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SOME COMMON FIXED POINT THEOREMS ON TRICOMPLEX VALUED PARAMETRIC METRIC SPACE

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Abstract. The goal of this manuscript is to present the concept of tricomplex-valued parametric metric space. In this space, some common fixed-point theorems are introduced. Our results unify and generalize many papers in the same direction. Moreover, some illustrative examples are presented to support the theoretical results. Finally, our results are applied to find the existence of a solution to a linear system of equations.

1. Introduction and preliminaries

Serge [1] made a pioneering attempt at improving special algebras. He conceptualized the commutative generalization of complex numerals as bicomplex numerals, tricomplex numerals, etc. as elements of an infinite set of algebra. Subsequently, during the 1930s, other investigators also contributed to this field [2, 3, 4]. But unfortunately, the next fifty years failed to witness any advancement. After that, Price [5] improved the function theory and bicomplex algebra. Afterward, renewed interest in this subject finds few more significant applications in different branches of mathematical sciences as well as other fields of science and technology. Also, one can see the attempts in [6]. An impressive body of work has been developed by a number of researchers. Among them an important work on the basic functions of bicomplex numerals has been done by Luna-Elizaarrarás et al. [7] and Choi et al. [8]. They proved few CFPTs (common fixed point theorems) in connection with two weakly compatible maps in bicomplex valued metric spaces. Jebril [9] proved some CFPTs under rational contractions for a pair of maps in bicomplex valued metric spaces. In 2021, Beg et al. [10] proved the following fixed point theorems on bicomplex valued metric spaces. In 2022, Gunaseelan et al. [11] introduced the concept of tricomplex valued metric space and proved CFPTs. In 2022, Ramaswamy et al. [12] proved fixed point results in tricomplex-valued metric spaces using control functions. Very recently, many papers concerned with the field of fixed points and their applications have been published, for more details see [13, 14, 15, 16, 17, 18].

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In this study, we use the $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ as the family of real, complex, bicomplex and tricomplex numerals respectively. Price [5] defined the tricomplex number as:

$$\ddot{\Re} = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 j_1 + \mathfrak{a}_5 i_3 + \mathfrak{a}_6 j_2 + \mathfrak{a}_7 j_3 + \mathfrak{a}_8 i_4,$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8 \in \mathbb{C}_0$, and independent units $i_1, i_2, i_3, i_4, j_1, j_2, j_3$ are s. t (such that) $i_1^2 = i_4^2 = -1, i_4 = i_1 j_3 = i_1 i_2 i_3, j_2 = i_1 i_3 = i_3 i_1, j_2^2 = 1, j_1 = i_1 i_2 = i_2 i_1$ and $j_1^2 = 1$, $\mathbb{C}_3$ be the set of tricomplex number is defined as:

$$\mathbb{C}_3 = \{\ddot{\Re} : \ddot{\Re} = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 j_1 + \mathfrak{a}_5 i_3 + \mathfrak{a}_6 j_2 + \mathfrak{a}_7 j_3 + \mathfrak{a}_8 i_4, \\ \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_3 = \{\ddot{\Re} : \ddot{\Re} = z_1 + i_3 z_2, z_1, z_2 \in \mathbb{C}_2\},$$

where $z_1 = \mathfrak{a}_1 + \mathfrak{a}_2 i_2 \in \mathbb{C}_2$ and $z_2 = \mathfrak{a}_3 + \mathfrak{a}_4 i_2 \in \mathbb{C}_2$. If $\ddot{\Re} = z_1 + i_3 z_2$ and $\vartheta = \mathfrak{w}_1 + i_3 \mathfrak{w}_2$ be any two tricomplex numerals then the sum is $\ddot{\Re} \pm \vartheta = (z_1 + i_3 z_2) \pm (\mathfrak{w}_1 + i_3 \mathfrak{w}_2) = z_1 \pm \mathfrak{w}_1 + i_3(z_2 \pm \mathfrak{w}_2)$ and the product is $\ddot{\Re}.\vartheta = (z_1 + i_3 z_2)(\mathfrak{w}_1 + i_3 \mathfrak{w}_2) = (z_1 \mathfrak{w}_1 - z_2 \mathfrak{w}_2) + i_3(z_1 \mathfrak{w}_2 + z_2 \mathfrak{w}_1)$.

There are four idempotent elements in $\mathbb{C}_3$, they are $0, 1, e_1 = \frac{1+j_3}{2}, e_2 = \frac{1-j_3}{2}$ out of which e_1 and e_2 are nontrivial s. t $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every tricomplex number $z_1 + i_3 z_2$ can be uniquely be expressed as the combination of e_1 and e_2, namely

$$\ddot{\Re} = z_1 + i_3 z_2 = (z_1 - i_2 z_2) e_1 + (z_1 + i_2 z_2) e_2.$$

Here $\ddot{\Re}$ is saying that the idempotent representation of a tricomplex number and its coefficients $\ddot{\Re}_1 = (z_1 - i_2 z_2)$ and $\ddot{\Re}_2 = (z_1 + i_2 z_2)$ are idempotent elements of the bicomplex numeral $\ddot{\Re}$.

An element $\ddot{\Re} = z_1 + i_3 z_2 \in \mathbb{C}_3$ is called invertible if $\exists$ another element $\vartheta \in \mathbb{C}_3$ s. t $\ddot{\Re}\vartheta = 1$ and ϑ is called inverse(multiplicative) of $\ddot{\Re}$. Consequently $\ddot{\Re}$ is called the inverse(multiplicative) of ϑ. An element which has an inverse in $\mathbb{C}_3$ is called a nonsingular element of $\mathbb{C}_3$ and an element which does not have an inverse in $\mathbb{C}_3$ is called a singular element of $\mathbb{C}_3$.

An element $\ddot{\Re} = z_1 + i_3 z_2 \in \mathbb{C}_3$ is non-singular if and only if $|z_1^2 + z_2^2| \neq 0$ and singular if and only if $|z_1^2 + z_2^2| = 0$.

The inverse of $\ddot{\Re}$ is define by

$$\ddot{\Re}^{-1} = \vartheta = \frac{z - i_3 z_2}{z_1^2 + z_2^2}.$$

The norm $||.||$ of $\mathbb{C}_3$ is a positive real valued function and $||.|| : \mathbb{C}_3 \to \mathbb{C}_0^+$ is define by

$$\begin{aligned} ||\ddot{\Re}|| = ||z_1 + i_3 z_2|| &= \{|z|^2 + |z|^2\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_2 z_2)|^2 + |(z_1 + i_2 z_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= (\mathfrak{a}_1^2 + \mathfrak{a}_2^2 + \mathfrak{a}_1^2 + \mathfrak{a}_3^2 + \mathfrak{a}_4^2 + \mathfrak{a}_5^2 + \mathfrak{a}_6^2 + \mathfrak{a}_7^2 + \mathfrak{a}_8^2)^{\frac{1}{2}}, \end{aligned}$$

where $\ddot{\Re} = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 j_1 + \mathfrak{a}_5 i_3 + \mathfrak{a}_6 j_2 + \mathfrak{a}_7 j_3 + \mathfrak{a}_8 i_4 = z_1 + i_3 z_2 \in \mathbb{C}_3$. A norm linear space is the linear space $\mathbb{C}_3$ with respect to a defined norm; also, $\mathbb{C}_3$ is complete; as a result, $\mathbb{C}_3$ is the Banach space. If $\ddot{\Re}, \vartheta \in \mathbb{C}_3$ then $||\ddot{\Re}\vartheta|| \leq 2||\ddot{\Re}||||\vartheta||$ true instead of $||\ddot{\Re}\vartheta|| \leq ||\ddot{\Re}||||\vartheta||$, hence $\mathbb{C}_3$ is not the Banach algebra.

The partial order relation $\preceq_{i_3}$ on $\mathbb{C}_3$ is defined as follows: Let $\mathbb{C}_3$ be the family of tricomplex numerals and $\ddot{\Re} = z_1 + i_3 z_2$, $\vartheta = \mathfrak{w}_1 + i_3 \mathfrak{w}_2 \in \mathbb{C}_3$ then $\ddot{\Re} \preceq_{i_3} \vartheta$ if and only if $z_1 \preceq \mathfrak{w}_1$ and $z_2 \preceq \mathfrak{w}_2$, i.e., $\ddot{\Re} \preceq_{i_3} \vartheta$ if one of the given conditions is hold:

(a) $z_1 = \mathfrak{w}_1$, $z_2 = \mathfrak{w}_2$,
(b) $z_1 \prec \mathfrak{w}_1$, $z_2 = \mathfrak{w}_2$,
(c) $z_1 = \mathfrak{w}_1$, $z_2 \prec \mathfrak{w}_2$, and
(d) $z_1 \prec \mathfrak{w}_1$, $z_2 \prec \mathfrak{w}_2$,

Specifically, we may write $\ddot{\Re} \precnsim_{i_3} \vartheta$ if $\ddot{\Re} \preceq_{i_3} \vartheta$ and $\ddot{\Re} \neq \vartheta$, that is, one of (b),(c) and (d) is hold, and we will write $\ddot{\Re} \prec_{i_3} \vartheta$ if only (d) is hold.
For any two tricomplex numerals $\ddot{\Re}, \vartheta \in \mathbb{C}_2$. The following can be validated:

(1) $\ddot{\Re} \preceq_{i_3} \vartheta \Rightarrow ||\ddot{\Re}|| \leq ||\vartheta||$,
(2) $||\ddot{\Re} + \vartheta|| \leq ||\ddot{\Re}|| + ||\vartheta||$,
(3) $||\mathfrak{a}\ddot{\Re}|| = \mathfrak{a}||\ddot{\Re}||$, where $\mathfrak{a}$ is a non negative real number,
(4) $||\ddot{\Re}\vartheta|| \leq 2||\ddot{\Re}||||\vartheta||$ and the equality holds only when at least one of $\ddot{\Re}$ and ϑ is nonsingular,
(5) $||\ddot{\Re}^{-1}|| = ||\ddot{\Re}||^{-1}$ if $\ddot{\Re}$ is a nonsingular,
(6) $||\frac{\ddot{\Re}}{\vartheta}|| = \frac{||\ddot{\Re}||}{||\vartheta||}$, if ϑ is a nonsingular.

Now, we review some fundamental ideas and notations that will be utilized in the sequel.

Definition 1.1. [11] Let Ψ be a nonempty set where as $\mathbb{C}_3$ be the family of tricomplex numerals. Assume that the map $\ddot{\delta} : \Psi \times \Psi \to \mathbb{C}_3$, satisfies the following conditions:

(A1) $0 \preceq_{i_3} \ddot{\delta}(, \varepsilon)\ \forall\ ,\ \varrho, \varepsilon \in \Psi$ and $\ddot{\delta}(\varrho, \varepsilon) = 0$ if and only if $\varrho = \varepsilon$;
(A2) $\ddot{\delta}(\varrho, \varepsilon) = \ddot{\delta}(\varepsilon, \varrho)\ \forall\ \ \varrho, \varepsilon \in \Psi$;
(A3) $\ddot{\delta}(\varrho, \varepsilon) \leq \ddot{\delta}(\varrho, z) + \ddot{\delta}(z, \varepsilon)\ \forall\ \ \varrho, \varepsilon, z \in \Psi$.

Then $\ddot{\delta}$ is called the tricomplex valued metric on Ψ, and $(\Psi, \ddot{\delta}_{\mathfrak{b}})$ is called the tricomplex valued metric space.

Definition 1.2. Let Ψ be a nonempty set whereas $\mathbb{C}_3$ be the set of tricomplex numerals. Suppose that the mapping $\ddot{\delta} : \Psi \times \Psi \times (0, \infty) \to \mathbb{C}_3$, satisfies the following conditions:

(A1) $0 \preceq_{i_3} \ddot{\delta}(\varrho, \varepsilon, \mathsf{p})$, for all $\varrho, \varepsilon \in \Psi$ and $\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) = 0$ if and only if $\varrho = \varepsilon$;
(A2) $\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) = \ddot{\delta}(\varepsilon, \varrho, \mathsf{p})\ \forall\ \ \varrho, \varepsilon \in \Psi$ and $\mathsf{p} > 0$;
(A3) $\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) \leq \ddot{\delta}(\varrho, z, \mathsf{p}) + \ddot{\delta}(z, \varepsilon, \mathsf{p})$, $\forall\ \ \varrho, \varepsilon, z \in \Psi$ and $\mathsf{p} > 0$.

Then $\ddot{\delta}$ is called the TCVP on Ψ, and $(\Psi, \ddot{\delta}_{\mathfrak{b}})$ is called the TCVPS.

Example 1.3. Let $\Psi = [0, 1]$ and $\ddot{\delta} : \Psi \times \Psi \times (0, \infty) \to \mathbb{C}_3$ be defined by $\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) = \mathsf{p}|\varrho - \varepsilon|e^{i_3 \frac{\pi}{6}}$, $\forall\ \ \mathsf{p} > 0$. Then $(\Psi, \ddot{\delta})$ is a TCVPS.

Definition 1.4. Let $(\Psi, \ddot{\delta})$ be a TCVPS and $\mathfrak{B} \subseteq \Psi$,

(B1) $\mathfrak{a} \in \mathfrak{B}$ is said to be an interior point of $\mathfrak{B}$ whenever for each $0 \prec_{i_3} \mathfrak{s} \in \mathbb{C}_3$ s. t

$$\mathtt{N}(\mathfrak{a}, \mathfrak{s}, \mathsf{p}) \subseteq \mathfrak{B},$$

where $\mathtt{N}(\mathfrak{a}, \mathfrak{s}, \mathsf{p}) = \{\varepsilon \in \Psi : \ddot{\delta}(\mathfrak{a}, \varepsilon, \mathsf{p}) \preceq_{i_3} \mathfrak{s}\}$ and $\mathsf{p} > 0$.

(B2) A point $\varrho \in \Psi$ is called limit point of $\mathfrak{B}$ whenever there is $0 \prec_{i_3} \mathfrak{s} \in \mathbb{C}_3$,

$$\mathtt{N}(\varrho, \mathfrak{s}, \mathsf{p}) \cup (\mathfrak{B} - \Psi) \neq \emptyset,$$

for all $\mathsf{p} > 0$.

(B3) A subset $\mathcal{A} \subseteq \Psi$ is said to be a open whenever every element of $\mathcal{A}$ is an interior point of $\mathcal{A}$. A subset $\mathfrak{B} \subseteq \Psi$ is called closed whenever each limit point of $\mathfrak{B}$ belongs to $\mathfrak{B}$. The family

$$\mathcal{F} = \{\mathtt{N}(\varrho, \mathfrak{s}, \mathsf{p}) : \varrho \in \Psi, 0 \prec_{i_3} \mathfrak{s}, \mathsf{p} > 0\}$$

is a sub-basis for a topology on Ψ. We denote this tricomplex topology by τ_{tps}. Indeed, the topology τ_{tps} is Hausdorff.

Definition 1.5. Let $(\Psi, \ddot{\delta})$ be a TCVPS. A sequence $\{\varrho_{\mathfrak{k}}\}$ in Ψ is said to be a convergent to $\varrho \in \Psi$ if for each $0 \prec_{i_3} \ni \in \mathbb{C}_3$ there exists $\mathfrak{k}_0 \in \mathbb{N}$ s. t $\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p}) \prec_{i_3} \mathsf{q}$, $\forall\ \mathfrak{k} \geq \mathfrak{k}_0$, $\mathsf{p} > 0$.

Lemma 1.6. *Let $(\Psi, \ddot{\delta})$ be a TCVPS. A sequence $\{\varrho_{\mathfrak{k}}\} \in \Psi$ is converges to $\varrho \in \Psi$ if and only if $\lim_{\mathfrak{k}\to\infty} ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p})|| = 0$.*

Proof. Let $\{\varrho_{\mathfrak{k}}\}$ is a convergent sequence and converges to a point ϱ and let $\ni > 0$ be any real number. Suppose

$$\mathsf{q} = \frac{\ni}{\sqrt{8}} + i_1\frac{\ni}{\sqrt{8}} + i_2\frac{\ni}{\sqrt{8}} + j_1\frac{\ni}{\sqrt{8}} + i_3\frac{\ni}{\sqrt{8}} + j_2\frac{\ni}{\sqrt{8}} + j_3\frac{\ni}{\sqrt{8}} + i_4\frac{\ni}{\sqrt{8}}.$$

Then $0 \prec_{i_3} \mathsf{q} \in \mathbb{C}_3$ and for this q there is a natural number $\mathfrak{k}_0 \in \mathbb{N}$ s. t $\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p}) \prec_{i_3} \mathsf{q}$, $\forall\ \mathfrak{k} \geq \mathfrak{k}_0$. Therefore

$$||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p}) < ||\mathsf{q}|| = \ni,\ \forall \mathfrak{k} \geq \mathfrak{k}_0.$$

Hence $\lim_{\mathfrak{k}\to\infty} ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p})|| = 0$.

Conversely, let $\lim_{\mathfrak{k}\to\infty} ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p})|| = 0$. Then for each $0 \prec_{i_3} \mathsf{q} \in \mathbb{C}_3$, there exists a real number $\ni > 0$ s. t for all $\ddot{\mathfrak{R}} \in \mathbb{C}_3$,

$$||\ddot{\mathfrak{R}}|| < \ni \Rightarrow \ddot{\mathfrak{R}} \prec_{i_3} \mathsf{q}.$$

Then, for this $\ni > 0$, there exists $\mathfrak{k}_0 \in \mathbb{N}$ s. t

$$||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p})|| < \ni,\ \forall\ \mathfrak{k} \geq \mathfrak{k}_0.$$

Therefore,

$$\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho, \mathsf{p}) \prec_{i_3} \mathsf{q},\ \forall\ \mathfrak{k} \geq \mathfrak{k}_0.$$

Hence $\{\varrho_{\mathfrak{k}}\}$ converges to a point ϱ. □

Definition 1.7. Let $(\Psi, \ddot{\delta})$ be a TCVPS. A sequence $\{\varrho_{\mathfrak{k}}\}$ in Ψ is said to be a Cauchy sequence in $(\Psi, \ddot{\delta})$ if for any $0 \prec_{i_3} \ni \in \mathbb{C}_3$, there exists $h \in \mathbb{N}$ s. t $\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p}) \prec_{i_3} \ni, \ \forall\ \mathfrak{k}, \mathsf{n} \in \mathbb{N}$, $\mathsf{p} > 0$ and $\mathfrak{k}, \mathsf{n} \geq h$.

Definition 1.8. Let $(\Psi, \ddot{\delta})$ be a TCVPS, and $\{\varrho_{\mathfrak{k}}\}$ be any sequence in Ψ. Then, if every Cauchy sequence in Ψ is convergent in Ψ then $(\Psi, \ddot{\delta})$ is called a complete TCVPS.

Lemma 1.9. *Let $(\Psi, \ddot{\delta})$ be a TCVPS and $\{\varrho_{\mathfrak{k}}\}$ be a sequence in Ψ. Then $\{\varrho_{\mathfrak{k}}\}$ is a Cauchy sequence in Ψ if and only if* $\lim_{\mathfrak{k}\to\infty} ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p})|| = 0$.

Proof. Let $\{\varrho_{\mathfrak{k}}\}$ is a Cauchy sequence in Ψ and $\ni > 0$ be any real number. Suppose

$$\mathsf{q} = \frac{\ni}{\sqrt{8}} + i_1\frac{\ni}{\sqrt{8}} + i_2\frac{\ni}{\sqrt{8}} + j_1\frac{\ni}{\sqrt{8}} + i_3\frac{\ni}{\sqrt{8}} + j_2\frac{\ni}{\sqrt{8}} + j_3\frac{\ni}{\sqrt{8}} + i_4\frac{\ni}{\sqrt{8}}.$$

Then $0 \prec_{i_3} \mathsf{q} \in \mathbb{C}_2^+$ and for this q there is a natural number $\mathfrak{k}_0 \in \mathbb{N}$ s. t $\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p}) \prec_{i_3} \mathsf{q}, \ \forall\ \ \mathfrak{k} > \mathfrak{k}_0$. Therefore,

$$||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p})|| < ||\mathsf{q}|| = \ni, \ \forall\ \ \mathfrak{k} > \mathfrak{k}_0, \text{ and } \mathsf{p} > 0.$$

And this implies,

$$\lim_{\mathfrak{k}\to\infty} ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p})|| = 0.$$

Conversely, let $\lim_{\mathfrak{k}\to\infty} ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p})|| = 0$. Then for each $0 \prec_{i_3} \mathsf{q} \in \mathbb{C}_3^+$, there exists a real number $\ni > 0$ s. t, $\forall\ \ddot{\Re} \in \mathbb{C}_2^+$,

$$||\ddot{\Re}|| < \ni \Rightarrow \ddot{\Re} \prec_{i_3} \mathsf{q}.$$

Then, for this $\ni > 0$, there exists a natural number $\mathfrak{k}_0 \in \mathbb{N}$ s. t

$$||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p})|| < \ni, \ \forall\ \mathfrak{k} > \mathfrak{k}_0 \text{ and } \mathsf{p} > 0.$$

Therefore,

$$\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+\mathsf{n}}, \mathsf{p}) \prec_{i_3} \mathsf{q}, \ \forall\ \mathfrak{k} > \mathfrak{k}_0, \text{ and } \mathsf{p} > 0.$$

Hence $\{\varrho_{\mathfrak{k}}\}$ is a Cauchy sequence. □

Definition 1.10. Let $\mathcal{S}$ and $\top$ be self maps of non-void set Ψ. A point $\varrho \in \Psi$ is said to be a CFP of $\mathcal{S}$ and $\top$ if $\varrho = \mathcal{S}\varrho = \top\varrho$.

2. Main results

In this part, we use the rational type contraction condition to illustrate the CFP theorem in TCVPS.

Theorem 2.1. *Let $(\Psi, \ddot{\delta})$ be a complete TCVPS and a self map $\mathcal{S}$ and $\top$ are satisfies:*

$$\ddot{\delta}(\mathcal{S}\varrho, \top\varepsilon, \mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho, \mathcal{S}\varrho, \mathsf{p})\ddot{\delta}(\varepsilon, \top\varepsilon, \mathsf{p}) + \gamma\ddot{\delta}(\varepsilon, \mathcal{S}\varrho, \mathsf{p})\ddot{\delta}(\varrho, \top\varepsilon, \mathsf{p})}{1 + \ddot{\delta}(\varrho, \varepsilon, \mathsf{p})}$$

for all $\varrho, \varepsilon \in \Psi$ and $\mathsf{p} > 0$, where μ, ν, γ are non-negative reals with $\mu+2\nu+2\gamma < 1$, then $\mathcal{S}$ and $\top$ have a UCFP (unique common fixed point).

Proof. Let ϱ_0 be an arbitrary point in Ψ and define $\varrho_{2\mathfrak{k}+1} = \mathcal{S}\varrho_{2\mathfrak{k}}$, $\varrho_{2\mathfrak{k}+2} = \top\varrho_{2\mathfrak{k}+1}, \mathfrak{k} = 0, 1, 2, \cdots$. Then

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p}) &= \ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) \\ &+ \frac{\nu\ddot{\delta}(\varrho_{2\mathfrak{k}}, \mathcal{S}\varrho_{2\mathfrak{k}}, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \gamma\ddot{\delta}(\varrho_{2\mathfrak{k}}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varrho_{2\mathfrak{k}}, \mathsf{p})}{1 + \ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})}.\end{aligned}$$

Since $\varrho_{2\mathfrak{k}+1} = \mathcal{S}\varrho_{2\mathfrak{k}}$ implies $\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varrho_{2\mathfrak{k}}, \mathsf{p}) = 0$, therefore

$$\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})}{1 + \ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})},$$

which implies that

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| \leq \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})|| + \frac{2\nu||\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||}{||1 + \ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}\mathsf{p})||}.$$

Since $||1 + \ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})|| > ||\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||$, therefore

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| \leq \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})|| + 2\nu||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||,$$

so that

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| \leq \frac{\mu}{1-2\nu}||\ddot{\delta}(\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||.$$

Also,

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+3}, \mathsf{p}) &= \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1}, \int\varrho_{2\mathfrak{k}+2}, \mathsf{p}) \\ &= \ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}+2}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) \\ &\preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) \\ &+ \frac{\nu\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \mathcal{S}\varrho_{2\mathfrak{k}+2}, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \gamma\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varrho_{2\mathfrak{k}+2}, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p})}{1 + \ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})}.\end{aligned}$$

Since $\varrho_{2\mathfrak{k}+2} = \top\varrho_{2\mathfrak{k}+1}$ implies $\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) = 0$, therefore

$$\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+3}, \mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \mathcal{S}\varrho_{2\mathfrak{k}+2}, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p})}{1 + \ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})},$$

which implies that

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+3}, \mathsf{p})|| \leq \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})|| + \frac{2\nu||\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+3}, \mathsf{p})||||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||}{||1 + \ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||}.$$

As $||1 + \ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})|| > ||\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||$, therefore

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+3}, \mathsf{p})|| \leq \frac{\mu}{1-2\nu}||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||.$$

Putting $h = \frac{\mu}{1-2\nu}$, we have ($\forall\ \mathfrak{k}$)

$$||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+1}, \mathsf{p})|| \leq h||\ddot{\delta}(\varrho_{\mathfrak{k}-1}, \varrho_{\mathfrak{k}}, \mathsf{p})|| \leq h^2||\ddot{\delta}(\varrho_{\mathfrak{k}-2}, \varrho_{\mathfrak{k}-1}, \mathsf{p})|| \leq \cdots \leq h^{\mathfrak{k}}||\ddot{\delta}(\varrho_0, \varrho_1, \mathsf{p})||.$$

Therefore, for any $\mathsf{n} > \mathfrak{k}$, we have

$$\begin{aligned}||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathsf{n}}, \mathsf{p})|| &\leq ||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathfrak{k}+1}, \mathsf{p})|| + ||\ddot{\delta}(\varrho_{\mathfrak{k}+1}, \varrho_{\mathfrak{k}+2}, \mathsf{p})|| + ||\ddot{\delta}(\varrho_{\mathfrak{k}+2}, \varrho_{\mathfrak{k}+3}, \mathsf{p})|| \cdots + ||\ddot{\delta}(\varrho_{\mathfrak{m}-1}, \varrho_{\mathsf{n}}, \mathsf{p})||\\
&\leq [h^{\mathfrak{k}} + h^{\mathfrak{k}+1} + h^{\mathfrak{k}+2} + \cdots + h^{\mathfrak{m}-1}]||\ddot{\delta}(\varrho_0, \varrho_1, \mathsf{p})||\\
&\leq \left[\frac{h^{\mathfrak{k}}}{1-h}\right]||\ddot{\delta}(\varrho_0, \varrho_1, \mathsf{p})||,\end{aligned}$$

which implies that

$$||\ddot{\delta}(\varrho_{\mathfrak{k}}, \varrho_{\mathsf{n}}, \mathsf{p})|| \leq \left[\frac{h^{\mathfrak{k}}}{1-h}\right]||\ddot{\delta}(\varrho_0, \varrho_1, \mathsf{p})|| \to 0 \text{ as } \mathfrak{k} \to \infty.$$

From Lemma 1.9, the sequence $\{\varrho_{\mathfrak{k}}\}$ is Cauchy. Since Ψ is complete, $\exists\ \varpi \in \Psi$ s. t $\varrho_{\mathfrak{k}} \to \varpi$ as $\mathfrak{k} \to \infty$. On the contrary, assume that $\varpi \neq \mathcal{S}\varpi$, thus $0 \prec_{i_3} z = \ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})$ and henceforth we can have

$$\begin{aligned}z = \ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p}) &\preceq_{i_3} \ddot{\delta}(\varpi, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varpi, \mathsf{p})\\
&\preceq_{i_3} \ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p}) + \mu\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})\\
&+ \frac{\nu\ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \gamma\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varpi, \mathsf{p})\ddot{\delta}(\varpi, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p})}{1 + \ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})}\\
&\preceq_{i_3} \ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p}) + \mu\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})\\
&+ \frac{\nu z.\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \gamma\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varpi, \mathsf{p})\ddot{\delta}(\varpi, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p})}{1 + \ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})}.\end{aligned}$$

Also, for every $\mathfrak{k}$, we have

$$\begin{aligned}||\ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})|| &\leq ||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| + \mu||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||\\
&+ \frac{\nu z||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| + 2\gamma||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varpi)||||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||}{||1 + \ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||}.\end{aligned}$$

As $\mathfrak{k} \to \infty$, we have

$$||\ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})|| = 0,$$

which is a contradiction so that $\varpi = \mathcal{S}\varpi$. Similarly, one can also show that $\varpi = \top\varpi$.

To prove the uniqueness of CFP, let ϖ^*(in Ψ) be another CFP of $\mathcal{S}$ and $\top$ i.e.$\varpi^* = \mathcal{S}\varpi^* = \top\varpi^*$. Then

$$\begin{aligned}\ddot{\delta}(\varpi, \varpi^*, \mathsf{p}) = \ddot{\delta}(\mathcal{S}\varpi, \top\varpi^*, \mathsf{p}) &\preceq_{i_3} \mu\ddot{\delta}(\varpi, \varpi^*, \mathsf{p})\\
&+ \frac{\nu\ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})\ddot{\delta}(\varpi^*, \top\varpi^*, \mathsf{p}) + \gamma\ddot{\delta}(\varpi^*, \mathcal{S}\varpi, \mathsf{p})\ddot{\delta}(\varpi, \top\varpi^*, \mathsf{p})}{1 + \ddot{\delta}(\varpi, \varpi^*, \mathsf{p})}\\
&= \mu\ddot{\delta}(\varpi, \varpi^*, \mathsf{p}) + \frac{\gamma\ddot{\delta}(\varpi^*, \varpi, \mathsf{p})\ddot{\delta}(\varpi, \varpi^*, \mathsf{p})}{1 + \ddot{\delta}(\varpi, \varpi^*, \mathsf{p})},\end{aligned}$$

which implies that

$$||\ddot{\delta}(\varpi, \varpi^*, \mathsf{p})|| \leq \mu||\ddot{\delta}(\varpi, \varpi^*, \mathsf{p})|| + \frac{2\gamma||\ddot{\delta}(\varpi^*, \varpi, \mathsf{p})||||\ddot{\delta}(\varpi, \varpi^*, \mathsf{p})||}{||1 + \ddot{\delta}(\varpi, \varpi^*, \mathsf{p})||}.$$

Since $||1+\ddot{\delta}(\varpi,\varpi^*,\mathsf{p})|| > ||\ddot{\delta}(\varpi,\varpi^*,\mathsf{p})||$, therefore

$$||\ddot{\delta}(\varpi,\varpi^*,\mathsf{p})|| \leq (\mu+2\gamma)||\ddot{\delta}(\varpi,\varpi^*,\mathsf{p})||,$$

which is a contradiction, thus $\varpi=\varpi^*$ (as $\mu+2\gamma<1$). Hence, complete the proof. $\square$

By taking $\mathcal{S}=\top$ in Theorem 2.1, one conclude that the given results :

Corollary 2.2. *Let $(\Psi,\ddot{\delta})$ be a complete TCVPS and a self map $\top:\Psi\to\Psi$ which satisfies:*

$$\ddot{\delta}(\top\varrho,\top\varepsilon,\mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho,\varepsilon,\mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho,\top\varrho,\mathsf{p})\ddot{\delta}(\varepsilon,\top\varepsilon,\mathsf{p})+\gamma\ddot{\delta}(\varepsilon,\top\varrho,\mathsf{p})\ddot{\delta}(\varrho,\top\varepsilon,\mathsf{p})}{1+\ddot{\delta}(\varrho,\varepsilon,\mathsf{p})}$$

for all $\varrho,\varepsilon\in\Psi$ and $\mathsf{p}>0$, where μ,ν,γ are non-negative reals with $\mu+2\nu+2\gamma<1$, then $\top$ has a UFP (unique fixed point).

Theorem 2.3. *Let $(\Psi,\ddot{\delta})$ be a complete TCVPS, and a map $\mathcal{S},\top:\Psi\to\Psi$ satisfy the inequality*

$$\ddot{\delta}(\mathcal{S}\varrho,\top\varepsilon,\mathsf{p}) \preceq_{i_3} \begin{cases} \mu\ddot{\delta}(\varrho,\varepsilon,\mathsf{p})+\nu\frac{\ddot{\delta}(\varrho,\mathcal{S}\varrho,\mathsf{p})\ddot{\delta}(\varepsilon,\top\varepsilon,\mathsf{p})+\ddot{\delta}(\varepsilon,\mathcal{S}\varrho,\mathsf{p})\ddot{\delta}(\varrho,\top\varepsilon,\mathsf{p})}{\ddot{\delta}(\mathcal{S}\varrho,\varrho,\mathsf{p})+\ddot{\delta}(\top\varepsilon,\varepsilon,\mathsf{p})} & \\ +\gamma\frac{\ddot{\delta}(\varrho,\mathcal{S}\varrho,\mathsf{p})\ddot{\delta}(\varrho,\top\varepsilon,\mathsf{p})+\ddot{\delta}(\varepsilon,\mathcal{S}\varrho,\mathsf{p})\ddot{\delta}(\varepsilon,\top\varepsilon,\mathsf{p})}{\ddot{\delta}(\mathcal{S}\varrho,\varepsilon,\mathsf{p})+\ddot{\delta}(\top\varepsilon,\varrho,\mathsf{p})}, & \text{if } \mathcal{D}\neq 0,\ \mathcal{D}_1\neq 0 \\ 0, \quad \text{if } \mathcal{D}=0 \text{ or } \mathcal{D}_1=0 & \end{cases} \tag{2.1}$$

for every $\varrho,\varepsilon\in\Psi$ and $\mathsf{p}>0$, where $\mathcal{D}=\ddot{\delta}(\mathcal{S}\varrho,\varrho,\mathsf{p})+\ddot{\delta}(\top\varepsilon,\varepsilon,\mathsf{p})$ and $\mathcal{D}_1=\ddot{\delta}(\mathcal{S}\varrho,\varepsilon,\mathsf{p})+\ddot{\delta}(\top\varepsilon,\varrho,\mathsf{p})$ and μ,ν,γ are non-negative reals with $\mu+2\nu+\gamma<1$. Then $\mathcal{S}$ and $\top$ have a UCFP.

Proof. Consider ϱ_0 be an arbitrary point in Ψ. Define $\varrho_{2\mathfrak{k}+1}=\mathcal{S}\varrho_{2\mathfrak{k}}$ and $\varrho_{2\mathfrak{k}+2}=\top\varrho_{2\mathfrak{k}+1}$, $\mathfrak{k}=0,1,2,\cdots$ Now, we will classify the following cases.
Case I : if (for $\mathfrak{k}=0,1,2,\cdots$)$\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}},\mathsf{p})+\ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})\neq 0$ and $\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})\neq 0$, then

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p}) &= \ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}},\top\varrho_{2\mathfrak{k}+1},\mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})\\ &+\nu\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}},\mathcal{S}\varrho_{2\mathfrak{k}},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\mathcal{S}\varrho_{2\mathfrak{k}},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})}{\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}},\mathsf{p})+\ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})}\\ &+\gamma\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}},\mathcal{S}\varrho_{2\mathfrak{k}},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\mathcal{S}\varrho_{2\mathfrak{k}},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\top\varrho_{2\mathfrak{k}+1})}{\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})}.\end{aligned}$$

Since $\varrho_{2\mathfrak{k}+1}=\mathcal{S}\varrho_{2\mathfrak{k}}$ and $\varrho_{2\mathfrak{k}+2}=\top\varrho_{2\mathfrak{k}+1}$, therefore

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p}) &\preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})\\ &+\nu\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})}\\ &+\gamma\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}},\mathsf{p})},\end{aligned}$$

or

$$\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p}) + \nu\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})} + \gamma\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}},\mathsf{p})},$$

which implies that

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})|| \le \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})|| + 2\nu\frac{||\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})||||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||}{||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})||} + \gamma||\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})||.$$

Since $||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})|| > ||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}},\mathsf{p})||$, therefore

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})|| \le \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})|| + 2\nu||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})|| + \gamma||\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})||,$$

so that

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})|| \le \frac{\mu+\gamma}{1-2\nu}||\ddot{\delta}(\varrho_{2\mathfrak{k}},\varrho_{2\mathfrak{k}+1},\mathsf{p})||.$$

Also,

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p}) &= \ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}+2},\top\varrho_{2\mathfrak{k}+1},\mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})\\ &+ \nu\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\mathcal{S}\varrho_{2\mathfrak{k}+2},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\mathcal{S}\varrho_{2\mathfrak{k}+2},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})}{\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})}\\ &+ \gamma\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\mathcal{S}\varrho_{2\mathfrak{k}+2},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\mathcal{S}\varrho_{2\mathfrak{k}+2},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})}{\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})}.\end{aligned}$$

Since $\varrho_{2\mathfrak{k}+3} = \mathcal{S}\varrho_{2\mathfrak{k}+2}$ and $\varrho_{2\mathfrak{k}+2} = \top\varrho_{2\mathfrak{k}+1}$, we get

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p}) &\preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})\\ &+ \nu\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+3},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+3},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})}\\ &+ \gamma\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+3},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+3},\varrho_{2\mathfrak{k}+1},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+2},\mathsf{p})},\end{aligned}$$

or

$$\begin{aligned}\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p}) &\preceq_{i_3} \mu\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})\\ &+ \nu\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+3},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})}\\ &+ \gamma\frac{\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+3},\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})}{\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+3},\mathsf{p})},\end{aligned}$$

which implies that

$$\begin{aligned}||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})|| \leq \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})|| \\ + 2\nu\frac{||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})||||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||}{||\ddot{\delta}(\varrho_{2\mathfrak{k}+3},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})||} \\ + \gamma||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||.\end{aligned}$$

Since $||\ddot{\delta}(\varrho_{2\mathfrak{k}+3},\varrho_{2\mathfrak{k}+2},\mathsf{p})+\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})|| > ||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||$, therefore

$$\begin{aligned}||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})|| \leq \mu||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})|| \\ + 2\nu\frac{||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3},\mathsf{p})||||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||}{||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+1},\mathsf{p})||} \\ + \gamma||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||,\end{aligned}$$

or

$$||\ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varrho_{2\mathfrak{k}+3}\mathsf{p})|| \leq \frac{\mu+\gamma}{1-2\nu}||\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+2},\mathsf{p})||.$$

Now, with $h=\frac{\mu+\gamma}{1-2\nu}$, we have ($\forall$ $\mathfrak{k}$)

$$||\ddot{\delta}(\varrho_{\mathfrak{k}},\varrho_{\mathfrak{k}+1},\mathsf{p})|| \leq h||\ddot{\delta}(\varrho_{\mathfrak{k}-1},\varrho_{\mathfrak{k}},\mathsf{p})|| \leq \cdots \leq h^{\mathfrak{k}}||\ddot{\delta}(\varrho_0,\varrho_1,\mathsf{p})||.$$

So, for any $\mathsf{n} > \mathfrak{k}$, we have

$$\begin{aligned}||\ddot{\delta}(\varrho_{\mathfrak{k}},\varrho_{\mathsf{n}},\mathsf{p})|| &\leq ||\ddot{\delta}(\varrho_{\mathfrak{k}},\varrho_{\mathfrak{k}+1},\mathsf{p})|| + ||\ddot{\delta}(\varrho_{\mathfrak{k}+1},\varrho_{\mathfrak{k}+2},\mathsf{p})|| + \cdots + ||\ddot{\delta}(\varrho_{\mathfrak{m}-1},\varrho_{\mathsf{n}},\mathsf{p})|| \\ &\leq [h^{\mathfrak{k}} + h^{\mathfrak{k}+1} + \cdots + h^{\mathfrak{m}-1}]||\ddot{\delta}(\varrho_0,\varrho_1,\mathsf{p})|| \\ &\leq \left[\frac{h^{\mathfrak{k}}}{1-h}\right]||\ddot{\delta}(\varrho_0,\varrho_1,\mathsf{p})||,\end{aligned}$$

and henceforth

$$||\ddot{\delta}(\varrho_{\mathfrak{k}},\varrho_{\mathsf{n}},\mathsf{p})|| \leq \left[\frac{h^{\mathfrak{k}}}{1-h}\right]||\ddot{\delta}(\varrho_0,\varrho_1,\mathsf{p})|| \to 0 \text{ as } \mathfrak{k} \to \infty.$$

From Lemma 1.9, we conclude that $\{\varrho_{\mathfrak{k}}\}$ is a Cauchy sequence. Since Ψ is a complete, then there exists $\varpi \in \Psi$ s. t $\varrho_{\mathfrak{k}} \to \varpi$ as $\mathfrak{k} \to \infty$. Now, we assert that $\varpi = \mathcal{S}\varpi$, otherwise $0 \prec_{i_3} z = \ddot{\delta}(\varpi,\mathcal{S}\varpi,\mathsf{p})$ and we obtain

$$\begin{aligned}z =&\ddot{\delta}(\varpi,\mathcal{S}\varpi,\mathsf{p}) \preceq_{i_3} \ddot{\delta}(\varpi,\top\varrho_{2\mathfrak{k}+1},\mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\mathcal{S}\varpi,\mathsf{p}) \\ \preceq_{i_3}&\ddot{\delta}(\varpi,\varrho_{2\mathfrak{k}+2},\mathsf{p}) + \mu\ddot{\delta}(\varpi,\varrho_{2\mathfrak{k}+1},\mathsf{p}) \\ &+ \nu\frac{\ddot{\delta}(\varpi,\mathcal{S}\varpi,\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\top\varrho_{2\mathfrak{k}+1},\mathsf{p}) + \ddot{\delta}(\varrho_{2\mathfrak{k}+1},\mathcal{S}\varpi,\mathsf{p})\ddot{\delta}(\varpi,\top\varrho_{2\mathfrak{k}+1})}{\ddot{\delta}(\mathcal{S}\varpi,\varpi,\mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1},\varrho_{2\mathfrak{k}+1},\mathsf{p})} \\ &+ \gamma\frac{\ddot{\delta}(\varpi,\mathcal{S}\varpi,\mathsf{p})\ddot{\delta}(\varpi,\top\varrho_{2\mathfrak{k}+1},\mathsf{p}) + \ddot{\delta}(\varrho_{2\mathfrak{k}+1},\mathcal{S}\varpi,\mathsf{p})\ddot{\delta}(\varrho_{2\mathfrak{k}+1},\top\varrho_{2\mathfrak{k}+1},\mathsf{p})}{\ddot{\delta}(\mathcal{S}\varpi,\varrho_{2\mathfrak{k}+1},\mathsf{p}) + \ddot{\delta}(\varrho_{2\mathfrak{k}+2},\varpi,\mathsf{p})},\end{aligned}$$

which implies that

$$\begin{aligned}||z|| = ||\ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})|| \leq ||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| + \mu||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p})|| \\ + \nu\frac{z||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| + 2||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varpi, \mathsf{p})||||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||}{||\ddot{\delta}(\mathcal{S}\varpi, \varpi, \mathsf{p}) + \ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})||} \\ + \gamma\frac{z||\ddot{\delta}(\varpi, \varrho_{2\mathfrak{k}+2}, \mathsf{p})|| + 2||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \mathcal{S}\varpi, \mathsf{p})||||\ddot{\delta}(\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+2}, \mathsf{p})||}{||\ddot{\delta}(\mathcal{S}\varpi, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \ddot{\delta}(\varrho_{2\mathfrak{k}+2}, \varpi, \mathsf{p})||},\end{aligned}$$

a contradiction, so that $||z|| = ||\ddot{\delta}(\varpi, \mathcal{S}\varpi, \mathsf{p})|| = 0$ i.e. $\varpi = \mathcal{S}\varpi$. It follows, similarly, that $\varpi = \top\varpi$.

We, show that $\mathcal{S}$ and $\top$ have a UCFP. Assume that ϖ^* in Ψ is an another CFP of $\mathcal{S}$ and $\top$. Then we get

$$\mathcal{S}\varpi^* = \top\varpi^* = \varpi^*.$$

Since $\mathcal{D} = \ddot{\delta}(\mathcal{S}\varpi, \varpi, \mathsf{p}) + \ddot{\delta}(\top\varpi^*, \varpi^*, \mathsf{p}) = 0$, therefore by definition of contraction condition

$$\ddot{\delta}(\varpi, \varpi^*, \mathsf{p}) = \ddot{\delta}(\mathcal{S}\varpi, \top\varpi^*, \mathsf{p}) = 0,$$

so that $\varpi = \varpi^*$, which conclude that the uniqueness of CFP.

Case II : if $\left(\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}}, \mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+1}, \mathsf{p})\right)$ $\times\left(\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}}, \mathsf{p})\right) = 0$ (for any $\mathfrak{k}$) implies $\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) = 0$. Now, if $\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}}, \mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) = 0$, then $\varrho_{2\mathfrak{k}} = \mathcal{S}\varrho_{2\mathfrak{k}} = \varrho_{2\mathfrak{k}+1} = \top\varrho_{2\mathfrak{k}} = \varrho_{2\mathfrak{k}+2}$. So that, we get $\varrho_{2\mathfrak{k}+1} = \mathcal{S}\varrho_{2\mathfrak{k}} = \varrho_{2\mathfrak{k}}$, so there exists $\mathfrak{k}_1$ and n_1 s. t $\mathfrak{k}_1 = \mathcal{S}\mathsf{n}_1 = \mathsf{n}_1$. Using previous arguments, one can also prove that there exist $\mathfrak{k}_2$ and n_2 s. t $\mathfrak{k}_2 = \top\mathsf{n}_2 = \mathsf{n}_2$. As $\ddot{\delta}(\mathcal{S}\mathsf{n}_1, \mathsf{n}_1, \mathsf{p}) + \ddot{\delta}(\top\mathsf{n}_2, \mathsf{n}_2, \mathsf{p}) = 0$, (due to definition) implies $\ddot{\delta}(\mathcal{S}\mathsf{n}_1, \top\mathsf{n}_2, \mathsf{p}) = 0$, thus $\mathfrak{k}_1 = \mathcal{S}\mathsf{n}_1 = \top\mathsf{n}_2 = \mathfrak{k}_2$ which in turn yields that $\mathfrak{k}_1 = \mathcal{S}\mathsf{n}_1 = \mathcal{S}\mathfrak{k}_1$. Similarly, we have $\mathfrak{k}_2 = \top\mathfrak{k}_2$. As $\mathfrak{k}_1 = \mathfrak{k}_2$, implies $\mathcal{S}\mathfrak{k}_1 = \top\mathfrak{k}_1 = \mathfrak{k}_1$, therefore $\mathfrak{k}_1 = \mathfrak{k}_2$, is a CFP of $\mathcal{S}$ and $\top$.

Now, we show that $\mathcal{S}$ and $\top$ have UCFP. Consider $\mathfrak{k}_1^*$ in Ψ is an another CFP of $\mathcal{S}$ and $\top$. Then we get

$$\mathcal{S}\mathfrak{k}_1^* = \top\mathfrak{k}_1^* = \mathfrak{k}_1^*.$$

Since $\mathcal{D} = \ddot{\delta}(\mathcal{S}\mathfrak{k}_1, \mathfrak{k}_1, \mathsf{p}) + \ddot{\delta}(\top\mathfrak{k}_1^*, \mathfrak{k}_1^*, \mathsf{p}) = 0$, therefore

$$\ddot{\delta}(\mathfrak{k}_1, \mathfrak{k}_1^*, \mathsf{p}) = \ddot{\delta}(\mathcal{S}\mathfrak{k}_1, \top\mathfrak{k}_1^*, \mathsf{p}) = 0.$$

This implies that $\mathfrak{k}_1^* = \mathfrak{k}_1$.

If $\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \varrho_{2\mathfrak{k}+1}, \mathsf{p}) + \ddot{\delta}(\top\varrho_{2\mathfrak{k}+1}, \varrho_{2\mathfrak{k}}, \mathsf{p}) = 0$, implies that $\ddot{\delta}(\mathcal{S}\varrho_{2\mathfrak{k}}, \top\varrho_{2\mathfrak{k}+1}, \mathsf{p}) = 0$, then the proof can be finished on the before lines. □

Letting $\mathcal{S} = \top$, we obtain the below results:

Corollary 2.4. *Consider* $(\Psi, \ddot{\delta})$ *be a complete TCVPS and a map* $\top : \Psi \to \Psi$ *satisfies :*

$$\ddot{\delta}(\top\varrho, \top\varepsilon, \mathsf{p}) \preceq_{i_3} \begin{cases} \mu\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) + \nu\frac{\ddot{\delta}(\varrho,\top\varrho,\mathsf{p})\ddot{\delta}(\varepsilon,\top\varepsilon,\mathsf{p})+\ddot{\delta}(\varepsilon,\top\varrho,\mathsf{p})\ddot{\delta}(\varrho,\top\varepsilon,\mathsf{p})}{\ddot{\delta}(\top\varrho,\varrho,\mathsf{p})+\ddot{\delta}(\top\varepsilon,\varepsilon,\mathsf{p})} & \\ +\gamma\frac{\ddot{\delta}(\varrho,\top\varrho,\mathsf{p})\ddot{\delta}(\varrho,\top\varepsilon)+\ddot{\delta}(\varepsilon,\top\varrho,\mathsf{p})\ddot{\delta}(\varepsilon,\top\varepsilon,\mathsf{p})}{\ddot{\delta}(\top\varrho,\varepsilon,\mathsf{p})+\ddot{\delta}(\top\varepsilon,\varrho,\mathsf{p})}, & \text{if } \mathcal{D} \neq 0,\ \mathcal{D}_1 \neq 0 \\ 0, & \text{if } \mathcal{D} = 0 \text{ or } \mathcal{D}_1 = 0 \end{cases} \tag{2.2}$$

for every $\varrho, \varepsilon \in \Psi$, *where* $\mathcal{D} = \ddot{\delta}(\top\varrho, \varrho, \mathsf{p}) + \ddot{\delta}(\top\varepsilon, \varepsilon, \mathsf{p})$ *and* $\mathcal{D}_1 = \ddot{\delta}(\top\varrho, \varepsilon, \mathsf{p}) + \ddot{\delta}(\top\varepsilon, \varrho, \mathsf{p})$ *and* μ, ν, γ *are non-negative reals with* $\mu + 2\nu + \gamma < 1$. *Then* $\top$ *has UFP.*

Example 2.5. Let $\Psi = \{0, \frac{1}{3}, 3\}$, define a map $\ddot{\delta} : \Psi \times \Psi \times (0, \infty) \to \mathbb{C}_3$ by $\ddot{\delta}(\varrho, \varepsilon) = (1 + i_3)\mathsf{p}|\varrho - \varepsilon|$, $\forall$, $\varrho, \varepsilon \in \Psi$ and $\mathsf{p} > 0$. Then $(\Psi, \ddot{\delta})$ is a complete TCVPS. Now, we define a self map $\top : \Psi \to \Psi$ by

$$\top(0) = 0,\ \top\left(\frac{1}{3}\right) = 0 \text{ and } \top(3) = \frac{1}{3}.$$

Consider $\mu = \frac{1}{6}$, $\nu = \frac{1}{5}$ and $\mu = \frac{1}{9}$, then $\mu + 2\nu + 2\gamma = \frac{1}{6} + \frac{2}{5} + \frac{2}{9} < 1$. Then, clearly

$$0 = \ddot{\delta}(\top\varrho, \top\varepsilon, \mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho, \top\varrho, \mathsf{p})\ddot{\delta}(\varepsilon, \top\varepsilon, \mathsf{p}) + \gamma\ddot{\delta}(\varepsilon, \top\varrho, \mathsf{p})\ddot{\delta}(\varrho, \top\varepsilon, \mathsf{p})}{1 + \ddot{\delta}(\varrho, \varepsilon, \mathsf{p})}.$$

Therefore, all the axioms of Corollary 2.2 are hold. Hence, 0 is the UFP of $\top$.

Example 2.6. Consider $\Psi = \mathfrak{B}(0, \mathfrak{s})$, $\mathfrak{s} > 1$, $\forall$ $\varrho, \varepsilon \in \Psi$, Define $\ddot{\delta} : \Psi \times \Psi \times (0, \infty) \to \mathbb{C}_3$ by :

$$\ddot{\delta}(\varrho(\varsigma), \varepsilon(\varsigma), \mathsf{p}) = \frac{i_3\mathsf{p}}{2\pi}\left|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma}\right|,$$

a complete TCVPS, where $\mathfrak{E}$ is a closed path in Ψ with containing 0. We show that $\ddot{\delta}$ is a tricomplex metric space. In this,

$$\begin{aligned} \ddot{\delta}(\varrho(\varsigma), \varepsilon(\varsigma), \mathsf{p}) &= \frac{i_3\mathsf{p}}{2\pi}\left|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma}\right| \\ &= \frac{i_3\mathsf{p}}{2\pi}\left|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{z(\varsigma)}{\varsigma} + \int_{\mathfrak{E}} \frac{z(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma}\right| \\ &\preceq_{i_3} \frac{i_3\mathsf{p}}{2\pi}\left|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{z(\varsigma)}{\varsigma}\right| + \frac{i_3\mathsf{p}}{2\pi}\left|\int_{\mathfrak{E}} \frac{z(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma}\right| \\ &= \ddot{\delta}(\varrho(\varsigma), z(\varsigma), \mathsf{p}) + \ddot{\delta}(z(\varsigma), \varepsilon(\varsigma), \mathsf{p}). \end{aligned}$$

Now, we define the map $\mathcal{S}, \top : \Psi \to \Psi$ follows as:

$$\mathcal{S}\varrho(\varsigma) = \varsigma,\ \top\varepsilon(\varsigma) = e^{\varsigma} - 1.$$

The maps $\mathcal{S}$ and $\top$ are analytical which is using the Cauchy integral formula, we obtain

$$\ddot{\delta}(\mathcal{S}\varrho(\varsigma), \top\varepsilon(\varsigma), \mathsf{p}) = \frac{i_3\mathsf{p}}{2\pi}\left|\int_{\mathfrak{E}} \frac{\varsigma}{\varsigma} - \int_{\mathfrak{E}} \frac{e^{\varsigma} - 1}{\varsigma}\right| = 0,$$

$$\ddot{\delta}(\varrho(\varsigma), \varepsilon(\varsigma), \mathsf{p}) = \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma}\bigg|,$$

$$\begin{aligned}\ddot{\delta}(\varrho(\varsigma), \mathcal{S}\varrho(\varsigma), \mathsf{p}) &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varsigma}{\varsigma}\bigg| \\ &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma}\bigg|,\end{aligned}$$

$$\begin{aligned}\ddot{\delta}(\varepsilon(\varsigma), \top\varepsilon(\varsigma), \mathsf{p}) &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{e^{\varsigma}-1}{\varsigma}\bigg| \\ &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varepsilon(\varsigma)}{\varsigma}\bigg|,\end{aligned}$$

$$\begin{aligned}\ddot{\delta}(\varrho(\varsigma), \mathcal{S}\varepsilon(\varsigma), \mathsf{p}) &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{\varsigma}{\varsigma}\bigg| \\ &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma}\bigg|,\end{aligned}$$

$$\begin{aligned}\ddot{\delta}(\varrho(\varsigma), \top\varepsilon(\varsigma), \mathsf{p}) &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma} - \int_{\mathfrak{E}} \frac{e^{\varsigma}-1}{\varsigma}\bigg| \\ &= \frac{i_3\mathsf{p}}{2\pi}\bigg|\int_{\mathfrak{E}} \frac{\varrho(\varsigma)}{\varsigma}\bigg|.\end{aligned}$$

Clearly,

$$\ddot{\delta}(\mathcal{S}\varrho, \top\varepsilon, \mathsf{p}) \preceq_{i_3} \mu\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho, \mathcal{S}\varrho, \mathsf{p})\ddot{\delta}(\varepsilon, \top\varepsilon, \mathsf{p}) + \gamma\ddot{\delta}(\varepsilon, \mathcal{S}\varrho, \mathsf{p})\ddot{\delta}(\varrho, \top\varepsilon, \mathsf{p})}{1 + \ddot{\delta}(\varrho, \varepsilon, \mathsf{p})},$$

for all $\varrho, \varepsilon \in \Psi$ and $\mathsf{p} > 0$, where μ, ν, γ are non-negative reals with $\mu+2\nu+2\gamma < 1$. Hence, all the axioms of Theorem 2.1 are hold, then the maps $\mathcal{S}$ and $\top$ have a UCFP in Ψ.

3. An application

In this part, we apply Corollary 2.2 to discuss the existence solution to linear integral equation.

Theorem 3.1. *Let $\Psi = \mathbb{C}^{\mathfrak{k}}$ be a TCVPS with*

$$\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) = \mathsf{p}\sum_{i=1}^{\mathfrak{k}} \big(|\varrho_i - \varepsilon_i| + i_3|\varrho_i - \varepsilon_i|\big),$$

where $\varrho, \varepsilon \in \Psi$ and $\mathsf{p} > 0$. If

$$\sum_{j=1}^{\mathfrak{k}} |\mu_{ij}| \preceq_{i_3} \mu < 1,\ \forall\ i = 1, 2, ...\mathfrak{k},$$

then the linear system equations below

$$\begin{cases} \mathfrak{b}_1 = \mathfrak{a}_{11}\varrho_1 + \mathfrak{a}_{12}\varrho_2 + \cdots + \mathfrak{a}_{1\mathfrak{k}}\varrho_{\mathfrak{k}} \\ \mathfrak{b}_2 = \mathfrak{a}_{21}\varrho_1 + \mathfrak{a}_{22}\varrho_2 + \cdots + \mathfrak{a}_{2\mathfrak{k}}\varrho_{\mathfrak{k}} \\ \quad \vdots \\ \mathfrak{b}_{\mathfrak{k}} = \mathfrak{a}_{\mathfrak{k}1}\varrho_1 + \mathfrak{a}_{\mathfrak{k}2}\varrho_2 + \cdots + \mathfrak{a}_{\mathfrak{k}\mathfrak{k}}\varrho_{\mathfrak{k}} \end{cases}$$

has a unique solution.

Proof. Define $\top : \Psi \to \Psi$ as

$$\top(\varrho) = \mathcal{A}\varrho + \mathfrak{b},$$

here $\varrho = (\varrho_1, \varrho_2, \varrho_3, \ldots.\varrho_{\mathfrak{k}}) \in \mathbb{C}^{\mathfrak{k}}$, $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \ldots.\mathfrak{b}_{\mathfrak{k}}) \in \mathbb{C}^{\mathfrak{k}}$ and

$$\mathcal{A} = \begin{pmatrix} \mathfrak{a}_{11} & \mathfrak{a}_{11} & \cdots & \mathfrak{a}_{1\mathfrak{k}} \\ \mathfrak{a}_{21} & \mathfrak{a}_{22} & \cdots & \mathfrak{a}_{2\mathfrak{k}} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{a}_{\mathfrak{k}1} & \mathfrak{a}_{\mathfrak{k}2} & \cdots & \mathfrak{a}_{\mathfrak{k}\mathfrak{k}} \end{pmatrix}.$$

Now,

$$\begin{aligned} \ddot{\delta}(\top(\varrho), \top(\varepsilon), \mathsf{p}) &= \mathsf{p}\sum_{j=1}^{\mathfrak{k}} (|\mu_{ij}(\varrho_j - \varepsilon_j)| + i_3|\mu_{ij}(\varrho_j - \varepsilon_j)|) \\ &\preceq_{i_3} \mathsf{p}\sum_{j=1}^{\mathfrak{k}} |\mu_{ij}| \left(\sum_{j=1}^{\mathfrak{k}} (|(\varrho_j - \varepsilon_j)| + i_3|(\varrho_j - \varepsilon_j)|) \right) \\ &\preceq_{i_3} \mathsf{p}\mu \sum_{j=1}^{\mathfrak{k}} (|(\varrho_j - \varepsilon_j)| + i_3|(\varrho_j - \varepsilon_j)|) \\ &= \mu\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) \\ &= \mu\ddot{\delta}(\varrho, \varepsilon, \mathsf{p}) + \frac{\nu\ddot{\delta}(\varrho, \top\varrho, \mathsf{p})\ddot{\delta}(\varepsilon, \top\varepsilon, \mathsf{p}) + \gamma\ddot{\delta}(\varepsilon, \top\varrho, \mathsf{p})\ddot{\delta}(\varrho, \top\varepsilon, \mathsf{p})}{1 + \ddot{\delta}(\varrho, \varepsilon, \mathsf{p})}. \end{aligned}$$

Therefore, all the axioms of Corollary 2.2 are hold with $\mu = \frac{1}{6}, \nu = 0, \gamma = 0$, $\mu + 2\nu + 2\gamma < 1$. Hence, there is unique solution to the linear system of equations. □

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