

## REFINEMENT OF THE CAUCHY-SCHWARTZ INEQUALITY WITH REFINEMENTS AND GENERALIZATIONS OF THE NUMERICAL RADIUS TYPE INEQUALITIES FOR OPERATORS

VUK STOJILJKOVIĆ<sup>1</sup> AND SILVESTRU SEVER DRAGOMIR<sup>2\*</sup>

ABSTRACT. Motivated by the results previously reported, the current work aims at developing new numerical radius upper bounds of Hilbert space operators by offering new improvements to the well-known Cauchy-Schwarz inequality. In particular, a novel Lemma (3.1) is given, which is utilized to further generalize several vector and numerical radius type inequalities, as well as previously given extensions of the Cauchy-Schwarz inequality. Specifically, (2.5) (2.8) (1.6) have been generalized by (4.3) (4.1) (4.2).

### 1. INTRODUCTION

Let  $B(\mathcal{H})$  be the Banach algebra of all bounded linear operators defined on a complex Hilbert space  $(\mathcal{H}; \langle \cdot, \cdot \rangle)$  with the identity operator  $1_{\mathcal{H}}$  in  $B(\mathcal{H})$ . A thing to note is that  $B(\mathcal{H})$  is not a Hilbert space. The numerical range of  $A \in B(\mathcal{H})$ , denoted by  $W(A)$ , is the image of the unit sphere of  $\mathcal{H}$  under the mapping  $x \mapsto \langle Ax, x \rangle$ . A relevant and important concept is a numerical radius, which is the supremum of the absolute values of all numbers in  $W(A)$ , that is

$$w(A) = \sup_{x \in \mathcal{H}, \|x\|=1} |\langle Ax, x \rangle|.$$

The absolute value of an operator  $S$  is defined as  $|S| = (S^*S)^{\frac{1}{2}}$ , which is always a non-negative operator. In 1952, Kato [11] proved the following celebrated generalization of Schwarz inequality for any operator  $T \in B(H)$ :

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle, \quad (1.1)$$

for any  $x, y \in \mathcal{H}, \alpha \in [0, 1]$ . In order to generalize this result, in 1994 Furuta [10] obtained the following result:

$$|\langle T|T|^{\alpha+\beta-1} x, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2\beta} y, y \rangle, \quad (1.2)$$

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*Date:* Received: Nov 30, 2023; Accepted: Jan 14, 2024.

\* Corresponding author: Vuk Stojiljković.

2010 *Mathematics Subject Classification.* Primary 47A30 ; Secondary 47A63 .

*Key words and phrases.* Numerical radius, Norm, Inequalities.

for any  $x, y \in \mathcal{H}$  and  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \geq 1$ . It is well known that  $w(\cdot)$  forms a norm on  $B(\mathcal{H})$  which is equivalent with the usual operator norm

$$\|A\| = \sup_{\|x\|=1} \langle Ax, Ax \rangle^{\frac{1}{2}},$$

for  $A \in B(\mathcal{H})$ . More precisely, the following sharp two-sided inequality holds

$$\frac{1}{2} \|A\| \leq w(A) \leq \|A\|, \quad (1.3)$$

where the sharpness holds when  $A^2 = 0$  and  $A$  is normal for the first and second inequalities respectively. Another important fact for the numerical radius is the power inequality which states

$$w(A^n) \leq w^n(A)$$

for any  $A \in B(\mathcal{H})$  and  $n \in \mathbb{N}$ . In [12], Kittaneh substantially improved the upper bound in (1.3) by showing that if  $A \in B(\mathcal{H})$ , then

$$w(A) \leq \frac{1}{2} \||A| + |A^*|\| \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{\frac{1}{2}} \right). \quad (1.4)$$

The following inequalities for  $w^2(\cdot)$  have been obtained in [13]

$$\frac{1}{4} \||A|^2 + |A^*|^2\| \leq w^2(A) \leq \frac{1}{2} \||A|^2 + |A^*|^2\|. \quad (1.5)$$

Kittaneh and Moradi [14] refined the inequality (1.5) for  $w^2(\cdot)$ , which holds for any  $A \in B(\mathcal{H})$

$$w^2(A) \leq \frac{1}{6} \||A|^2 + |A^*|^2\| + \frac{1}{3} w(A) \||A| + |A^*|\|. \quad (1.6)$$

Furthermore, Kittaneh et al. [9] established some inequalities that can be presented as

$$w^\alpha(J) \leq \frac{1}{2} \||J|^{2\alpha s} + |J^*|^{2\alpha(1-s)}\| \quad (1.7)$$

and

$$w^{2\alpha}(J) \leq \|s|J|^{2\alpha} + (1-s)|J^*|^{2\alpha}\|, \quad (1.8)$$

where  $J \in B(\mathcal{H})$ ,  $0 \leq s \leq 1$ , and  $\alpha \geq 1$ .

Inequalities for  $w^4(\cdot)$  were obtained by Abu-Omar et al. [1]

$$w^4(T) \leq \frac{1}{4} w^2(T^2) + \frac{1}{4} w(T^2) \|P\| + \frac{1}{16} \|P\|^2. \quad (1.9)$$

The following refinement of the  $w^4(\cdot)$  was obtained by Bhunia et al. [6]

$$w^4(T) \leq \frac{1}{4} w^2(T^2) + \frac{1}{8} w(T^2 P + P T^2) + \frac{1}{16} \|P\|^2, \quad (1.10)$$

where  $P = T^* T + T T^*$ .

Recently, Alomari [4] obtained refinements of (1.6), showing the following for  $T \in B(\mathcal{H})$ ,

$$\begin{aligned} w^2(T) &\leq \frac{1}{12} \||T| + |T^*|\|^2 + \frac{1}{3} w(T) \||T| + |T^*|\| \\ &\leq \frac{1}{6} \||T|^2 + |T^*|^2\| + \frac{1}{3} w(T) \||T| + |T^*|\|. \end{aligned} \quad (1.11)$$

Alomari also obtained

$$\begin{aligned} w^p(T) &\leq \frac{1}{2}\beta \left\| |T|^{2p\alpha} + |T^*|^{2p(1-\alpha)} \right\| + \frac{1}{\sqrt{2}}(1-\beta)w^{\frac{p}{2}}(T) \left\| |T|^{2p\alpha} + |T^*|^{2p(1-\alpha)} \right\|^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left\| |T|^{2p\alpha} + |T^*|^{2p(1-\alpha)} \right\|, \end{aligned} \quad (1.12)$$

for all  $p \geq 1$  and  $0 \leq \alpha, \beta \leq 1$ . For more results of this type, consult his paper [4].

Motivated by the previously described findings, the current work attempts to provide fresh refinements to the well-known Cauchy-Schwarz inequality in order to develop new numerical radius upper bounds of Hilbert space operators. Specifically, novel Lemma (3.1) is provided which generalizes previously given refinements of the Cauchy-Schwarz inequality, and is further used to generalize various vector and numerical radius type inequalities. In particular, inequalities (2.5) (2.8) (1.6) have been generalized by (4.3) (4.1) (4.2). Also, various corollaries of the obtained four operator inequality (3.4) have been given which showcase that our inequality indeed generalizes and sharpens the inequalities given by Dragomir in his paper [7]. More about Hilbert space inequalities can be seen in the following papers [16, 17, 18].

Moreover, the paper is structured as follows. First section is the introductory one, the second section concerns itself with the requirements of the sequel. Third section provides proof of the Lemma and showcases the generalization. Fourth section concerns itself with obtaining refinements in the numerical radius sense.

## 2. PRELIMINARIES

**Lemma 2.1.** *(Cauchy Schwartz) Inequality in the Hilbert space  $\mathcal{H}$  states that for any  $u, v \in \mathcal{H}$  we have the inequality*

$$|\langle u, v \rangle| \leq \|u\| \|v\| \quad (2.1)$$

*with equality if and only if the vectors  $u$  and  $v$  are linearly dependent in  $\mathcal{H}$ .*

**Lemma 2.2.** *(Hölder Mc-Carty inequality [15]). Let  $A \in B(\mathcal{H})$ ,  $A \geq 0$  and let  $x \in \mathcal{H}$  be any unit vector. Then we have*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \text{ for } r \geq 1, \quad (2.2)$$

$$\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \text{ for } 0 < r \leq 1. \quad (2.3)$$

The next result concerns with non-negative convex functions and can be found in [2].

**Lemma 2.3.** *Let  $f$  be a non-negative convex function on  $[0, +\infty)$  and  $A, B \in B(\mathcal{H})$  be positive operators. Then*

$$\left\| f\left(\frac{A+B}{2}\right) \right\| \leq \left\| \frac{f(A) + f(B)}{2} \right\|. \quad (2.4)$$

Another important fact about the numerical radius upper bounds that are of our interest are due to Dragomir in [8].

**Lemma 2.4.** *Let  $A, B \in B(\mathcal{H})$  and  $r \geq 1$ , then*

$$w^r(B^*A) \leq \frac{1}{2} \left\| |A|^{2r} + |B|^{2r} \right\|. \quad (2.5)$$

Dragomir further gave the refinement of the two operator inequality

$$w^2(B^*A) \leq \frac{1}{6} \left\| |A|^4 + |A^*|^4 \right\| + \frac{1}{3} w(B^*A) \left\| |A|^2 + |A^*|^2 \right\|. \quad (2.6)$$

We will make use of the following four-operator type inequality given by Dragomir [7].

**Lemma 2.5.** *Let  $A, B, C, D \in B(\mathcal{H})$ . Then for  $x, y \in \mathcal{H}$  we have the inequality*

$$|\langle DCBAx, x \rangle|^2 \leq \langle A^*|B|^2Ax, x \rangle \langle D|C^*|D^*y, y \rangle. \quad (2.7)$$

**Lemma 2.6.** ([7], eq. 31) *Let  $A, B, C, D \in B(\mathcal{H})$ . Then we have*

$$\|DCBA\|^2 \leq \|A^*|B|^2A\| \|D|C^*|^2D^*\|. \quad (2.8)$$

### 3. MAIN RESULTS

We give our first result, a Lemma which is instrumental in the development of the later results.

**Lemma 3.1.** *Let  $u, v \in \mathcal{H}$  and let  $k \geq 1$ . Let  $J$  be a set such that  $(0, 1) \subset J \subset \mathbb{R}$ . Let  $h$  be a mapping such that  $h : J \rightarrow \mathbb{R}^+$ , such that the following holds  $h(\theta) + h(1 - \theta) = 1$ . Then the following inequality holds*

$$|\langle u, v \rangle|^{2k} \leq h(\theta) \|u\|^{2k} \|v\|^{2k} + h(1 - \theta) |\langle u, v \rangle|^k \|u\|^k \|v\|^k \leq \|u\|^{2k} \|v\|^{2k}. \quad (3.1)$$

*Proof.* Consider the following,

$$\begin{aligned} |\langle u, v \rangle|^{2k} &= h(\theta) |\langle u, v \rangle|^k |\langle u, v \rangle|^k + h(1 - \theta) |\langle u, v \rangle|^k |\langle u, v \rangle|^k \\ &\leq h(\theta) \|u\|^{2k} \|v\|^{2k} + h(1 - \theta) |\langle u, v \rangle|^k \|u\|^k \|v\|^k \\ &\leq \|u\|^{2k} \|v\|^{2k} (h(\theta) + h(1 - \theta)) = \|u\|^{2k} \|v\|^{2k}. \end{aligned}$$

□

The following inequalities are corollaries of (3.1).  
The following Lemma was given by Alomari [4].

**Corollary 3.2.** *For all vectors  $u$  and  $v$  in an inner product space, we have*

$$|\langle u, v \rangle|^2 \leq (1 - \theta) |\langle u, v \rangle| \|u\| \|v\| + \theta \|u\|^2 \|v\|^2 \leq \|u\|^2 \|v\|^2$$

for all  $\theta \in [0, 1]$ .

*Proof.* Setting  $k = 1$ ,  $h(\theta) = \theta$  and letting  $\theta \in [0, 1]$ , we obtain the stated inequality. □

**Corollary 3.3.** *Setting  $h(\theta) = \theta$  and  $\theta = \frac{\alpha}{\alpha+1}$ ,  $\alpha \geq 0$ ,  $k = 1$  we obtain the Lemma given by Al-Dolat and Jaradat [3], namely*

$$|\langle u, v \rangle|^2 \leq \frac{1}{\alpha+1} \|u\| \|v\| |\langle u, v \rangle| + \frac{\alpha}{\alpha+1} \|u\|^2 \|v\|^2 \leq \|u\|^2 \|v\|^2. \quad (3.2)$$

**Corollary 3.4.** *Setting  $h(\theta) = \frac{1}{2}(\frac{1}{2} + \theta)$  and  $k = 1$ , we obtain the following refinement of the Cauchy-Schwartz inequality*

$$|\langle u, v \rangle|^2 \leq \frac{1}{2} \left( \frac{1}{2} + \theta \right) \|u\|^2 \|v\|^2 + \frac{1}{2} \left( \frac{3}{2} - \theta \right) |\langle u, v \rangle| \|u\| \|v\| \leq \|u\|^2 \|v\|^2. \quad (3.3)$$

The following Theorem is a refinement of the celebrated four operator Schwartz type inequality given by Dragomir [7] (2.7).

**Theorem 3.5.** *Let  $A, B, C, D \in B(\mathcal{H})$  and  $h$  such that (3.1) holds. Then for  $x, y \in \mathcal{H}$  and  $k \geq 1$  we have the inequality*

$$\begin{aligned} |\langle DCBAx, y \rangle|^{2k} &\leq h(\theta) \langle A^* | B|^2 Ax, x \rangle^k \langle D | C^*|^2 D^* y, y \rangle^k \\ &+ h(1 - \theta) |\langle DCBAx, y \rangle|^k \sqrt[k]{\langle A^* | B|^2 Ax, x \rangle} \sqrt[k]{\langle D | C^*|^2 D^* y, y \rangle} \\ &\leq \langle A^* | B|^2 Ax, x \rangle^k \langle D | C^*|^2 D^* y, y \rangle^k. \end{aligned} \quad (3.4)$$

*Proof.* Using (3.1) and setting  $u = BAx, v = C^* D^* y$ , then simplifying

$$\begin{aligned} \|BAx\|^2 &= \langle BAx, BAx \rangle = \langle A^* B^* BAx, x \rangle = \langle A^* | B|^2 Ax, x \rangle, \\ \|C^* D^* y\|^2 &= \langle D | C^*|^2 D^* y, y \rangle \end{aligned}$$

which when put in (3.1), we obtain the desired inequality.  $\square$

**Corollary 3.6.** *Setting  $k = 1$  and connecting the left and right hand side, we obtain the inequality given by Dragomir (2.7).*

We show that our Theorem generalizes the one given by Alomari et al.[5].

**Corollary 3.7.** *Let  $M, N, U, V \in B(\mathcal{H})$ ,  $\lambda \in [0, 1]$ , and  $\alpha \geq 1$ . Then, we have*

$$\begin{aligned} |\langle VMNUa, b \rangle|^{2\alpha} &\leq \lambda \langle (U^* | N|^2 U)^\alpha a, a \rangle \langle (V | M^*|^2 V^*)^\alpha b, b \rangle \\ &+ (1 - \lambda) |\langle VMNUa, b \rangle|^\alpha \sqrt[\alpha]{\langle (U^* | N|^2 U)^\alpha a, a \rangle \langle (V | M^*|^2 V^*)^\alpha b, b \rangle} \\ &\leq \langle (U^* | N|^2 U)^\alpha a, a \rangle \langle (V | M^*|^2 V^*)^\alpha b, b \rangle. \end{aligned} \quad (3.5)$$

*Proof.* Utilizing (2.2) on the first and second term on the middle expression in the inequality (3.4), we obtain the middle term in the inequality. The last term is obtained by utilizing (2.2) on the first term, then utilizing (2.7) and (2.2) on the second term together, then we use the fact that  $h(\theta) + h(1 - \theta) = 1$ , from which we obtain the right hand side. Connecting the left hand side, the middle term obtained and the right hand side, we deduce the desired inequality with respect to  $h(\theta)$ . Finally, setting  $h(\theta) = \theta, \theta \in [0, 1]$ , we deduce the desired inequality.  $\square$

In particular, the obtained inequality (3.4) refines also the Furuta's inequality (1.2).

**Corollary 3.8.** *The Furuta inequality for  $\alpha, \beta \geq 0$  with  $\alpha + \beta \geq 1$  is a particular case of (3.4),*

$$\begin{aligned} |\langle T | T|^{\alpha+\beta-1} x, y \rangle|^{2k} &\leq h(\theta) \langle |T|^{2\alpha} x, x \rangle^k \langle |T^*|^{2\beta} y, y \rangle^k \\ &+ h(1 - \theta) |\langle T | T|^{\alpha+\beta-1} x, y \rangle|^k \sqrt[k]{\langle |T|^{2\alpha} x, x \rangle} \sqrt[k]{\langle |T^*|^{2\beta} y, y \rangle} \\ &\leq \langle |T|^{2\alpha} x, x \rangle^k \langle |T^*|^{2\beta} y, y \rangle^k. \end{aligned} \quad (3.6)$$

*Proof.* Let  $T = U|T|$  be the polar decomposition of the operator  $T$ , where  $U$  is partial isometry and the kernel  $N(U) = N(|T|)$ . If we take  $D = U, C = |T|^\beta, B = I$  and  $A = |T|^\alpha$  then we have

$$\begin{aligned} DCBA &= U|T|^\beta|T|^\alpha = U|T||T|^{\alpha+\beta-1} = T|T|^{\alpha+\beta-1}, \\ A^*|B|^2A &= |T|^\alpha|T|^\alpha = |T|^{2\alpha} \end{aligned}$$

and

$$D|C^*|^2D^* = U|T|^{2\beta}U^* = |T^*|^{2\beta}$$

where we have used the following property for the polar decomposition of  $T$ ,  $|T^*|^{2\beta} = U|T|^{2\beta}U^*, \beta > 0$ .

Setting  $k = 1$ , we obtain the Furuta's inequality by connecting the left and right hand side.  $\square$

The following corollary gives a sharper estimate to the one given by Dragomir [7].

**Corollary 3.9.** *For any operator  $T \in B(\mathcal{H})$  and any  $\alpha, \beta \geq 1$  we have the inequality*

$$\begin{aligned} |\langle T|T|^{\beta-1}T|T|^{\alpha-1}x, y \rangle|^{2k} &\leq h(\theta) \langle |T|^{2\alpha}x, x \rangle^k \langle |T^*|^{2\beta}y, y \rangle^k \\ &+ h(1-\theta) |\langle T|T|^{\beta-1}T|T|^{\alpha-1}x, y \rangle|^k \sqrt[k]{\langle |T|^{2\alpha}x, x \rangle} \sqrt[k]{\langle |T^*|^{2\beta}y, y \rangle} \\ &\leq \langle |T|^{2\alpha}x, x \rangle^k \langle |T^*|^{2\beta}y, y \rangle^k. \end{aligned} \quad (3.7)$$

*Proof.* Let  $T = U|T|$  be the polar decomposition of the operator  $T$ , where  $U$  is partial isometry and the kernel  $N(U) = N(|T|)$ . If we take  $D = U, C = |T|^\beta, B = U$  and  $A = |T|^\alpha$ , then we have

$$\begin{aligned} DCBA &= U|T|^\beta U|T|^\alpha = U|T||T|^{\beta-1}U|T||T|^{\alpha-1} = T|T|^{\beta-1}T|T|^{\alpha-1}, \\ A^*|B|^2A &= |T|^\alpha U^*U|T|^\alpha = |T|^{\alpha-1}|T|U^*U|T||T|^{\alpha-1} \\ &= |T|^{\alpha-1}T^*T|T|^{\alpha-1} = |T|^{\alpha-1}|T|^2|T|^{\alpha-1} = |T|^{2\alpha} \end{aligned}$$

and

$$D|C^*|^2D^* = U|T|^{2\beta}U^* = |T^*|^{2\beta},$$

which when incorporated into the obtained (3.4) inequality, we obtain the desired result.  $\square$

**Corollary 3.10.** *Setting  $\alpha = \beta = 1, k = 1$  in (3.7) we obtain the following inequality, which is sharper than the one given by Dragomir [7] (equation 13),*

$$\begin{aligned} |\langle T^2x, y \rangle|^2 &\leq h(\theta) \langle |T|^2x, x \rangle \langle |T^*|^2y, y \rangle \\ &+ h(1-\theta) |\langle T^2x, y \rangle| \sqrt{\langle |T|^2x, x \rangle} \sqrt{\langle |T^*|^2y, y \rangle} \\ &\leq \langle |T|^2x, x \rangle \langle |T^*|^2y, y \rangle. \end{aligned} \quad (3.8)$$

The following inequality refines the inequality given by Dragomir in his paper [7], namely equation (21) in the paper.

**Corollary 3.11.** *For any operator  $T \in B(\mathcal{H})$  and any  $\gamma, \delta \geq 0$  we have the inequality*

$$\begin{aligned} |\langle |T|^\gamma T^2 |T|^\gamma x, y \rangle|^{2k} &\leq h(\theta) \langle |T|^{2+2\delta} x, x \rangle^k \langle |T^*| |T|^\gamma |^2 y, y \rangle^k \\ &+ h(1-\theta) |\langle |T|^\gamma T^2 |T|^\gamma x, y \rangle|^k \sqrt[k]{\langle |T|^{2+2\delta} x, x \rangle} \sqrt[k]{\langle |T^*| |T|^\gamma |^2 y, y \rangle} \\ &\leq \langle |T|^{2+2\delta} x, x \rangle^k \langle |T^*| |T|^\gamma |^2 y, y \rangle^k, \end{aligned} \quad (3.9)$$

for any  $x, y \in \mathcal{H}$ .

*Proof.* If we take  $D = |T|^\gamma, C = T, B = T$  and  $A = |T|^\delta$  then we have

$$\begin{aligned} DCBA &= |T|^\gamma T^2 |T|^\gamma, \\ A^* |B|^2 A &= |T|^\gamma |T|^2 |T|^\gamma = |T|^{2+2\delta} \end{aligned}$$

and

$$\begin{aligned} D |C^*|^2 D^* &= |T|^\gamma |T|^2 |T|^\delta = |T|^\gamma T T^* |T|^\gamma \\ &= |T|^\gamma T (|T|^\gamma T)^* = \left| (|T|^\gamma T)^* \right|^2 = |T^*| |T|^\gamma |^2, \end{aligned}$$

which when plugged in (3.4), we obtain the desired inequality.  $\square$

The corollaries given showcase the applicability of the obtained Lemma (3.1).

#### 4. NUMERICAL RADIUS TYPE INEQUALITIES

**Theorem 4.1.** *Let  $A, B, C, D \in B(\mathcal{H}), k \geq 1$  and  $h$  such that it fulfills the condition of (3.1). Then we have*

$$\begin{aligned} \|DCBA\|^{2k} &\leq h(\theta) \|A^* |B|^2 A\|^k \|D |C^*|^2 D^*\|^k \\ &+ h(1-\theta) \|DCBA\|^k \|A^* |B|^2 A\|^{\frac{k}{2}} \|D |C^*|^2 D^*\|^{\frac{k}{2}} \leq \|A^* |B|^2 A\|^k \|D |C^*|^2 D^*\|^k. \end{aligned} \quad (4.1)$$

*Proof.* Taking the supremum over all unit vectors  $x, y \in \mathcal{H}$  in (3.4), we get

$$\begin{aligned} \|DCBA\|^{2k} &= \sup_{\|x\|=1, \|y\|=1} |\langle DCBAx, y \rangle|^{2k} \\ &\leq \sup_{\|x\|=1, \|y\|=1} \left( h(\theta) \langle A^* |B|^2 Ax, x \rangle^k \langle D |C^*|^2 D^* y, y \rangle^k \right. \\ &\quad \left. + h(1-\theta) |\langle DCBAx, y \rangle|^k \sqrt[k]{\langle A^* |B|^2 Ax, x \rangle} \sqrt[k]{\langle D |C^*|^2 D^* y, y \rangle} \right) \\ &\leq \sup_{\|x\|=1, \|y\|=1} \left( \langle A^* |B|^2 Ax, x \rangle^k \langle D |C^*|^2 D^* y, y \rangle^k \right). \end{aligned}$$

Considering the middle term in the last inequality

$$\begin{aligned} &\sup_{\|x\|=1, \|y\|=1} \left( h(\theta) \langle A^* |B|^2 Ax, x \rangle^k \langle D |C^*|^2 D^* y, y \rangle^k \right. \\ &\quad \left. + h(1-\theta) |\langle DCBAx, y \rangle|^k \sqrt[k]{\langle A^* |B|^2 Ax, x \rangle} \sqrt[k]{\langle D |C^*|^2 D^* y, y \rangle} \right) \end{aligned}$$

and using basic properties of the supremum we obtain the left hand side of the desired inequality. Using (2.8) on the second term in the obtained inequality

$$h(\theta) \|A^*|B|^2A\|^k \|D|C^*|^2D^*\|^k \\ + h(1-\theta) \|DCBA\|^k \|A^*|B|^2A\|^{\frac{k}{2}} \|D|C^*|^2D^*\|^{\frac{k}{2}}$$

and using the properties of the mapping  $h$ , we obtain the right hand side of the desired inequality. Combining everything, we obtain the desired inequality.  $\square$

**Corollary 4.2.** *The previously obtained inequality refines the Dragomir's result given in [7] (eq. (31)). Setting  $k = 1$  and connecting left and right hand side, we obtain the inequality given by Dragomir, namely*

$$\|DCBA\|^2 \leq \|A^*|B|^2A\| \|D|C^*|^2D^*\|.$$

**Theorem 4.3.** *Let  $A \in B(\mathcal{H})$  and  $h$  be such that the conditions from (3.1) are fulfilled, then for  $k \geq 2, \alpha \in [0, 1]$  the following inequality holds*

$$w(A)^k \leq \frac{h(\theta)}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \| + \frac{h(1-\theta)}{2} w^{\frac{k}{2}}(A) \| |A|^{\alpha k} + |A^*|^{k(1-\alpha)} \| \quad (4.2) \\ \leq \frac{1}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \|.$$

*Proof.* Let  $x \in \mathcal{H}$  be any unit vector, consider the following

$$|\langle Ax, x \rangle|^k = h(\theta) |\langle Ax, x \rangle|^k + h(1-\theta) |\langle Ax, x \rangle|^k \\ \leq h(\theta) \langle |A|^{2\alpha} x, x \rangle^{\frac{k}{2}} \langle |A^*|^{2(1-\alpha)} x, x \rangle^{\frac{k}{2}} + h(1-\theta) |\langle Ax, x \rangle|^{\frac{k}{2}} \langle |A|^{2\alpha} x, x \rangle^{\frac{k}{4}} \langle |A^*|^{2(1-\alpha)} x, x \rangle^{\frac{k}{4}} \quad (1.1) \\ \leq \frac{h(\theta)}{2} (\langle |A|^{2\alpha} x, x \rangle^k + \langle |A^*|^{2(1-\alpha)} x, x \rangle^k) + \frac{h(1-\theta)}{2} |\langle Ax, x \rangle|^{\frac{k}{2}} (\langle |A|^{2\alpha} x, x \rangle^{\frac{k}{2}} + \langle |A^*|^{2(1-\alpha)} x, x \rangle^{\frac{k}{2}}) (AG) \\ \leq \frac{h(\theta)}{2} (\langle |A|^{2\alpha k} x, x \rangle + \langle |A^*|^{2k(1-\alpha)} x, x \rangle) + \frac{h(1-\theta)}{2} |\langle Ax, x \rangle|^{\frac{k}{2}} (\langle |A|^{\alpha k} x, x \rangle + \langle |A^*|^{k(1-\alpha)} x, x \rangle) \quad (2.2).$$

When we take sup over all the unit vectors, we obtain the left hand side of the inequality.

To obtain the right hand side of the inequality, start from the middle term in the inequality (4.2), then we proceed as follows

$$w(A)^k \leq \frac{h(\theta)}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \| + \frac{h(1-\theta)}{2} w^{\frac{k}{2}}(A) \| |A|^{\alpha k} + |A^*|^{k(1-\alpha)} \| \\ \leq \frac{h(\theta)}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \| + \frac{h(1-\theta)}{4} \| |A|^{\alpha k} + |A^*|^{k(1-\alpha)} \|^2 \quad (1.7) (s = \alpha) \\ \leq \frac{h(\theta)}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \| + \frac{h(1-\theta)}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \| = \frac{1}{2} \| |A|^{2\alpha k} + |A^*|^{2k(1-\alpha)} \|. \quad \square$$

*Remark 4.4.* Setting  $k = 2, h = I, \theta \in [0, 1], \alpha = \frac{1}{2}$  we obtain the refinement given by Al-Dolat et al. [3]. In particular, upper bounds of  $w^k(A)$  have been obtained for powers of  $k \geq 2$  for arbitrary function  $h$  that fulfills the condition given by (3.1), it also shows that our inequality is sharper than the one given by El-Haddad and Kittaneh (1.7). Refinement of (1.5) given by Kittaneh and Moradi (1.6) is obtained by setting  $h(\theta) = \theta, \theta = \frac{1}{3}, k = 2, \alpha = \frac{1}{2}$  in (4.2), see [14].

**Theorem 4.5.** *Let  $A, B \in B(\mathcal{H})$ . Let the mapping  $h$  fulfill the requirements of (3.1), then for any  $k \geq 1$  the following inequality holds*

$$\begin{aligned} w^{2k}(B^*A) &\leq \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{2} w^k(B^*A) \| |A|^{2k} + |B|^{2k} \| \quad (4.3) \\ &\leq \frac{1}{2} \| |A|^{4k} + |B|^{4k} \|. \end{aligned}$$

*Proof.* Setting  $B = I, C = I, D^* = B, y = x$  in (3.4), we obtain the following

$$\begin{aligned} |\langle B^*Ax, x \rangle|^{2k} &\leq h(\theta) \langle |A|^2x, x \rangle^k \langle |B|^2x, x \rangle^k \\ &\quad + h(1-\theta) |\langle B^*Ax, x \rangle|^k \sqrt[k]{\langle |A|^2x, x \rangle} \sqrt[k]{\langle |B|^2x, x \rangle}. \end{aligned}$$

Using AG inequality, we obtain

$$\leq \frac{h(\theta)}{2} (\langle |A|^2x, x \rangle^{2k} + \langle |B|^2x, x \rangle^{2k}) + \frac{h(1-\theta)}{2} |\langle B^*Ax, x \rangle|^k (\langle |A|^2x, x \rangle^k + \langle |B|^2x, x \rangle^k).$$

Now using (2.2), we obtain

$$\leq \frac{h(\theta)}{2} (\langle |A|^{4k}x, x \rangle + \langle |B|^{4k}x, x \rangle) + \frac{h(1-\theta)}{2} |\langle B^*Ax, x \rangle|^k (\langle |A|^{2k}x, x \rangle + \langle |B|^{2k}x, x \rangle).$$

Taking sup over all the unit vectors, we obtain the left hand side inequality.

To obtain the right hand side, notice the following

$$\begin{aligned} w^{2k}(B^*A) &\leq \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{2} w^k(B^*A) \| |A|^{2k} + |B|^{2k} \| \\ &\leq \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{4} \| |A|^{2k} + |B|^{2k} \|^2 \quad (2.5) \\ &\leq \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{4} \| (|A|^{2k} + |B|^{2k})^2 \| \\ &= \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{4} \left\| \left( \frac{2|A|^{2k} + 2|B|^{2k}}{2} \right)^2 \right\|. \end{aligned}$$

Now using the fact that  $f(x) = x^2$  is convex and positive, then by (2.4), we have

$$\begin{aligned} &\leq \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{4} \left\| \frac{(2|A|^{2k})^2 + (2|B|^{2k})^2}{2} \right\| \\ &= \frac{h(\theta)}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{h(1-\theta)}{2} \| |A|^{4k} + |B|^{4k} \| = \frac{1}{2} \| |A|^{4k} + |B|^{4k} \|. \end{aligned}$$

□

*Remark 4.6.* Setting  $h = I, \theta \in [0, 1]$  we obtain the inequality given by Alomari [4] (eq. 3.1), namely

$$\begin{aligned} w^{2k}(B^*A) &\leq \frac{\theta}{2} \| |A|^{4k} + |B|^{4k} \| + \frac{1-\theta}{2} w^k(B^*A) \| |A|^{2k} + |B|^{2k} \| \\ &\leq \frac{1}{2} \| |A|^{4k} + |B|^{4k} \|. \end{aligned}$$

In particular, the proof of the inequality given in Theorem is simpler than the one given by Alomari.

Setting  $h(\theta) = \theta, \theta = \frac{l}{1+l}, l \geq 0, k = 1$  we recover the inequality given by Al-Dolat et al. ([3], Corollary 2.7), namely we obtain

$$\begin{aligned} w^2(B^*A) &\leq \frac{1}{2l+2} \| |A|^4 + |B|^4 \| + \frac{l}{2l+2} w^k(B^*A) \| |A|^2 + |B|^2 \| \\ &\leq \frac{1}{2} \| |A|^4 + |B|^4 \|. \end{aligned} \quad (4.4)$$

**Corollary 4.7.** *Our inequality (4.3) refines the inequality given by Dragomir (2.5). In particular, for arbitrary function  $h$  that fulfills the conditions given by (3.1), (4.3) refines the inequality given by Dragomir.*

*Remark 4.8.* Setting  $h(\theta) = \theta, \theta = \frac{l}{1+l}, l = \frac{1}{2}, k = 1$ , we recover the inequality

$$w^2(B^*A) \leq \frac{1}{6} \| |A|^4 + |A^*|^4 \| + \frac{1}{3} w(B^*A) \| |A|^2 + |A^*|^2 \|$$

which was given by Dragomir (2.6).

The inequality obtained provides a refinement given by Dragomir (2.6) for arbitrary powers of  $k \geq 1$ .

## 5. CONCLUSION

Refinement of the Cauchy-Schwartz inequality with respect to the function  $h$  and powers of  $k$  has been given (3.1). As a consequence the four-operator inequality given by Dragomir has been refined (2.7), as well as other inequalities which were obtained by Dragomir in his paper [7]. We have generalized the inequality given by Dragomir (2.6) with respect to the powers of the operator obtained. Numerical radius type inequality (4.1) has been given which sharpens the inequality given by Dragomir in his paper [7], namely (2.8). We also have obtained the inequality (1.6) for powers of  $k \geq 2$ . Our inequality (4.2) refined the one given by El-Haddad and Kittaneh (1.7) for arbitrary function  $h$  such that the conditions from (3.1) hold. Question arises whether further generalizations of the obtained inequalities are possible, and whether sharper inequalities can be obtained.

## REFERENCES

1. A. Abu-Omar, F. Kittaneh, *Upper and lower bounds for the numerical radius with an application to involution operators*, Rocky Mountain J. Math. 45 (4) (2015) 1055–1065. <https://doi.org/10.1216/RMJ-2015-45-4-1055>
2. J. Aujla, F. Silva, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. 369 (2003), 217–233. [https://doi.org/10.1016/S0024-3795\(02\)00720-6](https://doi.org/10.1016/S0024-3795(02)00720-6)

3. M. Al-Dolat , I. Jaradat, *A refinement of the Cauchy-Schwarz inequality accompanied by new numerical radius upper bounds*. Filomat 2023 Volume 37, Issue 3, Pages: 971-977 <https://doi.org/10.2298/FIL2303971A>
4. M.W. Alomari, *On Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*. Ricerche mat (2022). <https://doi.org/10.1007/s11587-022-00689-2>
5. M.W. Alomari, M. Bakherad, M. Hajmohamadi, C. Chesneau, V. Leiva, C. Martin-Barreiro, *Improvement of Furuta's Inequality with Applications to Numerical Radius*. Mathematics 2023, 11, 36. <https://doi.org/10.3390/math11010036>
6. P. Bhunia, S. Bag , K. Paul, *Numerical radius inequalities and its applications in estimation of zeros of polynomials*, Linear Algebra and its Applications, Volume 573, 2019, Pages 166-177. <https://doi.org/10.1016/j.laa.2019.03.017>
7. S.S. Dragomir, *Some inequalities generalizing Kato's and Furuta's results*. Filomat 2014 Volume 28, Issue 1, Pages: 179-195. <https://doi.org/10.2298/FIL1401179D>
8. S.S. Dragomir. *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*. Sarajevo J. Math. 2009, 5, 269–278. <https://vuir.vu.edu.au/id/eprint/17652>
9. M. El-Haddad, F. Kittaneh, *Numerical radius inequalities for Hilbert space operators II*. Studia Math. 2007, 182, 133–140. <https://doi.org/10.4064/SM182-2-3>
10. T. Furuta, *An extension of the Heinz-Kato theorem*. Proc. Amer. Math. Soc. 120 (1994), no. 3, 785–787. <https://doi.org/10.1090/S0002-9939-1994-1169027-6>
11. T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. 125(1952), 208-212. <https://doi.org/10.1007/BF01343117>
12. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. 158 (2003), 11–17. <https://doi.org/10.4064/SM158-1-2>
13. F. Kittaneh, *Numerical radius inequalities for Hilbert space operators* Studia Mathematica 168.1 (2005): 73-80. <http://eudml.org/doc/284514>
14. F. Kittaneh, H.R. Moradi, *Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. 23 (2020), 1117–1125. <https://doi.org/doi.org/10.7153/mia-2020-23-85>
15. C. A. McCarthy: *cp*, Israel J. Math. 5 (1967), 249–271. <https://doi.org/10.1007/BF02771613>
16. V. Stojiljković, R. Ramaswamy, O.A.A. Abdelnaby, S. Radenović, *Some Refinements of the Tensorial Inequalities in Hilbert Spaces*. Symmetry 2023, 15, 925. <https://doi.org/10.3390/sym15040925>
17. V. Stojiljkovic. *Twice Differentiable Ostrowski Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces*. European Journal of Pure and Applied Mathematics, 16(3), (2023), 1421–1433. <https://doi.org/10.29020/nybg.ejpam.v16i3.4843>
18. V. Stojiljkovic. (2023). *'Twice Differentiable Ostrowski Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces'*, Electronic Journal of Mathematical Analysis and Applications, 11(2), pp. 1-15. <https://doi.org/10.21608/ejmaa.2023.199881.1014>

<sup>1</sup> FACULTY OF SCIENCES, UNIVERSITY OF NOVI SAD, SERBIA.  
Email address: [vuk.stojiljkovic999@gmail.com](mailto:vuk.stojiljkovic999@gmail.com)

<sup>2</sup> MATHEMATICS, COLLEGE OF SPORT HEALTH AND ENGINEERING, VICTORIA UNIVERSITY MELBOURNE CITY, VIC 8001, AUSTRALIA  
Email address: [Sever.Dragomir@vu.edu.au](mailto:Sever.Dragomir@vu.edu.au)