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SOME IMPORTANT PROPERTIES OF INVARIANT SUBMANIFOLDS OF AN ALMOST α-COSYMPLECTIC (k, μ, ν)-SPACE

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Abstract. The object of the paper is to investigate invariant submanifolds of an almost α-cosymplectic (k, μ, ν)-space. Then we have proved that the necessary and sufficient conditions for an invariant submanifold of an almost α-cosymplectic (k, μ, ν)-space to be totally geodesic under the some conditions. Consequently, we have obtained some interesting results invariant submanifolds of an almost α-cosymplectic (k, μ, ν)-space.

1. Introduction

It is well known that there exist contact metric manifolds M^{2n+1} whose curvature tensor R and the direction of the characteristic vector field ξ holds $R(u_1, u_2)\xi = 0$ for any vector fields on M^{2n+1}. Using a $D-$homothetic deformationn to a contact metric manifold with $R(u_1, u_2)\xi = 0$, we get a contact metric manifold satisfying the following special condition

$$R(u_1, u_2)\xi = \eta(u_2)(kI + \mu h)u_1 - \eta(u_1)(kI + \mu h)u_2, \tag{1.1}$$

where k, μ are constants and h is the self-adjoint $(1, 1)-$type tensor field. This condition is called $(k, \mu)-$nullity on M^{2n+1} [3]. T. Koufogiorgos and C. Tsichlias obtained a new class of 3-dimensional contact metric manifolds that k and μ are non-constant smooth functions [4].

S. I. Goldberg and K. Yano obtained integrability conditions for almost cosymplectic structures on almost contact manifolds.The simplest examples of almost cosymplectic manifolds are these structures of almost Kaehler manifolds, the real $\mathbb{R}$ line, and the circle S^1. Besides, they studied an almost cosymplectic manifold is cosymplectic only in the case it is locally flat [8].

İ. Küpeli Erken researched almost $\alpha-$cosymplectic manifolds. They studied, respectively, projectively flat, conformally flat and concircularly flat almost

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$\alpha-$cosymplectic manifolds (with the $\eta-$parallel tensor field ϕh). They devoted to properties of almost with the $\eta-$parallel tensor field ϕh [11].

Z. Olszak gave certain sufficient conditions for an almost contact metric structure to be almost cosymplectic. They proved that almost cosymplectic manifolds of non-zero constant sectional curvature do not exist in dimensions greater than three. However, such manifolds of zero sectional curvature (i. e. locally flat) exist and they were cosymplectic. Moreover they studied certain restrictions on the scalar curvature of almost cosymplectic manifolds which were conformally flat or of constant ϕ-sectional curvature [14].

In 2022, M. Atçeken studied at the invariant submanifolds of an almost α-cosymplectic (k, μ, ν)-space that matched certain geometric requirements so that $Q(\sigma, R) = 0$, $Q(S, \sigma) = 0$, $Q(S, \widetilde{\nabla}\sigma) = 0$, $Q(S, \widetilde{R} \cdot \sigma) = 0$, $Q(g, C \cdot R) = 0$ and $Q(S, C \cdot \sigma) = 0$. They showed that under certain circumstances, these conditions are identical to totally geodesic [2].

Motivated by the above studies, the aim of our article is to study invariant submanifolds of an almost α-cosymplectic (k, μ, ν)-space. Additionally, we show that some conditions for an invariant submanifold of an almost α-cosymplectic (k, μ, ν)-space to be totally geodesic. Then some characterizations are obtained and classifications have been made.

2. Preliminaries

An almost contact manifold is a 1-form η satisfying M^{2n+1}, an odd-dimensional manifold, a field ϕ of endomorphisms of the tangent spaces, a characteristic or Reeb vector field, and a vector field ξ

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

in which $I : TM^{2n+1} \to TM^{2n+1}$ denotes an identity mapping. Because of (2.1), it follows

$$\eta \circ \phi = 0, \ \phi\xi = 0, \ rank(\phi)=2n. \tag{2.2}$$

An almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ is noted to be normal if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denote the Nijenhuis tensor field of ϕ. Any almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ is known to have a Riemannian metric like that

$$g(\phi u_1, \phi u_2) = g(u_1, u_2) - \eta(u_1)\eta(u_2), \tag{2.3}$$

for all vector fields $u_1, u_2 \in \Gamma(TM)$ [5]. A metric of this type, g, is known as an equipped metric, and the structure (ϕ, η, ξ, g) and manifold $M^{2n+1}(\phi, \eta, \xi, g)$, associated with it, are known as an almost contact metric manifolds and are written as $M^{2n+1}(\phi, \eta, \xi, g)$. It is defined for $M^{2n+1}(\phi, \eta, \xi, g)$ to have a 2-form Φ. It is known as the fundamental form of $M^{2n+1}(\phi, \eta, \xi, g)$ when $\Phi(u_1, u_2) = g(\phi u_1, u_2)$. An almost contact metric manifold is referred to as a cosymplectic manifold if η and Φ are closed, that is, $d\eta = d\Phi = 0$ [6].

The definition of an almost α-cosymplectic manifold for every real number α is [9]

$$d\eta = 0, \; d\Phi = 2\alpha\eta \wedge \Phi. \tag{2.4}$$

The term $\alpha-$cosymplectic refers to a normal almost $\alpha-$cosymplectic manifold [15]. It's commonly known that the following equality holds for the tensor h on the contact metric manifold $M^{2n+1}(\phi, \eta, \xi, g)$, described by $2h = L_\xi \phi$,

$$\widetilde{\nabla}_{u_1}\xi = -\phi u_1 - \phi h u_1, \; h\phi + \phi h = 0, \; trh = tr\phi h = 0, \; h\xi = 0, \tag{2.5}$$

in this case, $\widetilde{\nabla}$ is the Levi-Civita connection on M^{2n+1} [1].

The following presented the notation of the $(k, \mu, \nu)-$contact metric manifold, which expands above generalized (k, μ)-spaces:

$$R(u_1, u_2)\xi = \eta(u_2)\left[kI + \mu h + \nu\phi h\right]u_1 + \eta(u_1)\left[kI + \mu h + \nu\phi h\right]u_2, \tag{2.6}$$

R is the Riemannian curvature tensor of M^{2n+1} and certain smooth functions k, μ and ν on M^{2n+1}, where u_1, u_2 are vector fields [10].

Lemma 2.1. *Given $M^{2n+1}(\phi, \eta, \xi, g)$ is an almost $\alpha-$cosymplectic $(k, \mu, \nu)-$space, so*

$$h^2 = (k + \alpha^2)\phi^2, \tag{2.7}$$

$$\xi(k) = 2(k + \alpha^2)(\nu - 2\alpha), \tag{2.8}$$

$$\begin{aligned} R(\xi, u_1)u_2 \;\; = \;\; & k[g(u_1, u_2)\xi - \eta(u_2)u_1] + \mu[g(hu_1, u_2)\xi - \eta(u_2)hu_1] \\ & +\nu[g(\phi h u_1, u_2)\xi - \eta(u_2)\phi h u_1], \end{aligned} \tag{2.9}$$

$$(\widetilde{\nabla}_{u_1}\phi)u_2 = g(\alpha\phi u_1 + hu_1, u_2)\xi - \eta(u_2)(\alpha\phi u_1 + hu_1), \tag{2.10}$$

$$\widetilde{\nabla}_{u_1}\xi = -\alpha\phi^2 u_1 - \phi h u_1, \tag{2.11}$$

for any vector fields u_1, u_2 on M^{2n+1} [5].

Let M be an immersed submanifold of $\widetilde{M}^{2n+1}$, which is an almost $\alpha-$cosymplectic (k, μ, ν)-space. We describe the tangent and normal subspaces of M in $\widetilde{M}$ by $\Gamma(TM)$ and $\Gamma(T^\perp M)$. After that, the Gauss and Weingarten formulas are provided, respectively, by

$$\widetilde{\nabla}_{u_1}u_2 = \nabla_{u_1}u_2 + \sigma(u_1, u_2), \tag{2.12}$$

and

$$\widetilde{\nabla}_{u_1}u_5 = -A_{u_5}u_1 + \nabla^\perp_{u_1}u_5 \tag{2.13}$$

for all $u_1, u_2 \in \Gamma(TM)$ and $u_5 \in \Gamma(T^\perp M)$, σ and A are referred to as the second fundamental form and shape operators of M, respectively, ∇ and $\nabla^\perp$ are the induced connections on M and $\Gamma(T^\perp M)$. $\Gamma(TM)$ stands for the set of differentiable vector fields on M. They are associated by

$$g(A_{u_5}u_1, u_2) = g(\sigma(u_1, u_2), u_5). \tag{2.14}$$

The second fundamental form σ is first covariant derivative is given by

$$(\widetilde{\nabla}_{u_1}\sigma)(u_2, u_3) = \nabla^\perp_{u_1}\sigma(u_2, u_3) - \sigma(\nabla_{u_1}u_2, u_3) - \sigma(u_2, \nabla_{u_1}u_3), \tag{2.15}$$

for all $u_1, u_2, u_3 \in \Gamma(TM)$. If $\widetilde{\widetilde{\nabla}}\sigma = 0$, the second fundamental form is parallel, which is considered to be submanifold.

The following Gauss equation results from denoting the Riemannian curvature tensor of the submanifold M by R.

$$\begin{aligned} \widetilde{R}(u_1, u_2)u_3 &= R(u_1, u_2)u_3 + A_{\sigma(u_1,u_3)}u_2 - A_{\sigma(u_2,u_3)}u_1 + (\widetilde{\nabla}_{u_1}\sigma)(u_2, u_3) \\ &\quad -(\widetilde{\nabla}_{u_2}\sigma)(u_1, u_3), \end{aligned} \tag{2.16}$$

for all $u_1, u_2, u_3 \in \Gamma(TM)$.

$\widetilde{R} \cdot \sigma$ is given by

$$\begin{aligned} (\widetilde{R}(u_1, u_2) \cdot \sigma)(u_4, u_5) &= R^{\perp}(u_1, u_2)\sigma(u_4, u_5) - \sigma(R(u_1, u_2)u_4, u_5) \\ &\quad -\sigma(u_4, R(u_1, u_2)u_5), \end{aligned} \tag{2.17}$$

where

$$R^{\perp}(u_1, u_2) = [\nabla^{\perp}_{u_1}, \nabla^{\perp}_{u_2}] - \nabla^{\perp}_{[u_1,u_2]},$$

indicate the normal bundle's Riemannian curvature tensor.

In fact, for the Riemannian manifold (M^{2n+1}, g), the W_1 curvature tensor is determined by

$$W_1(u_1, u_2)u_3 = R(u_1, u_2)u_3 + \frac{1}{2n}[S(u_2, u_3)u_1 - S(u_1, u_3)u_2], \tag{2.18}$$

for all $u_1, u_2, u_3 \in \Gamma(TM)$ [17].

Similarly, the tensor $W_1 \cdot \sigma$ is defined by

$$\begin{aligned} (W_1(u_1, u_2) \cdot \sigma)(u_4, u_5) &= R^{\perp}(u_1, u_2)\sigma(u_4, u_5) - \sigma(W_1(u_1, u_2)u_4, u_5) \\ &\quad -\sigma(u_4, W_1(u_1, u_2)u_5), \end{aligned} \tag{2.19}$$

for all $u_1, u_2, u_4, u_5 \in \Gamma(TM)$.

Furthermore, the W_8-curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$W_8(u_1, u_2)u_3 = R(u_1, u_2)u_3 - \frac{1}{2n}[S(u_2, u_3)u_1 - S(u_1, u_2)u_3] \tag{2.20}$$

for all $u_1, u_2, u_3 \in \Gamma(TM)$ [16].

On a semi-Riemannian manifold (M, g), for a $(0, k)-$type tensor field $(0, k)$-type tensor field T and $(0, 2)$-type tensor field A, $(0, k + 2)$-type tensor field Tachibana $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(u_{11}, u_{12}, ..., u_{1k}; u_1, u_2) &= -T((u_1 \wedge_A u_2)u_{11}, u_{12}, ..., u_{1k}) \\ &- \ T(u_{11}, (u_1 \wedge_A u_2)u_{13}, ..., u_{1k}) \\ &. \\ &. \\ &. \\ &- \ T(u_{11}, u_{12}, ..., (u_1 \wedge_A u_2)u_{1k}), \end{aligned} \tag{2.21}$$

for all $u_{11}, u_{12}, ..., u_{1k}, u_1, u_2 \in \chi(M)$, where

$$(u_1 \wedge_A u_2)u_3 = A(u_2, u_3)u_1 - A(u_1, u_3)u_2. \tag{2.22}$$

3. Invariant Submanifolds of an almost $\alpha-$Cosymplectic (k,μ,ν)-Space

Now, let M be an immersed submanifold of $\widetilde{M}^{2n+1}$ and M be an almost $\alpha-$cosymplectic $(k,\mu,\nu)-$space. If $\phi(T_{u_1}M)\subseteq T_{u_1}M$, for each point at $u_1\in M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ with respect to ϕ. Following, it will be clear that a submanifold that is invariant with regard to ϕ is likewise invariant with respect to h.

Proposition 3.1. *If ξ is tangent to M, then M is an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Hence, M is equivalent to the following equality;*

$$\begin{aligned} R(u_1,u_2)\xi &= k[\eta(u_2)u_1-\eta(u_1)u_2]+\mu[\eta(u_2)hu_1-\eta(u_1)hu_2] \\ &\quad +\nu[\eta(u_2)\phi hu_1-\eta(u_1)\phi hu_2] \end{aligned} \tag{3.1}$$

$$(\nabla_{u_1}\phi)u_2=g(\alpha\phi u_1+hu_1,u_2)\xi-\eta(u_2)(\alpha\phi u_1+hu_1) \tag{3.2}$$

$$\nabla_{u_1}\xi=-\alpha\phi^2u_1-\phi hu_1 \tag{3.3}$$

$$\phi\sigma(u_1,u_2)=\sigma(\phi u_1,u_2)=\sigma(u_1,\phi u_2),\quad \sigma(u_1,\xi)=0, \tag{3.4}$$

where ∇,σ and R stand for M's shape operator, Riemannian curvature tensor, and the induced Levi-Civita connection on M, respectively.

Proof. As the proof is a consequence of straightforward math, we omit it. □

We shall assume for the remainder of this work that M is an invariant submanifold of an $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. From (2.5), we have in this instance

$$\phi hu_1=-h\phi u_1, \tag{3.5}$$

for all $u_1\in\Gamma(TM)$, in other words M is also invariant in relation to the tensor field h.

Theorem 3.2. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g,W_1\cdot\sigma)=0$ if and only if M is either totally geodesic or $[4k^2+(\mu^2+\nu^2)(k+\alpha^2)]=0$.*

Proof. We suppose that $Q(g,W_1\cdot\sigma)=0$. This means that

$$(W_1(u_1,u_2)\cdot\sigma)((u_3\wedge_g u_6)u_4,u_5)+(W_1(u_1,u_2)\cdot\sigma)(u_4,(u_3\wedge_g u_6)u_5)=0,$$

for all $u_1,u_2,u_4,u_5,u_3,u_6\in\Gamma(TM)$, which implies that

$$\begin{aligned} &(W_1(u_1,u_2)\cdot\sigma)+(g(u_4,u_6)u_3-g(u_3,u_4)u_6,u_5)+(W_1(u_1,u_2)\cdot\sigma) \\ &+(u_4,g(u_5,u_6)u_3-g(u_3,u_5)u_6)=0. \end{aligned} \tag{3.6}$$

In (3.6), putting $u_2 = u_4 = u_3 = u_5 = \xi$ and using (2.18), (2.19),(3.1), we observe

$$\begin{aligned}
& (W_1(u_1,\xi)\cdot\sigma)(\eta(u_6)\xi - u_6,\xi) = (W_1(u_1,\xi)\cdot\sigma)(\eta(u_6)\xi,\xi) \\
& -(W_1(u_1,\xi)\cdot\sigma)(u_6,\xi) \\
= \; & R^{\perp}(u_1,\xi)\sigma(\eta(u_6)\xi,\xi) - \sigma(\eta(u_6)W_1(u_1,\xi)\xi,\xi) \\
& -\sigma(\eta(u_6)\xi, W_1(u_1,\xi)\xi) - R^{\perp}(u_1,\xi)\sigma(u_6,\xi) \\
& +\sigma(W_1(u_1,\xi)u_6,\xi) + \sigma(u_6, W_1(u_1,\xi)\xi) = 0. \qquad (3.7)
\end{aligned}$$

In view of (2.6) and (2.16), non-zero components of (3.7) vectors give us

$$\sigma(W_1(u_1,\xi)\xi, u_6) = \sigma(u_6, 2k(u_1 - \eta(u_1)\xi) + \mu h u_1 + \nu\phi h u_1) = 0. \qquad (3.8)$$

Also taking ϕu_1 instead of u_1 in (3.8) and by virtue of lemma 2.1 and proposition 1, we have

$$2k\sigma(u_6, hu_1) - \mu(k+\alpha^2)\sigma(u_1,u_6) - \nu(k+\alpha^2)\sigma(\phi u_1, u_6) = 0. \qquad (3.9)$$

(3.8) and (3.9) implies that

$$\left[4k^2 + (\mu^2+\nu^2)(k+\alpha^2)\right] = 0 \text{ or } \sigma = 0.$$

The proof is finished as a result. □

Theorem 3.3. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S, W_1\cdot\sigma) = 0$ if and only if M is either totally geodesic or $2nk\left[4k^2 + (k+\alpha^2)(\mu^2+k\nu^2)\right] = 0$.*

Proof. We believe that $Q(S, W_1\cdot\sigma) = 0$, which follows that

$$Q(S, W_1(u_1,u_2)\cdot\sigma)(u_4,u_5;u_3,u_6) = 0,$$

for all $u_1,u_2,u_4,u_5,u_3,u_6 \in \Gamma(TM)$, by virtue of (2.19) and (2.21), we obtain

$$\begin{aligned}
& S(u_3,u_4)(W_1(u_1,u_2)\cdot\sigma)(u_6,u_5) - S(u_6,u_4)(W_1(u_1,u_2)\cdot\sigma)(u_3,u_5) \\
& +S(u_3,u_5)(W_1(u_1,u_2)\cdot\sigma)(u_4,u_6) \\
& -S(u_6,u_5)(W_1(u_1,u_2)\cdot\sigma)(u_4,u_3) = 0. \qquad (3.10)
\end{aligned}$$

Expanding (3.10) and putting $u_2 = u_4 = u_3 = u_5 = \xi$, non-zero components is

$$2nk\sigma(u_6, W_1(u_1,\xi)\xi). \qquad (3.11)$$

As a result, by combining the previous equation and applying (2.20), we determine that

$$4nk^2\sigma(u_1,u_6) + 2nk\mu\sigma(u_6,\mu h u_1) + 2nk\nu\sigma(u_6,\phi h u_1) = 0. \qquad (3.12)$$

On the other hand, substituting ϕu_1 for u_1 and taking into account (2.7) and (3.4), we conclude that $2nk\left[4k^2 + (k+\alpha^2)(\mu^2+k\nu^2)\right]\sigma(u_1,u_6) = 0$,which follows that, $2nk\left[4k^2 + (k+\alpha^2)(\mu^2+k\nu^2)\right] = 0$ or $\sigma = 0$. □

Theorem 3.4. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g, W_8\cdot\sigma) = 0$ if and only if M is either totally geodesic or $\mu^2+\nu^2 = 0$.*

Proof. We suppose that $Q(g, W_8 \cdot \sigma) = 0$. This means that

$$(W_8(u_1, u_2) \cdot \sigma)((u_3 \wedge_g u_6)u_4, u_5) + (W_8(u_1, u_2) \cdot \sigma)(u_4, (u_3 \wedge_g u_6)u_5) = 0,$$

for all $u_1, u_2, u_4, u_5, u_3, u_6 \in \Gamma(TM)$, which implies that

$$\begin{aligned}&(W_8(u_1, u_2) \cdot \sigma) + (g(u_4, u_6)u_3 - g(u_3, u_4)u_6, u_5) + (W_8(u_1, u_2) \cdot \sigma)\\ &+(u_4, g(u_5, u_6)u_3 - g(u_3, u_5)u_6) = 0. \end{aligned} \tag{3.13}$$

In (3.13), putting $u_2 = u_4 = u_3 = u_5 = \xi$ and using (2.6), (2.20), we observe

$$\begin{aligned} & (W_8(u_1, \xi) \cdot \sigma)(\eta(u_6)\xi - u_6, \xi) = (W_8(u_1, \xi) \cdot \sigma)(\eta(u_6)\xi, \xi)\\ & -(W_8(u_1, \xi) \cdot \sigma)(u_6, \xi)\\ = \ & R^{\perp}(u_1, \xi)\sigma(\eta(u_6)\xi, \xi) - \sigma(\eta(u_6)W_8(u_1, \xi)\xi, \xi)\\ & -\sigma(\eta(u_6)\xi, W_8(u_1, \xi)\xi) - R^{\perp}(u_1, \xi)\sigma(u_6, \xi)\\ & +\sigma(W_8(u_1, \xi)u_6, \xi) + \sigma(u_6, W_8(u_1, \xi)\xi) = 0. \end{aligned} \tag{3.14}$$

In view of (2.17) and (2.20), non-zero components of (3.14) vectors give us

$$\sigma(W_8(u_1, \xi)\xi, u_6) = \sigma(u_6, \mu h u_1 + \nu \phi h u_1) = 0. \tag{3.15}$$

Substituting ϕu_1 for u_1 in (3.15) and considering the equations (2.1) and (2.7), then we get

$$\mu\sigma(u_6, h\phi u_1) + \nu\sigma(u_6, hu_1) = 0. \tag{3.16}$$

From (3.15) and (3.16), we conclude that

$$(\mu^2 + \nu^2)\sigma(u_6, hu_1) = 0$$

So, the proof is finished. □

Theorem 3.5. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then $Q(S, W_8 \cdot \sigma) = 0$ if and only if M is either totally geodesic or $\mu^2 + \nu^2 = 0$.*

Proof. Let us assume that $Q(S, W_8 \cdot \sigma) = 0$. It follows that

$$Q(S, W_0^*(u_1, u_2) \cdot \sigma)(u_4, u_5; u_3, u_6) = 0,$$

for all $u_1, u_2, u_4, u_5, u_3, u_6 \in \Gamma(TM)$, by virtue of (2.17) and (2.20), we deduce that

$$\begin{aligned}&S(u_3, u_4)(W_8(u_1, u_2) \cdot \sigma)(u_6, u_5) - S(u_6, u_4)(W_8(u_1, u_2) \cdot \sigma)(u_3, u_5)\\ &+S(u_3, u_5)(W_8(u_1, u_2) \cdot \sigma)(u_4, u_6)\\ &-S(u_6, u_5)(W_8(u_1, u_2) \cdot \sigma)(u_4, u_3) = 0. \end{aligned} \tag{3.17}$$

By setting $u_2 = u_4 = u_3 = u_5 = \xi$ in the last equation and non-zero components is

$$2nk\sigma(u_6, W_8(u_1, \xi)\xi). \tag{3.18}$$

On the other hand (3.18) can be written as follows:

$$2nk\mu\sigma(u_6, \mu h u_1) + 2nk\mu\nu\sigma(u_6, \phi h u_1) = 0. \tag{3.19}$$

In the same way, by using (3.15) and (3.16), we get $(\mu^2 + \nu^2)\sigma(hu_1, u_6) = 0$, this means that, $\mu^2 + \nu^2 = 0$ or $\sigma = 0$. This proves our assertion. □

Example 3.6. Let $M = \{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5,\ u_5 \neq \pm 1, 0\}$ and we take

$$\begin{aligned} e_1 &= (u_5+1)\frac{\partial}{\partial u_1}, \quad e_2 = \frac{1}{u_5-1}\frac{\partial}{\partial u_2}, \quad e_3 = \frac{1}{2}(u_5+1)^2\frac{\partial}{\partial u_3}, \\ e_4 &= \frac{5}{u_5-1}\frac{\partial}{\partial u_4}, \quad e_5 = (u_5^2-1)\frac{\partial}{\partial u_5} \end{aligned}$$

are linearly independent vector fields on M. We also definite $(1,1)-$type tensor field ϕ by $\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = e_4$, $\phi e_4 = -e_3$ and $\phi e_5 = 0$.

Furthermore, the Riemannian metric tensor g is given by

$$g(e_i, e_j) = \{1,\ i = j; \quad 0,\ \ i \neq j\}.$$

By direct computations, we can easily to see that

$$\phi^2 u_1 = -u_1 + \eta(u_1)\xi, \ \ \eta(u_1) = g(u_1, \xi)$$

and

$$g(\phi u_1, \phi u_2) = g(u_1, u_2) - \eta(u_1)\eta(u_2).$$

Thus $M^5(\phi, \xi, \eta, g)$ is a 5-dimensional almost contact metric manifold. From the Lie-operatory, we have the non-zero components

$$\begin{aligned} [e_1, e_5] &= -(u_5-1)e_1, \quad [e_2, e_5] = (u_5+1)e_2, \quad [e_3, e_5] = -(u_5-1)e_3, \\ [e_4, e_5] &= (u_5+1)e_4. \end{aligned}$$

Furthermore, By ∇, we denote the Levi-Civita connection on M, by using Koszul's formula, we can reach at the non-zero components

$$\begin{aligned} \nabla_{e_1} e_5 &= -(u_5-1)e_1, \quad \nabla_{e_2} e_5 = (u_5+1)e_2, \quad \nabla_{e_3} e_5 = -(u_5-1)e_3, \\ \nabla_{e_4} e_5 &= (u_5+1)e_4. \end{aligned}$$

Comparing the above relations with

$$\nabla_{u_1} e_5 = u_1 - \eta(u_1)e_5 - \phi h u_1,$$

we can observe

$$he_1 = -u_5 e_2, \quad he_2 = -u_5 e_1, \quad he_3 = -u_5 e_4, \quad he_4 = -u_5 e_3 \text{ and } \quad he_5 = 0.$$

By direct calculations, we get

$$\begin{aligned} R(e_1, e_5)e_5 &= ke_1 + \mu he_1 + \nu\phi he_1 = 2(u_5-1)e_1, \\ R(e_2, e_5)e_5 &= ke_2 + \mu he_2 + \nu\phi he_2 = -2u_5(u_5+1)e_2, \\ R(e_3, e_5)e_5 &= ke_3 + \mu he_3 + \nu\phi he_3 = 2(u_5+1)e_3, \end{aligned}$$

and

$$R(e_4, e_5)e_5 = ke_4 + \mu he_4 + \nu\phi he_4 = -2u_5(u_5+1)e_4,$$

which imply that $k = -(u_5^2+1)$, $\mu = 0$ and $\nu = 2 - \frac{1}{u_5} + u_5$.

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