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# STRUCTURAL STABILITY OF $p(x)$-LAPLACIAN KIND PROBLEMS WITH MAXIMAL MONOTONE GRAPHS AND NEUMANN TYPE BOUNDARY CONDITION

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Abstract. In this work, we study the convergence of a sequence of solutions of degenerate elliptic problems with variable coercivity and growth exponents. The functional setting involves Lebesgue and Sobolev spaces with variable exponent which varies also with $n$.

## 1. Introduction

Let $\Omega$ be an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $\eta$ is the outer unit normal to $\partial\Omega$. Our aim is to study the structural stability (i.e., the continuous dependence on coefficients) of solutions of the following elliptic problems

$$(Pb_n) : \begin{cases} \beta_n(u_n) - \operatorname{div} a_n(x, \nabla u_n) = f_n & \text{in } \Omega, \\ a_n(x, \nabla u_n).\eta = 0 & \text{on } \partial\Omega, \end{cases}$$

where $(\beta_n)_{n\in\mathbb{N}^*}$ is an associated sequence of the Yosida regularisation of a maximal monotone graph $\beta = \partial j$ defined in $\mathbb{R}$ such that $\mathrm{dom}(\beta)$ is bounded on $\mathbb{R}$, $\overline{\mathrm{dom}(\beta)} = [m, M] \subset \mathbb{R}$ with $m \leq 0 \leq M$ and $0 \in \beta(0)$; $(a_n(x,\xi))_{n\in\mathbb{N}^*}$ is a family of applications which verify the classical Leray-Lions hypotheses but with a variable summability exponent $p_n(.)$ converging in measure to some exponent $p(.)$ such that $1 < p_- \leq p_n(.) \leq p_+ < \infty$, $(f_n)_{n\in\mathbb{N}^*} \subset L^1(\Omega)$. The result is known in the case where $\beta_n = b$ (independent of $n$) is a continuous, onto and nondecreasing function such that $b(0) = 0$ (see [12]). But, one loses stability (strictly speaking) when one has maximal monotone graphs.

In this paper, we use a maximal monotone graph $\beta$ with a bounded domain that reveals a bounded Radon diffuse measure in order to take into account the border of the domain. We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measure in $\Omega$, equipped with its standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$. Given $\nu \in \mathcal{M}_b(\Omega)$, we say that $\nu$ is diffuse with respect to the capacity $W^{1,p(.)}(\Omega)$( $p(.)-$capacity for short) if $\nu(E) = 0$

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for every set $E$ such that $Cap_{p(.)}(E, \Omega) = 0$, where the Sobolev $p(.)-$capacity of $E$ is defined by

$$Cap_{p(.)}(E, \Omega) = \inf_{u \in S_{p(.)}(E)} \int_{\Omega} \left( |u|^{p(x)} + |\nabla u|^{p(x)} \right) dx,$$

with

$$S_{p(.)}(E) = \left\{ u \in W^{1,p(.)}(\Omega) : \, u \geq 1 \text{ in an open set containing } E \text{ and } u \geq 0 \text{ in } \Omega \right\}.$$

In the case where $S_{p(.)}(E) = \emptyset$, we set $Cap_{p(.)}(E, \Omega) = \infty$. The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$. Problems with variable exponents $p(x)$ and $p_n(x)$ were appeared and studied by Zhikov in the pioneering paper [19]. The use of $p(.)$-Laplacian operator in models of electrorheological and thermorheological fluids dynamics (see [6, 15, 16, 17]) and in models of image processing (see [5]) has significantly contributed to boost the fields. The structural stability results that we prove in this paper are useful, in particular, for the study of convergence of numerical approximations of the $p(x)-$Laplacian. Indeed, it is necessary, for such a numerical study, to approach $p(x)$ by some piecewise constant or piecewise polynomial functions $p_h(x)$, $h$ being the discretization parameter (see e.g. [11]).
Let us give the outline of the paper. In Section 2, we make some useful assumptions and preliminary for the sequel. In Section 3, we tackle the question of continuous dependence for renormalized solutions.

## 2. Assumptions and preliminary

The model problems for our study are

$$(Pb) : \begin{cases} b(u) - \operatorname{div} a(x, \nabla u) = f & \text{in } \Omega, \\ a(x, \nabla u).\eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is continuous, onto, non-decreasing such that $b(0) = 0$ and

$$(Pb^{Gr}) : \begin{cases} \beta(u) - \operatorname{div} a(x, \nabla u) \ni f & \text{in } \Omega, \\ a(x, \nabla u).\eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\beta$ is a maximal monotone graph $\beta = \partial j$ defined in $\mathbb{R}$ such that $\operatorname{dom}(\beta)$ bounded on $\mathbb{R}$, $\overline{\operatorname{dom}(\beta)} = [m, M] \subset \mathbb{R}$ with $m \leq 0 \leq M$ and $0 \in \beta(0)$, $a(.,.)$ is a Leray-Lions operator which involves variable exponents.
For every $\varepsilon > 0$, we consider the Yosida regularisation $\beta_\varepsilon$ of $\beta$ defined by

$$\beta_\varepsilon = \frac{1}{\varepsilon} \left( I - (I + \varepsilon \beta)^{-1} \right).$$

The function $\beta_\varepsilon$ is a nondecreasing, surjective, Lipschitz-continuous and verifies $\beta_\varepsilon(0) = 0$.

$$p : \overline{\Omega} \longrightarrow \mathbb{R} \text{ is a continuous function such that } 1 < p_- \leq p_+ < \infty, \tag{2.3}$$

where $p_- := \inf_{x \in \Omega} p(x)$ and $p_+ := \sup_{x \in \Omega} p(x)$.
$a(.,.) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$, the

strict monotonicity assumption

$$(a(x,\xi)-a(x,\eta)).(\xi-\eta)>0 \text{ for all } \xi,\eta\in\mathbb{R}^N \text{ such that } \xi\neq\eta \tag{2.4}$$

and the following growth and coercivity assumptions with respect to $\xi$.

$$|a(x,\xi)|\le C_1(\mathcal{M}(x)+|\xi|^{p(x)-1}), \tag{2.5}$$

$$a(x,\xi).\xi\ge C_2|\xi|^{p(x)}, \tag{2.6}$$

where $C_1$ and $C_2$ are positive constants, $\mathcal{M}$ is a nonnegative function in $L^{p'(.)}(\Omega)$ with $1/p(x)+1/p'(x)=1$.
As consequence to the continuity and the coercivity of $a(x,.)$, one has

$$a(x,0)=0 \text{ for a.e. } x\in\Omega. \tag{2.7}$$

For any exponent $p(.)$, we denote by $p'(.)$ its conjugate exponent such that $1/p(x)+1/p'(x)=1$ and by $p^*(.)$ its optimal Sobolev embedding exponent such that

$$p^*(.):=\begin{cases} Np(.)/(N-p(.)) & \text{if } p(.)<N,\\ \text{any real value} & \text{if } p(.)=N,\\ \infty & \text{if } p(.)>N.\end{cases}$$

For any given $k>0$, we define the truncation function $T_k:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$T_k(r)=\max(\min(r,k),-k).$$

One denotes by

$$sign_0(z)=\begin{cases} 1 & \text{if } z>0,\\ 0 & \text{if } z=0,\\ -1 & \text{if } z<0.\end{cases}$$

The truncation function $T_k$ has the following properties.

$$|T_k(z)|=\min(|z|,k),\ \lim_{k\to\infty}T_k(z)=z \text{ and } \lim_{k\to 0}\frac{1}{k}T_k(z)=sign_0(z).$$

For a Lebesgue measurable set $A\subset\Omega$, $\chi_A$ denotes its characteristic function and $meas(A)$ denotes its Lebesgue measure.
Let $u:\Omega\longrightarrow\mathbb{R}$ be a function and $k\in\mathbb{R}$, we write $\{|u|\le k\}$ for the set $\{x\in\Omega;\ |u(x)|\le k\}$, (respectively, $\ge,=,<,>$). We will also need to truncate vector-valued functions with the help of the following maps.

$$\text{for } m>0,\ h_m:\mathbb{R}^N\longrightarrow\mathbb{R}^N,\ h_m(\lambda)=\begin{cases}\lambda & \text{if } |\lambda|\le m,\\ \dfrac{m}{|\lambda|}\lambda & \text{if } |\lambda|>m.\end{cases} \tag{2.8}$$

One has the following property (see [1, Lemma 2.1]).

**Lemma 2.1.** *Let $h_m(.)$ be defined by (2.8) and $a(x,.)$ be monotone in the sense (2.4). Then, for all $\lambda\in\mathbb{R}^N$, the map $m\longmapsto a(x,h_m(\lambda)).h_m(\lambda)$ is nondecreasing and converges to $a(x,\lambda).\lambda$ as $m\to\infty$.*

The exponent $p(.)$ appearing in (2.5) and (2.6) depends on the spatial variable $x$ and then requires so to work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u:\Omega\longrightarrow\mathbb{R}$ for which the convex modular
$$\rho_{p(.)}(u):=\int_\Omega |u|^{p(x)}\mathrm{d}x$$
is finite. If the exponent is bounded, i.e., if $p_+<\infty$, then the expression
$$\|u\|_{p(.)}:=\|u\|_{L^{p(\cdot)}(\Omega)}:=\inf\left\{\lambda>0;\ \rho_{p(.)}\left(\frac{u}{\lambda}\right)\leq 1\right\}$$
defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $\left(L^{p(\cdot)}(\Omega),\|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1<p_-\leq p_+<\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$. Moreover, one has the Hölder type inequality
$$\left|\int_\Omega uv\mathrm{d}x\right|\leq\left(\frac{1}{p_-}+\frac{1}{(p')_-}\right)\|u\|_{p(.)}\|v\|_{p'(.)}\,, \tag{2.9}$$
for all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$.
$W^{1,p(\cdot)}(\Omega)$ denotes the space of all functions $u\in L^{p(\cdot)}(\Omega)$ such that their gradients $\nabla u$, taken in the sense of distributions, belong to $(L^{p(\cdot)}(\Omega))^N$. This space is a Banach space equipped with the following norm.
$$\|u\|_{1,p(.)}:=\|u\|_{W^{1,p(\cdot)}(\Omega)}:=\|u\|_{p(.)}+\|\nabla u\|_{p(.)}\,.$$
The space $\left(W^{1,p(\cdot)}(\Omega),\|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space. For more details on the generalized Lebesgue and Sobolev spaces, see [13]. In the sequel, we will use the same notation $L^{p(\cdot)}(\Omega)$ for the space $(L^{p(\cdot)}(\Omega))^N$ of vector-valued functions.
In manipulating the generalized Lebesgue and Sobolev spaces, the following lemma (cf.[10]) permits to pass from norm to convex modular and vice-versa.

**Lemma 2.2.** *If $u_n,u\in L^{p(\cdot)}(\Omega)$ and $p_+<\infty$, then the following properties hold true.*
*(i)* $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right)=1$, *if* $u\neq 0$.
*(ii)* $\rho_{p(.)}(u)<1$ (*respectively* $=1;>1$) $\iff \|u\|_{p(.)}<1$ (*respectively* $=1;>1$).
*(iii)* $\rho_{p(.)}(u)\leq 1\Longrightarrow \|u\|_{p(.)}^{p_+}\leq\rho_{p(.)}(u)\leq\|u\|_{p(.)}^{p_-}$.
*(iv)* $\rho_{p(.)}(u)\geq 1\Longrightarrow \|u\|_{p(.)}^{p_-}\leq\rho_{p(.)}(u)\leq\|u\|_{p(.)}^{p_+}$.
*(v)* $\|u_n\|_{p(.)}\to 0$ (*respectively* $\to\infty$) $\iff \rho_{p(.)}(u_n)\to 0$ (*respectively* $\to\infty$).

For a measurable function $u:\Omega\longrightarrow\mathbb{R}$, we introduce the function
$$\rho_{1,p(.)}(u):=\int_\Omega|u|^{p(x)}\mathrm{d}x+\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x.$$
Then, one has the following lemma (see [18]).

**Lemma 2.3.** *If $u_n,u\in W^{1,p(\cdot)}(\Omega)$ and $p_+<\infty$, then the following properties hold true.*
*(i)* $\rho_{1,p(.)}(u)<1$ (*respectively* $=1;>1$) $\iff \|u\|_{1,p(.)}<1$ (*respectively* $=1;>1$).
*(ii)* $\rho_{1,p(.)}(u)\leq 1\Longrightarrow \|u\|_{1,p(.)}^{p_+}\leq\rho_{1,p(.)}(u)\leq\|u\|_{1,p(.)}^{p_-}$.

*(iii)* $\rho_{1,p(.)}(u) \geq 1 \Longrightarrow \|u\|_{1,p(.)}^{p_-} \leq \rho_{1,p(.)}(u) \leq \|u\|_{1,p(.)}^{p_+}$.
*(iv)* $\|u_n\|_{1,p(.)} \to 0$ *(respectively* $\to \infty$*)* $\Longleftrightarrow \rho_{1,p(.)}(u_n) \to 0$ *(respectively* $\to \infty$*).*

One has below, imbedding result between Lebesgue and Sobolev spaces with varible exponent (see [8, 10]).

**Proposition 2.4.** *Let $p, q \in C(\overline{\Omega})$ with $p_- > 1$. Assume that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. Then, there is a compact imbedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$. In particular, there is a compact imbedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$.*

The following result (a corollary of Lebesgue dominated convergence theorem) is useful to prove strong convergence results.

**Lemma 2.5** (Lebesgue generalized convergence theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and $f$ a measurable function such that $f_n \to f$ a.e. in $\Omega$. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that for all $n \in \mathbb{N}$, $|f_n| \leq g_n$ a.e. in $\Omega$ and $g_n \to g$ in $L^1(\Omega)$. Then,*

$$\int_\Omega f_n \mathrm{d}x \longrightarrow \int_\Omega f \mathrm{d}x.$$

**Lemma 2.6.** *(Cf.[14]) Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of graphs (i.e. for $(x, y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$. We suppose that: $\forall n \geq 1$, $w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then $z \in dom(\beta)$.*

For the applications we have in mind, we will need the following theorem (cf. [1]) in which the results of *(ii)* and *(iii)* express measure convergence of some sequences.

**Theorem 2.7** (Young measures and nonlinear weak-* convergence)**.**

*(i)* *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and $(v_n)_{n\in\mathbb{N}}$ an equi-integrable sequence in $\Omega$ of functions to values in $\mathbb{R}^d$, $d \in \mathbb{N}$. Then, there exists a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$, weakly measurable in $x$ with respect to the Lebesgue measure on $\Omega$, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, one has*

$$\lim_{k\to\infty} \int_\Omega F(x, v_{n_k}(x))\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^d} F(x, \lambda)\mathrm{d}\nu_x(\lambda)\mathrm{d}x, \tag{2.10}$$

*whenever the sequence $(F(., v_n(.)))_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$. In particular,*

$$v(x) := \int_{\mathbb{R}^d} \lambda \mathrm{d}\nu_x(\lambda) \tag{2.11}$$

*is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ in $L^1(\Omega)$, as $k \to \infty$.*
*The family $(\nu_x)_{x\in\Omega}$ is called the Young measure generated by the subsequence $(v_{n_k})_{k\in\mathbb{N}}$.*

*(ii)* *If $\Omega$ is of finite measure, and $(\nu_x)_{x\in\Omega}$ is the Young measure generated by a sequence $(v_n)_{n\in\mathbb{N}}$, then*

$$\big(\nu_x = \delta_{v(x)} \text{ a.e. } x \in \Omega\big) \Longleftrightarrow (v_n \text{ converges in measure on } \Omega \text{ to } v \text{ as } n \to \infty).$$

(*iii*) *If* $\Omega$ *is of finite measure,* $(u_n)_{n\in\mathbb{N}}$ *generates a Dirac Young measure* $\left(\delta_{u(x)}\right)_{x\in\Omega}$ *on* $\mathbb{R}^{d_1}$*, and* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_2}$*, then the sequence* $((u_n,v_n))_{n\in\mathbb{N}}$ *generates the Young measure* $\left(\delta_{u(x)}\otimes\nu_x\right)_{x\in\Omega}$ *on* $\mathbb{R}^{d_1+d_2}$.

Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$, following the terminology of [9] we will say that $(v_n)_{n\in\mathbb{N}}$ nonlinear weak-* converges, and $(\nu_x)_{x\in\Omega}$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. Theorem 2.7$-(i)$ thus means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on $\Omega$.

## 3. Renormalized solutions

We define $\mathcal{T}^{1,p(.)}(\Omega)$ as the set of functions $u:\Omega\longrightarrow\mathbb{R}$ measurable such that $T_k(u)\in W^{1,p(.)}(\Omega)$, for any $k>0$.
The following proposition (see e.g. [2]) is useful because it allows us to give a sense to the definition of the solutions for problems (2.1) and (2.2).

**Proposition 3.1.** *Let* $u\in\mathcal{T}^{1,p(.)}(\Omega)$*. Then, there exists a unique measurable function* $v:\Omega\longrightarrow\mathbb{R}^N$ *such that*

$$\nabla T_k(u)=v\chi_{\{|u|<k\}},\ \textit{for all } k>0,$$

*where* $\chi_E$ *is the characteristic function of a measurable set* $E$*. The function* $v$ *is a generalized gradient and is denoted by* $\nabla u$ *(weak gradient of* $u$*). If, moreover,* $u$ *belongs to* $W^{1,p(.)}(\Omega)$*, then* $v$ *belongs to* $\left(L^{p(.)}(\Omega)\right)^N$ *and coincides with the standard distributional gradient of* $u$.

We define $\mathcal{T}^{1,p(.)}_{\mathcal{H}}(\Omega)$ (see [3]) as the set of functions $u\in\mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)\subset W^{1,p(.)}(\Omega)$ satisfying the following conditions.

(i) $u_n\longrightarrow u$ a.e. in $\Omega$;
(ii) $\nabla T_k(u_n)\longrightarrow\nabla T_k(u)$ in $L^1(\Omega)$ for any $k>0$.

The symbol $\mathcal{H}$ in the notation is related to the fact that we consider here homogeneous Neumann boundary condition.
We define also $\mathbb{S}$ as the set of $W^{1,\infty}$ functions $S:\mathbb{R}\longrightarrow\mathbb{R}$ having a compact support.
The following function,

$$\text{for } k>0,\ S_k: z\in\Omega\longmapsto\begin{cases}1 & \text{if } |z|\le k-1,\\ k-|z| & \text{if } k-1\le|z|\le k,\\ 0 & \text{if } |z|\ge k,\end{cases}\tag{3.1}$$

is an example of function in $\mathbb{S}$ that will be used a lot in the sequel. Note that this function is nonnegative with $suppS_k=[-k,k]$ and $suppS'_k$ is contained in $[-k,-k+1]\cup[k-1,k]$ and that the sequences $S_k$ and $S'_k$ are uniformly bounded by one.

**Definition 3.2.** A measurable function $u : \Omega \to \mathbb{R}$ is a renormalized solution to problem (2.1) if $T_k(u) \in W^{1,p(.)}(\Omega)$, $b(u) \in L^1(\Omega)$,

$$\lim_{k\to\infty} \int_{\{k<|u|<k+1\}} a(x, \nabla u).\nabla u \mathrm{d}x = 0, \tag{3.2}$$

and, for all $S \in \mathbb{S}$ and for all $\phi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big(S(u)a(x, \nabla u).\nabla\phi + S'(u)a(x, \nabla u).(\nabla u)\phi + b(u)S(u)\phi\Big)\mathrm{d}x = \int_\Omega fS(u)\phi \mathrm{d}x. \tag{3.3}$$

**Definition 3.3.** A solution of problem (2.2) is a couple $(u, b) \in \mathcal{T}^{1,p(.)}_{\mathcal{H}}(\Omega) \times L^1(\Omega)$, such that

$$\left[\begin{array}{l} u \in dom(\beta)\mathcal{L}^N - \text{a.e. in } \Omega,\ b \in \beta(u)\mathcal{L}^N - \text{a.e. in } \Omega, \\ \text{there exists } \mu \in \mathcal{M}^{p(.)}_b(\Omega) \text{ with } \mu \perp \mathcal{L}^N, \\ \ \mu^+ \text{ is concentrated on } \{u = M\},\ \mu^- \text{ is concentrated on } \{u = m\}, \end{array}\right.$$

and

$$\int_\Omega a(x, \nabla u).\nabla\phi \mathrm{d}x + \int_\Omega b\phi \mathrm{d}x + \int_\Omega \phi \mathrm{d}\mu = \int_\Omega f\phi \mathrm{d}x,\ \forall\phi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega). \tag{3.4}$$

One has the following results.

**Theorem 3.4.** *([12]) Assume (2.3) – (2.6) and $f \in L^1(\Omega)$. Then, there exists a unique renormalized solution to problem (2.1).*

**Theorem 3.5.** *([14]) Assume (2.3) – (2.6) and $f \in L^1(\Omega)$. Then, there exists a unique solution to problem (2.2).*

## 4. Continuous dependence for renormalized solutions

We consider elliptic problems

$$(Pb_n) : \begin{cases} \beta_n(u_n) - \operatorname{div} a_n(x, \nabla u_n) = f_n & \text{in } \Omega, \\ a_n(x, \nabla u_n).\eta = 0 & \text{on } \partial\Omega, \end{cases}$$

where the assumptions (2.3) – (2.6) are verified, for all $n \in \mathbb{N}^*$, with the diffusion flux functions $a_n(.,.)$, the exponents $p_n(.) : \overline{\Omega} \longrightarrow [p_-, p_+]$ and the nonnegative functions $\mathcal{M}_n$ in $L^{p'_n(.)}(\Omega)$ are such that the sequence $\left(\|\mathcal{M}_n\|_{L^{p'_n(.)}(\Omega)}\right)_{n\in\mathbb{N}^*}$ is uniformly bounded, with $C_1$, $C_2$, $p_+$ and $p_-$ independent of $n$. Moreover, $\beta_n = n\left(I - \left(I + \frac{1}{n}\beta\right)^{-1}\right)$, for $n \geq 1$.

Note that, for $f_n \in L^1(\Omega)$ and under assumptions (2.3) – (2.6), the problem $(Pb_n)$ admits a unique renormalized solution $u_n$ according to Theorem 3.4.

The purpose of this section is to prove that the sequence of solutions $(u_n)_{n\in\mathbb{N}^*}$ to problems $(Pb_n)$ converges to a function $u$ which is a solution of the limit problem (2.2) with the exponent $p(.)$ under the following convergence assumptions.

$$\left[\begin{array}{l}\text{for all bounded subset K of } \mathbb{R}^N,\\ \sup\limits_{\xi\in K}|a_n(.,\xi)-a(.,\xi)| \text{ converges to zero in measure on } \Omega,\end{array}\right. \tag{4.1}$$

where $a(x,\xi)$ verifies the assumptions (2.4) − (2.6) with the exponent $p(.)$ verifying (2.3) such that

$$p_n(.) \text{ converges to } p(.) \text{ in measure on } \Omega. \tag{4.2}$$

One assumes also that

$$f_n \text{ converges to } f \text{ weakly in } L^1(\Omega). \tag{4.3}$$

One further assumes that the exponents $p(.)$ and $p_n(.)$ verify the following log-Hölder continuity assumption.

$$\exists c>0,\ \forall x,y\in\overline{\Omega},\ x\neq y,\ -(\log|x-y|)|p(x)-p(y)|\leq c. \tag{4.4}$$

*Remark* 4.1. Note that several regularity results for Sobolev spaces with variable exponents can be obained thanks to log-Hölder continuity condition (4.4); in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(.)}(\Omega)$ (for more details, see [7]).

Now, through the theorem below, we establish a structural stability result for the renormalized solutions.

**Theorem 4.2.** *Under the assumptions (4.1) − (4.3), let $(u_n)_{n\in\mathbb{N}^*}$ be the sequence of renormalized solutions of the problems $(Pb_n)$ associated to $a_n(.,.)$, $f_n$ and the exponents $p_n$ with $a(.,.)$, $f$ and $p$ as limits of $a_n$, $f_n$ and $p_n(.)$ respectively in (4.1)–(4.3).*

*Assume also that the exponents $p(.)$, $p_n(.)$ satisfy the log-Hölder continuity assumption (4.4).*

*Then there exists a measurable function $u$ on $\Omega$ such that $u_n$, $\nabla u_n$ converge to $u$, $\nabla u$, respectively, a.e. in $\Omega$, as $n\to\infty$. The function $u$ is, in fact, a solution of the problem (2.2) associated to the diffusion flux $a(.,.)$ and the source term $f$.*

**Proof of Theorem 4.2.** We divide the proof into several steps. Throughout the proof, we reason up to an extracted subsequence of $(u_n)_{n\in\mathbb{N}^*}$ (still denoted $(u_n)_{n\in\mathbb{N}^*}$) and all constant independent of $n$ will be denoted by $C$.

**Lemma 4.3.** *(i) For all $k>0$, the sequence $\left(\|T_k(u_n)\|_{1,p_n(.)}\right)_{n\in\mathbb{N}^*}$ is bounded.*

*(ii) The sequence of renormalized solutions $(u_n)_{n\in\mathbb{N}^*}$ of the problems $(Pb_n)$ verifies, for $k>0$ large enough,*

$$meas(\{|u_n|>k\})\leq\frac{C\,\|f_n\|_{L^1}}{\min\left(\beta_n(k),|\beta_n(-k)|\right)}, \tag{4.5}$$

$$\sup_{n\in\mathbb{N}^*} meas(\{|u_n|>k\})\to 0,\ \text{ as } k\to\infty, \tag{4.6}$$

*and*

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}^*}\int_{\{k<|u_n|<k+1\}}|\nabla u_n|^{p_n(x)}\mathrm{d}x=0. \tag{4.7}$$

(*iii*) *There exists a measurable function $u$ on $\Omega$ such that $u \in dom(\beta)$ a.e. in $\Omega$ and, up to a subsequence, $T_k(u_n) \rightharpoonup T_k(u)$ in $W^{1,p_-}(\Omega)$, for all $k > 0$. Moreover, $u_n \to u$ a.e. in $\Omega$, $\nabla T_k(u_n)$ converges to a Young measure $(\nu_x^k)_{x\in\Omega}$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-$^*$ convergence and*

$$\nabla T_k(u) = \int_{\mathbb{R}^N} \lambda \mathrm{d}\nu_x^k(\lambda). \tag{4.8}$$

(*iv*) *For all $k > 0$, $|\lambda|^{p(x)}$ is integrable with respect to the measure $\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x$ on $\mathbb{R}^N \times \Omega$ and $T_k(u) \in W^{1,p(.)}(\Omega)$.*

*Proof.* (*i*) In the renormalized formulation (3.3) of the problem $(Pb_n)$, we choose $S = S_{h+k} \in \mathbb{S}$ defined in (3.1) with $h, k > 0$, $h$ large enough. Since $u_n$ is a renormalized solution of the problem $(Pb_n)$, it follows that $T_k(u_n) \in W^{1,p_n(.)}(\Omega) \cap L^\infty(\Omega)$ and so, one can take $\phi = T_k(u_n)$ as test function in the renormalized formulation (3.3) to obtain (as the term $\int_\Omega \beta_n(u_n)S(u_n)T_k(u_n)\mathrm{d}x$ is nonnegative),

$$\int_\Omega a_n(x, \nabla T_k(u_n)).\nabla T_k(u_n)\mathrm{d}x + \int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)T_k(u_n)\mathrm{d}x \le k\int_\Omega |f_n|\mathrm{d}x.$$

While $k$ is fixed, $h$ can be sent to infinity. By (3.2), the second integral vanishes, as $h \to \infty$, since

$$\left|\int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)T_k(u_n)\mathrm{d}x\right| \le k\int_{\{h+k-1<|u_n|<h+k\}} a_n(x, \nabla u_n).\nabla u_n\mathrm{d}x.$$

So, by using the coercivity condition (2.6), as $h \to \infty$ one has

$$C\int_\Omega |\nabla T_k(u_n)|^{p_n(x)}\mathrm{d}x \le k\int_\Omega |f_n|\mathrm{d}x.$$

Since the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges weakly to $f$ in $L^1(\Omega)$, then the right-hand side of this last inequality is uniformly bounded. So, one obtains

$$\int_\Omega |\nabla T_k(u_n)|^{p_n(x)}\mathrm{d}x \le Ck. \tag{4.9}$$

Moreover,

$$\begin{aligned}\int_\Omega |T_k(u_n)|^{p_n(x)}\mathrm{d}x &\le \int_\Omega k^{p_n(x)}\mathrm{d}x \\ &\le \max(k^{p_+}, k^{p_-})meas(\Omega).\end{aligned} \tag{4.10}$$

From (4.9) and (4.10), one deduces that the sequence $\rho_{1,p_n(.)}(T_k(u_n))$ is uniformly bounded. By the Lemma 2.3 and the fact that $p_n(.) \in [p_-, p_+]$, one has

$$\|T_k(u_n)\|_{1,p_n(.)} \le \max\left(\rho_{1,p_n(.)}(T_k(u_n))^{1/p_-}, \rho_{1,p_n(.)}(T_k(u_n))^{1/p_+}\right).$$

Therefore, the sequence $\|T_k(u_n)\|_{1,p_n(.)}$ is uniformly bounded.

$(ii)$ In the renormalized formulation (3.3) of problem $(Pb_n)$, one takes $S = S_k \in \mathbb{S}$ and $\phi = T_{\frac{1}{k}}(u_n)$ as test function, with $k > 0$ large enough, to get

$$\int_\Omega S(u_n)a_n(x, \nabla u_n).\nabla T_{\frac{1}{k}}(u_n)\mathrm{d}x + \int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x$$
$$+ \int_\Omega \beta_n(u_n)S(u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x = \int_\Omega f_n S(u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x$$

which gives

$$\int_\Omega a_n\left(x, \nabla T_{\frac{1}{k}}(u_n)\right).\nabla T_{\frac{1}{k}}(u_n)\mathrm{d}x + \int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x$$
$$+ \int_\Omega \beta_n(u_n)S(u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x \le \frac{1}{k}\|f_n\|_{L^1(\Omega)}$$

or again

$$k\int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x + \int_\Omega \beta_n(u_n)S(u_n)kT_{\frac{1}{k}}(u_n)\mathrm{d}x \le \|f_n\|_{L^1(\Omega)}.$$

The term $k\int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)T_{\frac{1}{k}}(u_n)\mathrm{d}x$ vanishes, as $k \to \infty$, due to (3.2). Note also that one has $kT_{\frac{1}{k}}(u_n) \to sign_0(u_n)$ as $k \to \infty$. So, by using Fatou's lemma, one gets, as $k \to \infty$,

$$\int_\Omega |\beta_n(u_n)|\mathrm{d}x \le \|f_n\|_{L^1(\Omega)}. \tag{4.11}$$

Therefore, for $k > 0$,

$$\int_{\{|u_n|>k\}} |\beta_n(u_n)|\mathrm{d}x \le \|f_n\|_{L^1(\Omega)}. \tag{4.12}$$

Since $|\beta_n(u_n)| \ge \min(\beta_n(k), |\beta_n(-k)|)$ on $\{|u_n| > k\}$, then the relation (4.12) gives

$$\min(\beta_n(k), |\beta_n(-k)|)meas(\{|u_n| > k\}) \le \|f_n\|_{L^1(\Omega)}$$

or again

$$meas(\{|u_n| > k\}) \le \frac{\|f_n\|_{L^1(\Omega)}}{\min(\beta_n(k), |\beta_n(-k)|)}. \tag{4.13}$$

Being weakly convergent in $L^1(\Omega)$, the sequence $(f_n)_{n\in\mathbb{N}^*}$ is bounded, so the right-hand side of (4.13) tends to zero as $k \to \infty$, then $meas(\{|u_n| > k\})$ tends to zero as $k \to \infty$ uniformly in $n$ and (4.6) is proved.

For the proof of (4.7), let's take $\phi = T_{k+1}(u_n) - T_k(u_n)$ as test function and $S \in \mathbb{S}$ such that $S = S_{k+2}$ in the renormalized formulation (3.3). The function $T_{k+1}(u_n) - T_k(u_n)$ has a support contained in the set $\{|u_n| \ge k\}$ and is bounded by one. One obtains by (2.6),

$$C\int_{\{k<|u_n|<k+1\}} |\nabla u_n|^{p_n(x)}\mathrm{d}x + \int_\Omega S'(u_n)a_n(x, \nabla u_n).(\nabla u_n)\phi\mathrm{d}x \le \int_{\{|u_n|\ge k\}} |f_n|\mathrm{d}x. \tag{4.14}$$

By the property (3.2) and by the equi-integrability of $f_n$, and according to (4.6), for $k \to \infty$, one deduces, from (4.14), the estimate (4.7).

$(iii)$ From Lemma 4.3 $-$ $(i)$, one gets

$$\begin{aligned}\|T_k(u_n)\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega |T_k(u_n)|^{p_-}\mathrm{d}x + \int_\Omega |\nabla T_k(u_n)|^{p_-}\mathrm{d}x \\ &\leq \int_\Omega \left(1+|T_k(u_n)|^{p_n(x)}\right)\mathrm{d}x + \int_\Omega \left(1+|\nabla T_k(u_n)|^{p_n(x)}\right)\mathrm{d}x \\ &\leq 2meas(\Omega) + \rho_{1,p_n(.)}(T_k(u_n)) \\ &\leq const(k).\end{aligned}$$

So, the sequence $T_k(u_n)$ is uniformly bounded in $W^{1,p_-}(\Omega)$. Therefore, up to a subsequence, one can assume that the sequence $T_k(u_n)$ converges to a certain function $\sigma_k$ weakly in $W^{1,p_-}(\Omega)$, and by the compact imbedding theorem of $W^{1,p_-}(\Omega)$ in $L^{p_-}(\Omega)$, one has $T_k(u_n)$ which converges strongly to $\sigma_k$ in $L^{p_-}(\Omega)$ and so a.e. in $\Omega$. Now, it remains to prove that $\sigma_k = T_k(u)$ a.e. in $\Omega$ where $u_n \to u$ a.e. in $\Omega$.
Let $s > 0$ and define the sets

$$E_n := \{|u_n| > k\}, \; E_m := \{|u_m| > k\} \text{ and } E_{n,m} := \{|T_k(u_n) - T_k(u_m)| > s\},$$

with $k > 0$.
One has $\{|u_n - u_m| > s\} \subset E_n \cup E_m \cup E_{n,m}$ which gives

$$meas(\{|u_n - u_m| > s\}) \leq meas(E_n) + meas(E_m) + meas(E_{n,m}).$$

Let $\varepsilon > 0$. According to (4.6), one can choose $k = k(\varepsilon)$ to get

$$meas(E_n) \leq \frac{\varepsilon}{3} \text{ and } meas(E_m) \leq \frac{\varepsilon}{3}.$$

Since $T_k(u_n)$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$. Hence, there exists $n_0 = n_0(\varepsilon, s) \in \mathbb{N}$ such that for all $n, m \geq n_0$,

$$meas(E_{n,m}) \leq \frac{1}{s^{p_-}} \int_\Omega |T_k(u_n) - T_k(u_m)|^{p_-}\mathrm{d}x \leq \frac{\varepsilon}{3}.$$

So,

$$meas(\{|u_n - u_m| > s\}) \leq \varepsilon, \text{ for all } n, m \geq n_0.$$

Finally, the sequence $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in measure. Hence, by extraction of subsequence, there exists a mesurable function $u$ such that $u_n \to u$ a.e. in $\Omega$. Since $T_k$ is continuous, then $T_k(u_n) \to T_k(u)$ a.e. in $\Omega$ and, by the uniqueness of the limit, one has $\sigma_k = T_k(u)$ a.e. in $\Omega$ because $T_k(u_n) \to \sigma_k$ a.e. in $\Omega$. From the Lemma 2.6, one deduces that, for all $k > 0$, $T_k(u) \in \mathrm{dom}(\beta)$ a.e. in $\Omega$ and as $\mathrm{dom}(\beta)$ is bounded, then $u \in \mathrm{dom}(\beta)$ a.e. in $\Omega$.
Also, the weak convergence of $T_k(u_n)$ to $T_k(u)$ in $W^{1,p_-}(\Omega)$ as $n \to \infty$ leads to the weak convergence of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ in $L^{p_-}(\Omega)$ as $n \to \infty$. Thanks to Theorem 2.7$-(i)$, $\nabla T_k(u_n)$ nonlinear weak-* converges to a Young measure $\left(\nu_x^k\right)_{x\in\Omega}$

and since its weak limit is $\nabla T_k(u)$, then $\nabla T_k(u)$ verifies the equality (4.8) according to (2.11).

$(iv)$ By assumption (4.2), $p_n \to p$ in measure on $\Omega$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $L^{p_-}(\Omega)$, then, according to Theorem 2.7$-(ii)$,$(iii)$, for all $k \in \mathbb{N}$, the sequence $(p_n, \nabla T_k(u_n))_{n\in\mathbb{N}^*}$ converges to the Young measure $\delta_{p(x)} \otimes d\nu_x^k$ on $\mathbb{R}\times\mathbb{R}^N$.
Let us now consider the Carathéodory function

$$F_m : (x, (\lambda_0, \lambda)) \in \Omega \times (\mathbb{R}\times\mathbb{R}^N) \longmapsto |h_m(\lambda)|^{\lambda_0},\ m \in \mathbb{N},$$

where $h_m$ is defined by (2.8). The sequence $(F_m(., (p_n(.), \nabla T_k(u_n))))_{n\in\mathbb{N}^*}$ is equi-integrable in $\Omega$ since it is uniformly bounded in $L^1(\Omega)$ according to (4.9). Then, we apply the nonlinear weak-* convergence property (2.10) to the function $F_m$ to get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega F_m(x,(p_n(x),\nabla T_k(u_n)(x)))\mathrm{d}x &= \int_\Omega\int_{\mathbb{R}\times\mathbb{R}^N} F_m(x,(\lambda_0,\lambda))\mathrm{d}\delta_{p(x)}(\lambda_0)\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x\\ &= \int_\Omega\int_{\mathbb{R}^N} F_m(x,(p(x),\lambda))\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x\\ &= \int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x)}\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x.\end{aligned}$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega F_m(x,(p_n(x),\nabla T_k(u_n)(x)))\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega |h_m(\nabla T_k(u_n))|^{p_n(x)}\mathrm{d}x\\ &\le \lim_{n\to\infty}\int_\Omega |\nabla T_k(u_n)|^{p_n(x)}\mathrm{d}x\\ &\le Ck,\end{aligned}$$

according to (4.9).
So,

$$\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x)}\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x \le Ck.$$

Since the sequence $(|h_m|)_{m\in\mathbb{N}}$ is increasing and $h_m(\lambda) \longrightarrow \lambda$ as $m\to\infty$, then by the monotone convergence theorem, we deduce that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{p(x)}\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x \le Ck.$$

By the formula (4.8) and Jensen inequality, one has

$$\int_\Omega |\nabla T_k(u)|^{p(x)}\mathrm{d}x = \int_\Omega \left|\int_{\mathbb{R}^N} \lambda \mathrm{d}\nu_x^k(\lambda)\right|^{p(x)}\mathrm{d}x \le \int_{\Omega\times\mathbb{R}^N} |\lambda|^{p(x)}\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x < Ck.$$

Hence, $\nabla T_k(u) \in L^{p(.)}(\Omega)$ and so $T_k(u) \in W^{1,p(.)}(\Omega)$. □

**Lemma 4.4.** *$(i)$ For all $k > 0$, the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}^*}$ such that $\mathcal{Y}_n^k(x) := a_n(x, \nabla T_k(u_n(x)))$, is equi-integrable in $\Omega$ and its weak limit $\mathcal{Y}^k \in L^{p'(.)}(\Omega)$*

*is such that*

$$\mathcal{Y}^k(x) := \int_{\mathbb{R}^N} a(x,\lambda)\mathrm{d}\nu_x^k(\lambda),\ a.e.\ x \in \Omega. \tag{4.15}$$

(*ii*) *For all* $\widehat{k} > k > 0$*, one has* $\mathcal{Y}^k = \mathcal{Y}^{\widehat{k}}\chi_{\{|u|<k\}}$.

*Proof.* (*i*) We first show that the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}^*}$, $\mathcal{Y}_n^k := a_n(x, \nabla T_k(u_n))$, is equi-integrable in $\Omega$. The assumption (2.5) applied on $a_n(.,.)$ with exponent $p_n(x)$ implies, for all measurable subset $E \subset \Omega$,

$$\begin{aligned}
\int_E |\mathcal{Y}_n^k|\mathrm{d}x \leq & C\int_E \left(1 + \mathcal{M}_n + |\nabla T_k(u_n)|^{p_n(x)-1}\right)\mathrm{d}x \\
\leq & C\int_E (1+\mathcal{M}_n)\,\mathrm{d}x + 2C\left\||\nabla T_k(u_n)|^{p_n(x)-1}\right\|_{L^{p_n'(\cdot)}(\Omega)}\|\chi_E\|_{L^{p_n(\cdot)}(\Omega)} \\
\leq & C\int_E (1+\mathcal{M}_n)\,\mathrm{d}x + C'\max\left(\left(\rho_{p_n}(\chi_E)\right)^{1/p_+}, \left(\rho_{p_n}(\chi_E)\right)^{1/p_-}\right) \\
\leq & C\int_E (1+\mathcal{M}_n)\,\mathrm{d}x + C'\max\left(meas(E)^{1/p_+}, meas(E)^{1/p_-}\right),
\end{aligned} \tag{4.16}$$

by using Hölder type inequality and Lemma 2.2, where $2C\left\||\nabla T_k(u_n)|^{p_n(x)-1}\right\|_{L^{p_n'(\cdot)}(\Omega)}$ is upper bounded by $C'$ according to (4.9).

The whole right-hand side of (4.16) tends to zero when $meas(E)$ tends to zero because the sequence $(\mathcal{M}_n)_{n\in\mathbb{N}^*}$ is equi-integrable in $\Omega$. So, the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}^*}$ is equi-integrable in $\Omega$. By Theorem 2.7$-(i)$, there exists a weak limit $\mathcal{Y}^k$ of the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}^*}$ in $L^1(\Omega)$.

In the following lines, we prove that the weak limit $\mathcal{Y}^k$ verifies the formula (4.15) and belongs to $L^{p'(\cdot)}(\Omega)$.

We put the set

$$R_n := \{x \in \Omega; |p(x) - p_n(x)| < 1/2\}$$

and we consider auxiliary functions $\tilde{\mathcal{Y}}_n^k := a\big(x, (\nabla T_k(u_n))\chi_{R_n}\big)$.

Let's show that the sequence $\left(\tilde{\mathcal{Y}}_n^k\right)_{n\in\mathbb{N}^*}$ is equi-integrable in $\Omega$. Indeed, we apply (2.5) with the exponent $p(.)$ on $a\,(.,.)$ to get

$$\int_E \tilde{\mathcal{Y}}_n^k\mathrm{d}x \leq C\int_E (1+\mathcal{M})\,\mathrm{d}x + C\int_{E\cap R_n}|\nabla T_k(u_n)|^{p(x)-1}\mathrm{d}x. \tag{4.17}$$

The first term of the right-hand side of (4.17) tends to zero when $meas(E)$ tends to zero. Also, for $x \in R_n$, one has $p(x) \leq p_n(x) + 1/2$ and, by using Hölder type inequality, it follows that

$$\begin{aligned}
\int_{E\cap R_n}|\nabla T_k(u_n)|^{p(x)-1}\mathrm{d}x \leq & \int_E \left(1 + |\nabla T_k(u_n)|^{p_n(x)-1/2}\right)\mathrm{d}x \\
\leq & meas(E) + C\left\||\nabla T_k(u_n)|^{p_n(x)-1/2}\right\|_{L^{(2p_n(\cdot))'}(\Omega)}\|\chi_E\|_{L^{2p_n(\cdot)}(\Omega)}.
\end{aligned} \tag{4.18}$$

But, by (4.9), one has

$$\rho_{(2p_n)'}\left(|\nabla T_k(u_n)|^{p_n(x)-1/2}\right)=\rho_{p_n}\left(\nabla T_k(u_n)\right)\le Ck. \tag{4.19}$$

Also, by Proposition 2.3, one has

$$\begin{aligned}\|\chi_E\|_{L^{2p_n(.)}} &\le \max\left(\left(\rho_{2p_n}\left(\chi_E\right)\right)^{1/(2p)_+},\left(\rho_{2p_n}\left(\chi_E\right)\right)^{1/(2p)_-}\right)\\ &\le \max\left((meas(E))^{1/(2p)_+},(meas(E))^{1/(2p)_-}\right).\end{aligned} \tag{4.20}$$

From (4.18) – (4.20), the second term of the right-hand side of (4.17) is uniformly small for $meas(E)$ small, and the equi-integrability of $\left(\tilde{\mathcal{Y}}_n^k\right)_{n\in\mathbb{N}}$ follows.

Now, we assert that, by extraction of a subsequence, the sequence $\tilde{\mathcal{Y}}_n^k$ converges weakly to some function $\tilde{\mathcal{Y}}^k$ in $L^1(\Omega)$ as $n\to\infty$ thanks to Theorem 2.7−$(i)$.

It remains to prove that $\mathcal{Y}^k=\tilde{\mathcal{Y}}^k$. For that, it is sufficient to prove that $\mathcal{Y}_n^k-\tilde{\mathcal{Y}}_n^k$ converges strongly to zero in $L^1(\Omega)$.

Indeed, let $\varepsilon>0$. By the Chebyshev inequality, one has

$$\begin{aligned}meas(\{|\nabla T_k(u_n)|>L\}) &\le \left(\int_\Omega |\nabla T_k(u_n)|\,dx\right)\Big/L\\ &\le \int_\Omega\left(1+|\nabla T_k(u_n)|^{p_n(x)}\right)\mathrm{d}x\Big/L\\ &\le \left(meas(\Omega)+Ck\right)/L,\end{aligned}$$

by inequality (4.9).

It follows that $\sup\limits_{n\in\mathbb{N}^*}(meas(\{|\nabla T_k(u_n)|>L\}))\to 0$ as $L\to\infty$. The sequence $\mathcal{Y}_n^k-\tilde{\mathcal{Y}}_n^k$ is equi-integrable in $\Omega$, so there exists $L_0=L_0(\varepsilon)$ such that for $L>L_0$, one has

$$\int_{\{|\nabla T_k(u_n)|>L\}}|\mathcal{Y}_n^k-\tilde{\mathcal{Y}}_n^k|\mathrm{d}x\le\varepsilon/4,\quad\text{for all } n\in\mathbb{N}. \tag{4.21}$$

By the assumption (4.1), one has for all $\sigma>0$,

$$\lim_{n\to\infty} meas\left(\left\{x\in\Omega;\sup_{|\lambda|\le L}|a_n(x,\lambda)-a(x,\lambda)|\ge\sigma\right\}\right)=0.$$

Hence, by the equi-integrability of $\mathcal{Y}_n^k-\tilde{\mathcal{Y}}_n^k$ on $\Omega$, there exists $n_0=n_0(\sigma,L_0)\in\mathbb{N}$ such that for all $n>n_0$,

$$\int_{\left\{x\in\Omega;\ \sup\limits_{|\lambda|\le L}|a_n(x,\lambda)-a(x,\lambda)|\ge\sigma\right\}}|\mathcal{Y}_n^k-\tilde{\mathcal{Y}}_n^k|\mathrm{d}x\le\varepsilon/4. \tag{4.22}$$

By definition, one has $\tilde{\mathcal{Y}}_n^k=a(x,\nabla T_k(u_n))$ on the set $R_n$ and we consider the following set

$$R_n^{L,\sigma}:=\left\{x\in R_n;\sup_{|\lambda|\le L}|a_n(x,\lambda)-a(x,\lambda)|<\sigma,|\nabla T_k(u_n)|\le L\right\}.$$

Since $|\nabla T_k(u_n)| \leq L$ on $R_n^{L,\sigma}$, then one has

$$|a_n(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u_n))| < \sigma \text{ on } R_n^{L,\sigma},$$

and so, for all $n$,

$$\int_{R_n^{L,\sigma}} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \mathrm{d}x \leq \sigma meas(\Omega) \leq \varepsilon/4, \tag{4.23}$$

by taking $\sigma = \sigma(\varepsilon) < \varepsilon/(4meas(\Omega))$.
Also, by (4.21) and (4.22), one has

$$\int_{R_n \setminus R_n^{L,\sigma}} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \mathrm{d}x \leq \varepsilon/2, \text{ for all } n > n_0(\sigma(\varepsilon), L(\varepsilon)). \tag{4.24}$$

Since $p_n(.)$ converges to $p(.)$ in measure on $\Omega$, one has

$$meas(\Omega \setminus R_n) = meas\big(\{|p - p_n| \geq 1/2\}\big)$$

which converges to zero as $n \to \infty$ and the equi-integrability of $\mathcal{Y}_n^k$ gives, for sufficiently large $n$,

$$\int_{\Omega \setminus R_n} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \mathrm{d}x = \int_{\Omega \setminus R_n} |\mathcal{Y}_n^k| \mathrm{d}x \leq \varepsilon/4. \tag{4.25}$$

Now, by using (4.23), (4.24) and (4.25), one gets, for $n > n_0(\sigma(\varepsilon), L(\varepsilon))$,

$$\int_{\Omega} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \mathrm{d}x \leq \varepsilon.$$

Hence, the sequence $\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k$ converges strongly to zero in $L^1(\Omega)$, as $n$ goes to infinity, and so, $\mathcal{Y}^k = \tilde{\mathcal{Y}}^k$.
Let us show the representation formula (4.15) for $\mathcal{Y}^k$. Since $meas(\Omega \setminus R_n) \to 0$ as $n \to \infty$, so, by the equi-integrability of $\nabla T_k(u_n)$ in $\Omega$, $\nabla T_k(u_n)(1 - \chi_{R_n})$ converges to zero as $n \to \infty$. Therefore, the sequence $\nabla T_k(u_n)\chi_{R_n}$ converges to the same Young measure $\nu_x^k$ as the sequence $\nabla T_k(u_n)$. Now, fix $\psi \in \mathcal{D}(\Omega)$ and let's consider the Carathéodory function $a(.,.).\psi$. Since the sequence $\tilde{\mathcal{Y}}_n^k = a(x, \nabla T_k(u_n)\chi_{R_n})$ is equi-integrable in $\Omega$, then we can use the nonlinear weak-* convergence property (2.10) to get

$$\lim_{n\to\infty} \int_{\Omega} a(x, \nabla T_k(u_n)\chi_{R_n}).\psi \mathrm{d}x = \int_{\Omega \times \mathbb{R}^N} a(x, \lambda).\psi \mathrm{d}\nu_x^k(\lambda) \mathrm{d}x. \tag{4.26}$$

Since $a(x, \nabla T_k(u_n)\chi_{R_n})$ converges weakly to $\tilde{\mathcal{Y}}^k$, (4.26) becomes

$$\int_{\Omega} \tilde{\mathcal{Y}}^k.\psi \mathrm{d}x = \int_{\Omega \times \mathbb{R}^N} a(x, \lambda).\psi \mathrm{d}\nu_x^k(\lambda) \mathrm{d}x = \int_{\Omega} \left( \int_{\mathbb{R}^N} a(x, \lambda) \mathrm{d}\nu_x^k(\lambda) \right).\psi \mathrm{d}x$$

which means that

$$\mathcal{Y}^k = \tilde{\mathcal{Y}}^k = \int_{\mathbb{R}^N} a(x, \lambda) \mathrm{d}\nu_x^k(\lambda) \text{ in } \mathcal{D}'(\Omega) \text{ and so, a.e. in } \Omega.$$

Now, we end the proof with $\mathcal{Y}^k \in L^{p'(.)}(\Omega)$. One uses Jensen inequality, the assumption (2.5) and Lemma 4.3$-(v)$ to obtain

$$\begin{aligned}\int_\Omega |\mathcal{Y}^k(x)|^{p'(x)}\mathrm{d}x &= \int_\Omega \left|\int_{\mathbb{R}^N} a(x,\lambda)\mathrm{d}\nu_x^k(\lambda)\right|^{p'(x)}\mathrm{d}x\\ &\leq \int_{\Omega\times\mathbb{R}^N} |a(x,\lambda)|^{p'(x)}\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x\\ &\leq \int_{\Omega\times\mathbb{R}^N} C(\mathcal{M}(x)+|\lambda|^{p(x)})\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x < \infty.\end{aligned}$$

$(ii)$ Let $\widehat{k} > k > 0$ and let's put $g_n^k := a_n(x, \nabla T_{\widehat{k}}(u_n))\chi_{\{|u|<k\}}$. According to $(i)$, $\left(g_n^k\right)_{n\in\mathbb{N}^*}$ converges weakly to $\mathcal{Y}^{\widehat{k}}\chi_{\{|u|<k\}}$ in $L^1(\Omega)$. If we prove that the sequence $\left(g_n^k\right)_{n\in\mathbb{N}^*}$ converges weakly to $\mathcal{Y}^k$ in $L^1(\Omega)$, we can conclude that $\mathcal{Y}^k = \mathcal{Y}^{\widehat{k}}\chi_{\{|u|<k\}}$, by the uniqueness of the limit. Let's put

$$h_n^k := a_n(x, \nabla T_{\widehat{k}}(u_n))\chi_{\{|u_n|<k\}}.$$

Since $\widehat{k} > k$, one has

$$T_k(u_n) \equiv T_k(T_{\widehat{k}}(u_n)),$$

and so

$$\nabla T_k(u_n) = \nabla T_{\widehat{k}}(u_n)\chi_{\left\{|T_{\widehat{k}}(u_n)|<k\right\}} = \nabla T_{\widehat{k}}(u_n)\chi_{\{|u_n|<k\}}.$$

Moreover, from assumption (2.7), one has $a_n(x,0)=0$ a.e. $x\in\Omega$. Hence,

$$a_n(x, \nabla T_{\widehat{k}}(u_n))\chi_{\{|u_n|<k\}} \equiv a_n(x, \nabla T_k(u_n))$$

and the sequence $\left(h_n^k\right)_{n\in\mathbb{N}^*}$ converges weakly to $\mathcal{Y}^k$ in $L^1(\Omega)$, according to $(i)$. Consider the sequence $\left(d_n^k\right)_{n\in\mathbb{N}^*}$ such that

$$d_n^k := g_n^k - h_n^k = a_n(x, \nabla T_{\widehat{k}}(u_n))\left(\chi_{\{|u|<k\}} - \chi_{\{|u_n|<k\}}\right).$$

The function $\chi_{(-k,k)}(.)$ is continuous on the image of $\Omega$ by $u(.)$ for a.e. $k > 0$. Indeed, one has $meas\left(\{|u|=k\}\right) = 0$ for a.e. $k>0$. Therefore, since $u_n$ converges to $u$ a.e. in $\Omega$, then

$$\chi_{\{|u_n|<k\}} = \chi_{(-k,k)}(u_n) \to \chi_{(-k,k)}(u) = \chi_{\{|u|<k\}} \text{ a.e. in } \Omega \text{ as } n\to\infty.$$

So,

$$d_n^k \to 0 \text{ a.e. in } \Omega.$$

Moreover, by $(i)$, the sequence $\left(d_n^k\right)_{n\in\mathbb{N}^*}$ is equi-integrable in $\Omega$. Hence, by Vitali's theorem, the sequence $\left(d_n^k\right)_{n\in\mathbb{N}^*}$ converges strongly to zero in $L^1(\Omega)$. Therefore, $g_n^k = h_n^k + d_n^k$ tends to $\mathcal{Y}^k$ weakly in $L^1(\Omega)$. So, this ends the proof of $(ii)$. □

**Lemma 4.5.** $(i)$ *For all $k>0$,*

$$\int_\Omega \mathcal{Y}^k.\nabla T_k(u)\mathrm{d}x \geq \liminf_{n\to\infty}\int_\Omega \mathcal{Y}_n^k.\nabla T_k(u_n)\mathrm{d}x \tag{4.27}$$

*and the "div-curl" inequality*

$$\int_{\Omega\times\mathbb{R}^N} \big(a(x,\lambda)-a(x,\nabla T_k(u))\big).\big(\lambda-\nabla T_k(u)\big)\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x \le 0 \tag{4.28}$$

*holds true.*

(*ii*) *For all* $k>0$,

$$\mathcal{Y}^k(x) = a(x,\nabla T_k(u(x))) \text{ for a.e. } x\in\Omega. \tag{4.29}$$

*Proof.* (*i*) Let $\psi\in C^\infty(\overline{\Omega})$. Since $p_n(.)$ is log-Hölder continuous, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p_n(.)}(\Omega)$. So, we can take $\psi$ as test function in the renormalized formulation (3.3) for $u_n$, to get

$$\begin{aligned}&\left|\int_\Omega \Big(\beta_n(u_n)S(u_n)\psi + S(u_n)\mathcal{Y}_n^M.\nabla\psi - f_nS(u_n)\psi\Big)\mathrm{d}x\right|\\ &\qquad\le \|\psi\|_{L^\infty}\int_\Omega |S'(u_n)|\mathcal{Y}_n^M.\nabla T_M(u_n)\mathrm{d}x,\end{aligned} \tag{4.30}$$

where $S\in\mathbb{S}$ with $suppS\subset[-M,M]$, $M>0$.

We are going to pass to the limit in (4.30), as $n$ tends to infinity. By Lemma 4.3$-$(*iii*), $u_n$ converges to $u$ a.e. in $\Omega$. Let's prove now that

$$\int_\Omega f_nS(u_n)\psi\mathrm{d}x \longrightarrow \int_\Omega fS(u)\psi\mathrm{d}x, \text{ as } n\to\infty. \tag{4.31}$$

One has

$$\int_\Omega f_nS(u_n)\psi\mathrm{d}x = \int_\Omega f_nS(u)\psi\mathrm{d}x + \int_\Omega f_n(S(u_n)-S(u))\psi\mathrm{d}x. \tag{4.32}$$

On one hand, one has $\int_\Omega f_nS(u)\psi\mathrm{d}x \longrightarrow \int_\Omega fS(u)\psi\mathrm{d}x$ since $f_n\rightharpoonup f$ in $L^1(\Omega)$. On the other hand, one has, for $R>0$,

$$\begin{aligned}\int_\Omega |f_n(S(u_n)-S(u))\psi|\mathrm{d}x =& \int_{\{|f_n|>R\}} |f_n(S(u_n)-S(u))\psi|\mathrm{d}x\\ &+\int_{\{|f_n|\le R\}} |f_n(S(u_n)-S(u))\psi|\mathrm{d}x\\ \le& 2\|\psi\|_{L^\infty}\|S\|_{L^\infty}\int_{\{|f_n|>R\}}|f_n|\mathrm{d}x\\ &+R\|\psi\|_{L^\infty}\int_\Omega |S(u_n)-S(u)|\mathrm{d}x.\end{aligned} \tag{4.33}$$

For $R>0$ fixed, the second term of the right-hand side of the inequality (4.33) tends to zero as $n\to\infty$. Indeed, because of the continuity of $S$ and the compactness of $suppS$, $S(u_n)$ converges strongly to $S(u)$ in $L^1(\Omega)$ by the Lebesgue dominated convergence theorem. By the Chebyshev inequality and since $f_n$ is bounded in $L^1(\Omega)$, one has

$$\sup_{n\in\mathbb{N}^*} meas(\{|f_n|>R\}) \le \frac{\sup_{n\in\mathbb{N}^*}\|f_n\|_1}{R} \le \frac{C}{R} \longrightarrow 0 \text{ as } R\longrightarrow\infty.$$

Since the sequence $f_n$ is equi-integrable in $\Omega$, then the first term in the right-hand side of (4.33) can be made as small as desired by the choice of $R$. Hence, the second term of the right-hand side of (4.32) tends to zero. And so, one deduces the convergence result (4.31).
Next, we prove that

$$\int_\Omega S(u_n)\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x \to \int_\Omega S(u)\mathcal{Y}^M.\nabla\psi \mathrm{d}x, \text{ as } n\to\infty. \tag{4.34}$$

Indeed, for $R>0$,

$$\int_\Omega S(u_n)\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x = \int_{\{|\nabla\psi|<R\}} S(u_n)\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x + \int_{\{|\nabla\psi|\geq R\}} S(u_n)\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x \tag{4.35}$$

For the first term of the right-hand side of (4.35), one has

$$\begin{aligned}\int_{\{|\nabla\psi|<R\}} S(u_n)\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x &= \int_{\{|\nabla\psi|<R\}} S(u)\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x\\ &\quad + \int_{\{|\nabla\psi|<R\}} (S(u_n)-S(u))\mathcal{Y}_n^M.\nabla\psi \mathrm{d}x. \end{aligned}\tag{4.36}$$

Since $\mathcal{Y}_n^M \rightharpoonup \mathcal{Y}^M$ in $L^{p'(\cdot)}(\Omega)$ by Lemma 4.4$-(i)$, then the first term of the right-hand side of (4.36) tends to $\int_{\{|\nabla\psi|<R\}} S(u)\mathcal{Y}^M.\nabla\psi \mathrm{d}x$ as $n\to\infty$.
For $\alpha>0$ fixed, we can rewrite the second term of the right-hand side of (4.36) as follows.

$$\begin{aligned}&\int_{\{|\nabla\psi|<R\}} |(S(u_n)-S(u))\mathcal{Y}_n^M.\nabla\psi|\mathrm{d}x\\ &= \int_{\{|\nabla\psi|<R\}\cap\{|\mathcal{Y}_n^M|\leq\alpha\}} |(S(u_n)-S(u))\mathcal{Y}_n^M.\nabla\psi|\mathrm{d}x\\ &\quad + \int_{\{|\nabla\psi|<R\}\cap\{|\mathcal{Y}_n^M|>\alpha\}} |(S(u_n)-S(u))\mathcal{Y}_n^M.\nabla\psi|\mathrm{d}x\\ &\leq \alpha R\int_\Omega |S(u_n)-S(u)|\mathrm{d}x + 2R\|S\|_{L^\infty}\int_{\{|\mathcal{Y}_n^M|>\alpha\}} |\mathcal{Y}_n^M|\mathrm{d}x. \end{aligned}\tag{4.37}$$

The sequence $\mathcal{Y}_n^M$ is equi-integrable in $\Omega$ and is bounded in $L^1(\Omega)$ as it converges weakly in $L^1(\Omega)$, so using the same argument which leads to assert that the right-hand side of (4.33) tends to zero, as $n\to\infty$, in the inequality (4.37), then the second term of the right-hand side of (4.36) tends to zero as $n\to\infty$. Thus, the first term of the right-hand side of (4.35) converges to $\int_{\{|\nabla\psi|<R\}} S(u)\mathcal{Y}^M.\nabla\psi \mathrm{d}x$ as $n\to\infty$.
For the second term of the right-hand side of (4.35), we note that, by Hölder type inequality,

$$\left|\int_{\{|\nabla\psi|\geq R\}} \mathcal{Y}_n^M.(\nabla\psi S(u_n))\mathrm{d}x\right| \leq C\|S\|_{L^\infty}\left\|\mathcal{Y}_n^M\right\|_{L^{p_n'(\cdot)}(\Omega)}\left\|\chi_{\{|\nabla\psi|\geq R\}}\nabla\psi\right\|_{L^{p_n(\cdot)}(\Omega)} \tag{4.38}$$

One has $\left\|\mathcal{Y}_n^M\right\|_{L^{p_n'(\cdot)}(\Omega)} \leq C$ by Lemma 4.3 and the growth assumption (2.5).
Since $\psi \in C^\infty(\overline{\Omega})$, one clearly has $meas(\{|\nabla\psi| \geq R\}) \to 0$ as $R \to 0$ since $|\nabla\psi|$ is bounded. Therefore,

$$\int_{\{|\nabla\psi|\geq R\}} |\nabla\psi|^{p_n(\cdot)}\mathrm{d}x \leq C meas(\{|\nabla\psi| \geq R\})$$

where $C$ is independent of $R$.
So, by Lemma 2.2$-(iii),(iv)$, $\sup\limits_n \left\|\chi_{\{|\nabla\psi|\geq R\}}\nabla\psi\right\|_{L^{p_n(\cdot)}(\Omega)}$ tends to zero as $R \to \infty$. Therefore, the second term of the right-hand side of (4.35) tends to zero as $R \to \infty$. Hence, as $n \to \infty$ and $R \to \infty$ in the equality (4.35), one deduces (4.34).
Moreover, by (4.11), $(\beta_n(u_n))_{n\in\mathbb{N}^*}$ is bounded in $L^1(\Omega)$, so

$$\left[\begin{array}{l} \text{there exists } z \in \mathcal{M}_b(\Omega) \text{ such that} \\ \beta_n(u_n) \longrightarrow z \text{ in } \mathcal{M}_b(\Omega), \text{ as } n \to \infty. \end{array}\right. \tag{4.39}$$

Thanks to convergences (4.31), (4.34) and (4.39), one obtains for $n$ large enough and for $\psi \in C^\infty(\overline{\Omega})$,

$$\begin{aligned} &\left|\int_\Omega S(u)\psi\mathrm{d}z + \int_\Omega \big(S(u)\mathcal{Y}^M.\nabla\psi - fS(u)\psi\big)\mathrm{d}x\right| \\ &\qquad \leq \|\psi\|_{L^\infty} \sup_{n\in\mathbb{N}^*}\int_\Omega |S'(u_n)|a_n(x,\nabla T_M(u_n)).\nabla T_M(u_n)\mathrm{d}x. \end{aligned} \tag{4.40}$$

It follows from (4.40) that $z \in \mathcal{M}_b^{p(\cdot)}(\Omega)$ ($C^\infty(\overline{\Omega})$ being dense in $W^{1,p(\cdot)}(\Omega)$).
Now, fix $k > 0$. By Lemma 4.3$-(iv)$, one has $T_k(u) \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. So, by the density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$, we can replace $\psi$ by $T_k(u)$ in (4.40).
Let's take $S = S_M$ defined in (3.1). According to Lemma 4.4$-(ii)$, for $M > k$, one has $\mathcal{Y}^k = \mathcal{Y}^M\chi_{\{|u|<k\}}$. Since $\nabla T_k(u) = 0$ outside $\{|u| < k\}$, then we can replace $\mathcal{Y}^M.\nabla T_k(u)$ by $\mathcal{Y}^k.\nabla T_k(u)$. Also, one has $suppS_M' \subset [-M,-M+1]\cup[M-1,M]$ and the sequence $S_M'$ is uniformly bounded i.e. $\|S_M'\|_{L^\infty(\Omega)} \leq C$, where $C$ is a positive constant independent of $M$. So, the term of the right-hand side of (4.40) is bounded by

$$\begin{aligned} &C\sup_{n\in\mathbb{N}^*}\int_{\{M-1\leq|u_n|\leq M\}} a_n(x,\nabla T_M(u_n)).\nabla T_M(u_n)\mathrm{d}x \\ &\quad\leq C\sup_{n\in\mathbb{N}^*}\int_{\{M-1\leq|u_n|\leq M\}}\left(\mathcal{M}_n|\nabla T_M(u_n)| + |\nabla T_M(u_n)|^{p_n(x)}\right)\mathrm{d}x \\ &\quad\leq C\sup_{n\in\mathbb{N}^*}\left\|\mathcal{M}_n\chi_{\{|u_n|\geq M-1\}}\right\|_{L^{p_n'(\cdot)}(\Omega)}\left\|\nabla T_M(u_n)\chi_{\{M-1\leq|u_n|\leq M\}}\right\|_{L^{p_n(\cdot)}(\Omega)} \\ &\qquad + C\sup_{n\in\mathbb{N}^*}\int_{\{M-1\leq|u_n|\leq M\}}|\nabla T_M(u_n)|^{p_n(x)}\mathrm{d}x, \end{aligned} \tag{4.41}$$

by using the growth condition on $a_n(.,.)$ and the Hölder type inequality.
Thanks to Lemma 2.2, the estimates (4.6) and (4.7), and the fact that $\mathcal{M}_n$ is equi-integrable, one has that the term of the right-hand side of (4.40) tends to zero when $M \to \infty$ in (4.41).

As $M \to \infty$ in (4.40), thanks to Lebesgue dominated convergence theorem, one obtains

$$\int_\Omega T_k(u)\mathrm{d}z + \int_\Omega \big(\mathcal{Y}^k.\nabla T_k(u) - fT_k(u)\big)\mathrm{d}x = 0. \tag{4.42}$$

Now, we consider the renormalized formulation (3.3) for $u_n$ where one takes $T_k(u_n)$ as test function and $S \in \mathbb{S}$ with $S = S_h$, to get

$$\begin{aligned}\int_\Omega \Big(S_h(u_n)\mathcal{Y}_n^k.\nabla T_k(u_n) + S_h'(u_n)a_n(x,\nabla u_n).(\nabla u_n)T_k(u_n) + \beta_n(u_n)S_h(u_n)T_k(u_n)\Big)\mathrm{d}x \\ = \int_\Omega f_nS_h(u_n)T_k(u_n)\mathrm{d}x. \end{aligned}\tag{4.43}$$

Now, we are going to pass to the limit in (4.43), as $h \to \infty$.
One uses the property (3.2) to pass to the limit, as $h \to \infty$, in the term containing the factor $S_h'(u_n)$ and, since $S_h$ is monotone in $h$, we use monotone convergence theorem to pass to the limit in the terms containing the factor $S_h(u_n)$, to get

$$\int_\Omega \Big(\mathcal{Y}_n^k.\nabla T_k(u_n) + \beta_n(u_n)T_k(u_n)\Big)\mathrm{d}x = \int_\Omega f_nT_k(u_n)\mathrm{d}x, \tag{4.44}$$

as $h \to \infty$ in (4.43).
Since $u_n$ converges to $u$ a.e. in $\Omega$, and also as $f_n \rightharpoonup f$ in $L^1(\Omega)$ and as $\|T_k\| < \infty$, arguing as in (4.32) and (4.33), it follows that

$$\int_\Omega f_nT_k(u_n)\mathrm{d}x = \int_\Omega f_nT_k(u)\mathrm{d}x + \int_\Omega f_n\left(T_k(u_n) - T_k(u)\right)\mathrm{d}x \to \int_\Omega fT_k(u)\mathrm{d}x,$$

as $n \to \infty$. In the sequel, since $\beta_n(u_n)T_k(u_n) \geq 0$, by Fatou's lemma, one deduces that

$$\int_\Omega T_k(u)\mathrm{d}z - \int_\Omega fT_k(u)\mathrm{d}x \leq \liminf_{n\to\infty}\left(\int_\Omega \left(\beta_n(u_n)T_k(u_n) - f_nT_k(u_n)\right)\mathrm{d}x\right).$$

So, from the inequality above and by using (4.44) and (4.42), one gets (4.27).
Now, let's go to the proof of the "div-curl" inequality (4.28). Thanks to Lemma 2.1, the sequence

$$\Big(a_n(x,h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))\Big)_{m>0}$$

is upper bounded by $\mathcal{Y}_n^k.\nabla T_k(u_n)$ since it converges while growing to $\mathcal{Y}_n^k.\nabla T_k(u_n)$, as $m \to \infty$. So, one has, by (4.27),

$$\int_\Omega \mathcal{Y}^k.\nabla T_k(u)\mathrm{d}x \geq \liminf_{n\to\infty}\int_\Omega a_n(x,h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))\mathrm{d}x, \text{ for all } m > 0.$$

Since $\int_\Omega \lambda \mathrm{d}\nu_x^k(\lambda)$ and $\int_\Omega a(x,\lambda)\mathrm{d}\nu_x^k(\lambda)$ are, respectively, the weak limits of $\nabla T_k(u_n)$ and $a_n(x,\nabla T_k(u_n))$, then using the nonlinear weak-* convergence property (2.10), it follows that

$$\lim_{n\to\infty}\int_\Omega a_n(x,h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))\mathrm{d}x = \int_{\Omega\times\mathbb{R}^N} a(x,h_m(\lambda)).h_m(\lambda)\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x.$$

Therefore,

$$\int_\Omega \mathcal{Y}^k.\nabla T_k(u)\mathrm{d}x \geq \int_{\Omega\times\mathbb{R}^N} a(x,h_m(\lambda)).h_m(\lambda)\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x.$$

Now, according to Lemma 2.1, we can apply the monotone convergence theorem on the sequence $(a(x,h_m(\lambda)).h_m(\lambda))_{m>0}$ to deduce that, as $m\to\infty$,

$$\int_\Omega \mathcal{Y}^k.\nabla T_k(u)\mathrm{d}x \geq \int_{\Omega\times\mathbb{R}^N} a(x,\lambda).\lambda\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x. \tag{4.45}$$

Now using the representation formulas (4.8) et (4.15), and the fact that $\nu_x^k(\lambda)$ is a probability measure on $\mathbb{R}^N$ for a.e. $x\in\Omega$, we find

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N} \big(a(x,\lambda)-a(x,\nabla T_k(u))\big).\big(\lambda-\nabla T_k(u)\big)\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x\\
&\quad= \int_{\Omega\times\mathbb{R}^N} a(x,\lambda).\lambda\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x - \int_\Omega\left(\int_{\mathbb{R}^N} a(x,\lambda)\mathrm{d}\nu_x^k(\lambda)\right)\nabla T_k(u)\mathrm{d}x\\
&\qquad -\int_\Omega a(x,\nabla T_k(u))\left(\int_{\mathbb{R}^N}\lambda\mathrm{d}\nu_x^k(\lambda)\right)\mathrm{d}x\\
&\qquad +\int_\Omega (a(x,\nabla T_k(u)).\nabla T_k(u))\left(\int_{\mathbb{R}^N}\mathrm{d}\nu_x^k(\lambda)\right)\mathrm{d}x\\
&\quad= \int_{\Omega\times\mathbb{R}^N} a(x,\lambda).\lambda\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x - \int_\Omega\left(\int_{\mathbb{R}^N} a(x,\lambda)\mathrm{d}\nu_x^k(\lambda)\right)\left(\int_{\mathbb{R}^N}\lambda\mathrm{d}\nu_x^k(\lambda)\right)\mathrm{d}x\\
&\quad= \int_{\Omega\times\mathbb{R}^N} a(x,\lambda).\lambda\mathrm{d}\nu_x^k(\lambda)\mathrm{d}x - \int_\Omega \mathcal{Y}^k.\nabla T_k(u)\mathrm{d}x.
\end{aligned} \tag{4.46}$$

From (4.46) and (4.45), we deduce (4.28).

$(ii)$ One proves that $\mathcal{Y}^k = a(x,\nabla T_k(u))$ a.e. in $\Omega$.
Thanks to the "div-curl" inequality (4.28) and the strict monotonicity assumption (2.4) on $a(x,.)$, one has

$$\Big(a(x,\lambda)-a(x,\nabla T_k(u))\Big).\Big(\lambda-\nabla T_k(u)\Big)\mathrm{d}\nu_x^k(\lambda) = 0 \text{ for a.e. } x\in\Omega,$$

and, subsequently for a.e. $x\in\Omega$, $\lambda=\nabla T_k(u)$ wrt the measure $\nu_x^k$ on $\mathbb{R}^N$. Since, by the representation formula (4.8), $\nabla T_k(u) = \int_\Omega \lambda\mathrm{d}\nu_x^k(\lambda)$, then the measure $\nu_x^k$ reduces to the Dirac measure $\delta_{\nabla T_k(u)}$. Now, from the representation formula (4.15) we can deduce (4.29). Indeed, one has

$$\mathcal{Y}^k(x) = \int_{\mathbb{R}^N} a(x,\lambda)\mathrm{d}\nu_x^k(\lambda) = \int_{\mathbb{R}^N} a(x,\lambda)\mathrm{d}\delta_{\nabla T_k(u(x))}(\lambda) = a(x,\nabla T_k(u(x))).$$

Moreover, the sequence $\nabla T_k(u_n)$ generates the Young measure $\nu_x^k = \delta_{\nabla T_k(u)}$ a.e. in $\Omega$. So, from Theorem 2.7$-(ii)$, $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in measure on $\Omega$ as $n\to\infty$. $\square$

**Lemma 4.6.** *For a.e.* $k>0$, $a_n(x,\nabla T_k(u_n)).\nabla T_k(u_n) \to a(x,\nabla T_k(u)).\nabla T_k(u)$ *strongly in* $L^1(\Omega)$.

*Proof.* By Lemma 4.5 $-(ii)$ and the convergence (4.1), up to a subsequence, one has $a_n(x,\nabla T_k(u_n)).\nabla T_k(u_n)$ converges to $a(x,\nabla T_k(u)).\nabla T_k(u)$ a.e. in $\Omega$. Since $a_n(x,\nabla T_k(u_n)).\nabla T_k(u_n)\geq 0$, by Fatou's lemma, one has

$$\int_\Omega a(x,\nabla T_k(u)).\nabla T_k(u)\mathrm{d}x \leq \liminf_{n\to\infty}\int_\Omega a_n(x,\nabla T_k(u_n)).\nabla T_k(u_n)\mathrm{d}x$$

and so, by (4.27), we have

$$\liminf_{n\to\infty}\int_\Omega a_n(x,\nabla T_k(u_n)).\nabla T_k(u_n)\mathrm{d}x = \int_\Omega a(x,\nabla T_k(u)).\nabla T_k(u)\mathrm{d}x.$$

Thus, by the Scheffé's theorem (see [20]), one deduces that $a_n(x,\nabla T_k(u_n)).\nabla T_k(u_n)$ converges to $a(x,\nabla T_k(u)).\nabla T_k(u)$ strongly in $L^1(\Omega)$, up to subsequence. □

**Lemma 4.7.** *The Radon-Nikodym decomposition with respect to Lebesgue measure $\mathcal{L}^N$ of the measure $z$,*

$$z = b\mathcal{L}^N+\mu,\ \text{with } \mu\perp\mathcal{L}^N,$$

*satisfies the following properties.*

$$\left[\begin{array}{l} b\in\beta(u)\ \mathcal{L}^N\text{-a.e. in } \Omega,\ b\in L^1(\Omega),\ \mu\in\mathcal{M}_b^{p(.)}(\Omega),\\ \mu^+ \text{ is concentrated on } [u=M],\\ \mu^- \text{ is concentrated on } [u=m]. \end{array}\right. \tag{4.47}$$

*Proof.* For every $\varepsilon>0$, we consider the Yosida regularisation $\beta_\varepsilon$ given by

$$\beta_\varepsilon = \frac{1}{\varepsilon}\big(I-(I+\varepsilon\beta)^{-1}\big).$$

We put

$$j_\varepsilon(s) = \min_{r\in\mathbb{R}}\left\{\frac{1}{\varepsilon}|s-r|^2+j(r)\right\}\ \forall s\in\mathbb{R},\ \forall\varepsilon>0.$$

According to Proposition 2.11 in [4], one has

$$\left[\begin{array}{l} dom(\beta)\subset dom(j)\subset\overline{dom(j)}\subset\overline{dom(\beta)}.\\ j_\varepsilon(s)=\frac{\varepsilon}{2}|\beta_\varepsilon|^2+j(J_\varepsilon) \text{ where } J_\varepsilon=(I+\varepsilon\beta)^{-1},\\ j_\varepsilon \text{ is convex, Frechet-differentiable and } \beta_\varepsilon=\partial j_\varepsilon,\\ j_\varepsilon\uparrow j \text{ as } \varepsilon\downarrow 0. \end{array}\right.$$

For all $\varepsilon>0$ and for all $s\in\mathbb{R}$, one has

$$j(s)\geq j_\varepsilon(s)\geq j_\varepsilon(u_\varepsilon)+(s-u_\varepsilon)\beta_\varepsilon(u_\varepsilon)\ \mathcal{L}^N\text{-a.e. in } \Omega,\ \forall s\in\mathbb{R}.$$

For all $\phi\in C_+^\infty(\overline{\Omega})$ and for all $S_k$ defined in (3.1), one has

$$\phi S_k(u_\varepsilon)j(s)\geq\phi S_k(u_\varepsilon)j_\varepsilon(s)\geq\phi S_k(u_\varepsilon)j_\varepsilon(u_\varepsilon)+(s-u_\varepsilon)\phi S_k(u_\varepsilon)\beta_\varepsilon(u_\varepsilon).$$

Let's fix $\varepsilon_0>0$ and take $0<\varepsilon<\varepsilon_0$. One has, for all $s\in\mathbb{R}$,

$$\int_\Omega\phi S_k(u_\varepsilon)j(s)\mathrm{d}x\geq\int_\Omega\phi S_k(u_\varepsilon)j_{\varepsilon_0}(u_\varepsilon)\mathrm{d}x+\int_\Omega(s-u_\varepsilon)\phi S_k(u_\varepsilon)\beta_\varepsilon(u_\varepsilon)\mathrm{d}x.$$

For $\varepsilon\to 0$, one has

$$\int_\Omega\phi S_k(u)j(s)\mathrm{d}x\geq\int_\Omega\phi S_k(u)j_{\varepsilon_0}(u)\mathrm{d}x+\int_\Omega(s-u)\phi S_k(u)\mathrm{d}z.$$

Now, as $\varepsilon_0 \to 0$, one gets

$$\int_\Omega \phi S_k(u)j(s)\mathrm{d}x \geq \int_\Omega \phi S_k(u)j(u)\mathrm{d}x + \int_\Omega (s-u)\phi S_k(u)\mathrm{d}z,$$

which implies

$$S_k(u)j(s) \geq S_k(u)j(u) + (s-u)S_k(u)z \text{ in } \mathcal{M}_b(\Omega),\ \forall s \in \mathbb{R}.$$

Since $z = b\mathcal{L}^N + \mu$, with $\mu \perp \mathcal{L}^N$, $b \in L^1(\Omega)$, one obtains

$$S_k(u)j(s) \geq S_k(u)j(u) + (s-u)S_k(u)b\ \mathcal{L}^N - \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R} \tag{4.48}$$

and

$$(s-u)S_k(u)\mu \leq 0 \text{ in } \mathcal{M}_b(\Omega),\ \forall s \in \overline{dom(j)}. \tag{4.49}$$

From (4.48), one deduces

$$j(s) \geq j(u) + (s-u)b\ \mathcal{L}^N - \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R}$$

and so $b \in \partial j(u) = \beta(u)\mathcal{L}^N-$a.e. in $\Omega$.
From (4.49), one has, for all $t \in \overline{dom(j)}$,

$$\mu \geq 0 \text{ in } [u \in (t,\infty) \cap supp(h)]$$

and

$$\mu \leq 0 \text{ in } [u \in (-\infty,t) \cap supp(h)].$$

In particular, one has

$$\mu([m \leq 0 \leq M]) = 0.$$

So, $\mu^+$ is concentrated on $[u = M]$ and $\mu^-$ is concentrated on $[u = m]$. □

**Lemma 4.8.** *$u$ verifies the renormalized formulation (3.4).*

*Proof.* Since $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(.)}(\Omega)$ and in $W^{1,p_n(.)}(\Omega)$ as $p$ and $p_n$ verify (4.4), we can take test functions in $C^\infty(\overline{\Omega})$. So, let $\psi \in C^\infty(\overline{\Omega})$ a test function for the renormalized formulation (3.3) for $u_n$. One has

$$\int_\Omega \big(S(u_n)a_n(x,\nabla u_n).\nabla\psi + S'(u_n)a_n(x,\nabla u_n).\nabla u_n\psi + b(u_n)S(u_n)\psi\big)\mathrm{d}x$$
$$= \int_\Omega f_n S(u_n)\psi\mathrm{d}x, \tag{4.50}$$

where $S \in \mathbb{S}$ with $suppS \subset [-M,M]$.
As $n \to \infty$ in (4.50), reasoning as above to pass from (4.30) to (4.40), we get the different limits given in (4.31), (4.34) and (4.39). So, we should direct especially our attention to the term

$$\int_\Omega S'(u_n)a_n(x,\nabla u_n).(\nabla u_n)\psi\mathrm{d}x = \int_\Omega S'(u_n)\mathcal{Y}_n^M.(\nabla T_M(u_n))\psi\mathrm{d}x.$$

The sequence $S'(u_n)$ is uniformly bounded and converges to $S'(u)$ a.e. in $\Omega$. Thanks to Lemma 4.6$-(i)$ and by using Lebesgue generalized convergence theorem, this term converges to

$$\int_\Omega S'(u)\mathcal{Y}^M.\nabla T_M(u)\psi\mathrm{d}x = \int_\Omega S'(u)(a(x,\nabla u).\nabla u)\psi\mathrm{d}x.$$

Therefore, one deduces from (4.50), as $n \to \infty$,

$$\int_\Omega S(u)\psi \mathrm{d}z + \int_\Omega S(u)a(x,\nabla u).\nabla\psi \mathrm{d}x + \int_\Omega S'(u)a(x,\nabla u).\nabla u\psi \mathrm{d}x = \int_\Omega fS(u)\psi \mathrm{d}\sigma, \tag{4.51}$$

for all $\psi \in C^\infty(\overline{\Omega})$.
As $u \in dom(\beta) \subset [m, M]$, one can take $S \in \mathbb{S}$ such that $S(u) = 1$ and $S'(u) = 0$ to get

$$\int_\Omega \psi \mathrm{d}z + \int_\Omega a(x,\nabla u).\nabla\psi \mathrm{d}x = \int_\Omega f\psi \mathrm{d}x, \tag{4.52}$$

for all $\psi \in C^\infty(\overline{\Omega})$.
Since $z = b\mathcal{L}^N + \mu$, then it becomes that

$$\int_\Omega b\psi \mathrm{d}x + \int_\Omega \psi \mathrm{d}\mu + \int_\Omega a(x,\nabla u).\nabla\psi \mathrm{d}x = \int_\Omega f\psi \mathrm{d}x, \text{ for all } \psi \in C^\infty(\overline{\Omega}).$$

Moreover, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(.)}(\Omega)$. So, the above equality holds true for all $\psi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and one gets (3.4), wich ends the proof of Theorem 4.2. □

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