be written as

$$B_3 + B_4e^q = 0$$

where, $B_3 = f^{n-1}(A_2f - nf')$, $\quad B_4 = (A_2 - A_0)(1 - C_1)\mathfrak{L}^{(k)}$.
Similar to subcase 1.1, we get $B_3 \equiv B_4 \equiv 0$. From $B_4 \equiv 0$, we must have $A_2 - A_0 = 0$. Since $C_1 \neq 1$ and $\mathfrak{L}^{(k)} \neq 0$, $\quad A_1 = A_2$. Hence, it becomes contradictory to $A_2 - A_1 \neq 0$.

Subcase 1.2: Suppose $\frac{(A_1-A)'}{A_1-A_0} + q' + \frac{\mathfrak{L}^{(k+1)}}{\mathfrak{L}^{(k)}} - A_3 = 0$, then

$$(A_1 - A_0)\mathfrak{L}^{(k)}e^q = C_2P_2(A_2 - A_1)e^{Q_2}, \quad C_2 \neq 0. \tag{3.53}$$

We claim that $C_2 = 1$, otherwise from (3.53) we have

$$\mathfrak{L}^{(k)}(z) = H(z)e^{u(z)} \tag{3.54}$$

where, $H(z) = \left(\frac{A_2(z-c)-A_1(z-c)}{A_1(z-c)-A_0(z-c)}\right) P_2(z - c)C_2$ and $u(z) = Q_2(z - c) - q(z - c)$.
Since $C_2 \neq 1$ we have $deg(u) = deg(Q_2)$, which is contradiction. Therefore $C_2 = 1$.
Substituting (3.54) in (3.46) we get

$$f^n = C_3P_1e^{Q_1}, \quad C_3 \neq 0. \tag{3.55}$$

We claim $C_3 = 1$. Otherwise, using (3.55) in (3.44) and after simplification we get,

$$e^q\mathfrak{L}^{(k)} - P_2e^{Q_2} = (1 - C_3)P_1e^{Q_1}. \tag{3.56}$$

Since $deg(Q_2) = deg(q) > deg(Q_1)$ and by Lemma 2.7 we get $(1 - C_3)P_1 = 0$. Since $P_1 \neq 0$, therefore we must have $C_3 = 1$.
Similarly we can prove another case as well.

Case 2: If $\rho(f) > max\{deg(Q_1), deg(Q_2)\}$, it follows from Lemma 2.6 and eqn (3.44) that,

$$\begin{aligned} T(r,e^q) &= m(r,e^q) \\ &= m\left(r, \frac{P_1e^{Q_1}+P_2e^{Q_2}-f^n}{\mathfrak{L}^{(k)}}\right)+S(r,f) \\ &\le m(r,e^{Q_1})+m(r,e^{Q_2})+(n+1)m(r,f)+S(r,f) \\ &\le (n+1)T(r,f)+S(r,f). \end{aligned}$$

Which intend to $deg(q) \le \rho(f)$.
Now, we will prove $deg(q) = \rho(f)$. Otherwise, if $deg(q) < \rho(f)$, we will exhibit the contradiction. Denote $M_1(z) = \mathfrak{L}^{(k)}e^q$ and $M_2(z) = P_1e^{Q_1} + P_2e^{Q_2}$ then $T(r, M_2) = S(r, f)$ and $T(r, M_1) = S(r, f)$. Equation (3.44) becomes $f^n = M_2(z)-M_1(z)$ and from Lemma 2.3 we get $m(r, f) = S(r, f)$ and $N(r, f) = S(r, f)$. Therefore $T(r, f) = S(r, f)$, Which is a contradiction.
$\therefore deg(q) = \rho(f) > max\{deg(Q_1), deg(Q_2)\}$.

Case 3: If $\rho(f) = max\{deg(Q_1), deg(Q_2)\}$, it follows from Lemma 2.6 and (3.44)

$$\begin{aligned} T(r,e^q) = m(r,e^q) &= m\left(r, \frac{P_1e^{Q_1}+P_2e^{Q_2}-f^n}{\mathfrak{L}^{(k)}}\right)+S(r,f) \\ &\le m\left(r, \frac{1}{\mathfrak{L}^{(k)}}\right)+m(r,P_1e^{Q_1}+P_2e^{Q_2})+m(r,f^n)+O(1) \\ &\le (n+1)T(r,f)+T(r,e^{Q_1})+T(r,e^{Q_2})+S(r,f). \end{aligned}$$

Which implies that

$$deg(q) \le \rho(f).$$

We now prove $deg(q) = \rho(f)$. Otherwise, if $deg(q) < \rho(f)$ and denoting $M_3(z) = \mathfrak{L}^{(k)}e^q$ then $T(r, M_3) = S(r, f)$ and (3.44) becomes

$$f^n + M_3 = P_1e^{Q_1} + P_2e^{Q_2}. \tag{3.57}$$

Differentiating (3.57) and eliminating e^{Q_1} and e^{Q_2} we get

$$A_1f^n - nf^{n-1}f' + G_1(z,f) = P_2A_3e^{Q_2} \tag{3.58}$$

$$A_2f^n - nf^{n-1}f' + G_2(z,f) = -P_1A_3e^{Q_1} \tag{3.59}$$

where $A_3 = A_1 - A_2$, $\quad G_1(z, f) = A_2M_3 - M_3'$ and $G_2(z, f) = A_1M_3 - M_3'$.
Again differentiating (3.58) and eliminating e^{Q_2}, we get

$$f^{n-2}\phi(z) = G_3(z,f) \tag{3.60}$$

where $G_3(z, f) = G_1A_4 - G_1'$, $\quad A_4 = \left(\frac{A_3'}{A_3} + A_2\right)$ and $\phi(z) = (A_1' - A_1A_4)f^2 + n(A_1-A_4)ff'-n(n-1)(f')^2-nff''$. Suppose $G_3 = 0$ then we have $G_1A_4-G_1' = 0$. If $G_1 = 0$ then $\mathfrak{L}^{(k)}e^q = C_4P_1e^{Q_1}$ and $degQ_1 = \rho(f)$. Since $deg(Q_2) = \rho(f) >$

$deg(Q_1)$, which is a contradiction. Therefore $G_3 \neq 0$ then we have $G_1A_4 - G_1' = 0 \Rightarrow G_1 = C_5A_3P_2e^{Q_2}$. Substituting this in (3.58)

$$f^{n-1}(A_1f - nf') = \left(\frac{1}{C_6} - 1\right) G_1, \quad C_6 \neq 0.$$

We get $A_1f - nf' = 0 \Rightarrow f^n = C_7P_1e^{Q_1}$. $C_7 \neq 0$ and $\rho(f) = deg(Q_1)$, which is a contradiction.
$\therefore G_3(z, f) \neq 0$ and it follows that $\phi(z) \neq 0$.
Consider

$$\phi(z) = l_1f^2 + l_2ff' + l_3(f')^2 + l_4ff'' \tag{3.61}$$

where, $l_1 = A_1' - A_1A_4$, $l_2 = n(A_1 - A_4)$, $l_3 = -n(n-1)$, $l_4 = -n$ and l_1, l_2 be meromorphic functions that are non-zero with $T(r, l_i) = S(r, f), i = 1, 2$.
We now consider the following cases.

Subcase 3.1: If f has finite number of zeros then, f is of the form $f(z) = S_1(z)e^{S_2(z)}$, where S_1, S_2 are polynomials, $S_1 \neq 0$ and $deg(S_1) = deg(Q_2), deg(S_2) > deg(q)$. Substituting $f(z)$ in (3.58), we get

$$\left[A_1S_1^n - nS_1^{n-1}(S_1' + S_1S_2')\right] e^{nS_2} + G_1(z, f) = A_3P_2e^{Q_2}. \tag{3.62}$$

If $A_1S_1^n - nS_1^{n-1}(S_1' + S_1S_2') = 0$ then $C_8P_1e^{Q_1} = S_1^ne^{nS_2}$, $c_8 \neq 0$ and since $deg(Q_1) < deg(S_2)$, it follows from Lemma 2.7 that $P_1 = 0$ which is illogical. Therefore $A_1S_1^n - nS_1^{n-1}(S_1' + S_1S_2') \neq 0$.
Suppose

$$S_2(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$$
$$\alpha_2 = b_nz^n + b_{n-1}z^{n-1} + \cdots + b_0$$

where $a_i, b_i 0 \leq i \leq n$ are constants and $a_nb_n \neq 0$.

$$S_1^{n-1}\left[A_1S_1 - n(S_1' + S_1S_2')\right] e^{na_n - b_nz^n + \cdots na_0 - b_0} + \left[A_1\mathfrak{L}^{(k)}e^q - (\mathfrak{L}^{(k)}e^q)'\right] \cdot e^{a_n - b_nz^n + \cdots + (a_0 - b_0)} = A_3P_2.$$

From (3.62), we get a contradiction.

Subcase 3.2 If f has infinitely many zeros, then from (3.61) we get simple zeros of f are infinite. On differentiating (3.61) we get

$$\phi'(z) = l_1'f^2 + (2l_1 + l_2')ff' + l_2(f')^2 + (2l_3 + l_4)f'f'' + l_2ff'' + l_4ff'''. \tag{3.63}$$

From (3.61) and (3.63), we obtain

$$\begin{aligned} &f'\left[(l_2\phi - l_3\phi')f' + \phi(2l_3 + l_4)f''\right] \\ &= f\left[(\phi'l_1 - \phi l_1')f + (\phi'l_2 - 2l_1\phi - \phi l_2')f' + (\phi'l_4 - l_2\phi)f'' - \phi l_4f''\right] \end{aligned} \tag{3.64}$$

If f has simple zero at z_0 and not the zero and pole of the coefficients of (3.64). Substituting z_0 in (3.64), we note that z_0 is zero of $(l_2\phi - l_3\phi')f' + \phi(2l_3 + l_4)f''$.
Let

$$\Psi(z) = \frac{(l_2\phi - l_3\phi')f' + \phi(2l_3 + l_4)f''}{f}. \tag{3.65}$$

Clearly $T(r,\Psi)=O(logr)$ by Lemma 2.8 we can conclude that Ψ is not irrational function. It follows from (3.65)

$$f''=\frac{\Psi f}{n(1-2n)\phi}-\left(\frac{l_2}{n(1-2n)}+\frac{n-1}{(1-2n)}\frac{\phi'}{\phi}\right)f' \tag{3.66}$$

By substituting (3.66) in (3.61)

$$\phi=v_1f^2+v_2ff'+v_3(f')^2 \tag{3.67}$$

where $v_1=l_1-\frac{\Psi}{(1-2n)\phi}$, $\quad v_2=\frac{(2-2n)l_2}{(1-2n)}+\frac{n(n-1)}{(1-2n)}\frac{\phi'}{\phi}$, $\quad v_3=-n(n-1)$, $\quad v_j, j=1,2$ are rational functions and

$$T(r,v_i)=S(r,f), i=1,2,3. \tag{3.68}$$

By the similar arguments from (3.61) and (3.67)

$$\left(v_3\frac{\phi'}{\phi}+v_2+v_3'\right)\eta-v_3\eta'=0 \tag{3.69}$$

where $\eta=v_2^2-4v_1v_3$ is a rational function.
Now we will have the following cases.

Subcase 3.2.1: If $\eta\neq 0$. Then (3.69) becomes

$$\frac{v_2}{v_3}=\frac{\eta'}{\eta}-\frac{\phi'}{\phi}-\frac{v_3'}{v_3}.$$

On integration, $e^{Q_1+Q_2}=\frac{\kappa}{P_1P_2A_3}\eta^{\frac{1-2n}{2}}\phi^{n-1}\in S(r,f)$, only when $Q_1=-Q_2$, which is a contradiction, since $deg(Q_1)<deg(Q_2)$.

Subcase 3.2.2: If $\eta=0$ then (3.67) becomes

$$\phi=v_3\left[f'+\frac{v_2}{2v_3}f\right]^2. \tag{3.70}$$

Let $\gamma=f'+\frac{v_2}{2v_3}f$, $\quad \gamma\neq 0$ and $T(r,\phi)=S(r,f)$, we have

$$T(r,\gamma)=S(r,f)\quad\text{and}\quad f'=\gamma-\frac{v_2}{2v_3}f. \tag{3.71}$$

Put (3.71) in (3.58) and (3.59)

$$\begin{aligned}\left(A_1+\frac{v_2}{2v_3}n\right)f^n-n\gamma f^{n-1}+G_1(z.f)&=A_3P_2e^{Q_2}\\ \left(A_2+\frac{v_2}{2v_3}n\right)f^n-n\gamma f^{n-1}+G_2(z.f)&=A_3P_1e^{Q_1}\end{aligned} \tag{3.72}$$

If $A_1 + \frac{v_2}{2v_3}n \equiv 0$ and $A_2 + \frac{v_2}{2v_3}n \equiv 0$ then we get $A_3 = 0$ which is absurd. Consequently, $\left(A_1 + \frac{v_2}{2v_3}n\right)\left(A_2 + \frac{v_2}{2v_3}n\right) \equiv 0$. Otherwise, set

$$H_1 = \left(A_1 + \frac{v_2}{2v_3}n\right) f^n - n\gamma f^{n-1} + G_1(z,f)$$
$$H_2 = \left(A_2 + \frac{v_2}{2v_3}n\right) f^n - n\gamma f^{n-1} + G_2(z,f).$$

Since $A_3 \neq 0$ and $P_2 \neq 0$ then from (3.72), we have $N(r, \frac{1}{G_t}) + N(r,f) = N(r, \frac{1}{A_3}) + N(r,f) = S(r,f), t = 1,2$. From (3.68) and (3.71) there exists two small functions Q_1, Q_2 of f such that

$$H_1 = \left(A_1 + \frac{v_2}{2v_3}n\right)(f - Q_1)^n = P_2 A_3 e^{Q_2} \tag{3.73}$$

and

$$H_2 = \left(A_2 + \frac{v_2}{2v_3}n\right)(f - Q_2)^n = -P_1 A_3 e^{Q_1}. \tag{3.74}$$

Based on Nevanlinna second fundamental theorem concerning to that $Q_1 = Q_2$ then from (3.73) and (3.74) we get

$$e^{Q_1 - Q_2} = -\frac{\left(A_2 + \frac{nv_2}{2v_3}\right)}{\left(A_1 + \frac{nv_2}{2v_3}\right)} \in S(r,f)$$

this is possible only when $Q_1 = Q_2$ which becomes contradiction to $deg(Q_1) < deg(Q_2)$.

$$\therefore \rho(f) = deg(q) = max\{deg(Q_1), deg(Q_2)\}.$$

□

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