Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 20 (2024) 107-119 https://doi.org/10.56947/amcs.v20.236

RESULTS OF COUPLED ITERATIVE FIXED POINT APPROXIMATION

K. RAUF[1] AND A. O. OYEKANMI[1*]

Abstract. Let X be a Banach space and F a subset of X. In this paper, we showed weak and strong convergence theorems for a Krasnoselskij-type iterative method to approximate the coupled solution of the nonexpansive operator $E : F \times F \to F$, where F is nonempty, depending on two variables. Furthermore, we obtained a fixed point for an almost-contraction operator (b, θ, L)-almost contraction operator $E : X + X \to X$ using the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$.

1. Introduction

The notion of fixed point theory has gained a lot of momentum recently in the area of mathematical analysis and its applications. To ensure the existence and uniqueness of a solution to periodic boundary value problems, the a need to show that the Krasnoselskij-type iterative method possesses weak and strong convergency. Bhaskar and Lakshmikantham [4], proved the existence and uniqueness of a coupled fixed point in the setting of partially ordered metric space. It is known that there is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding constructive type operator. Hence there is practical and theoretical interest in approximating fixed points of several contractive type operators.

Coupled fixed point results have been obtained by many researchers little work has been done on weakly nonexpansive operators. In this paper, we proved weak and strong convergence theorem for a Krasnoselskij-type iterative method to approximate the coupled solution of the nonexpansive operator.

study of the existence and approximation of the solutions of various nonlinear problems such as optimization problems, variation inequality problems, inclusion problems, equilibrium problems, and nonlinear functional equations. Aghajani and Arab [1], obtained some results on fixed points of (ψ, φ, θ)-contractive mappings in partially ordered metric spaces and application. Amini-Harandi [2], introduced some results on coupled and tripled fixed point theory in partially ordered metric space with application to the initial value problem. Aydi *et al.*

Date: Received: Nov 7, 2023; Accepted: Dec 26, 2023.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 06F15, 06F05; Secondary 20M05.

Key words and phrases. Nonexpansive, Coupled Fixed Point, Partially Ordered Normed Space, Weakly nonexpansive.

[3], established some results on coupled fixed point in cone metric spaces for $\overline{\omega}$-compatible mappings. Berinde [5], introduced iterative approximation of fixed points, and also in [6], Berinde generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric space. Berinde and Pacurar [7] introduced Krasnoselskij-type algorithm for variational inequality problems and fixed point problems in Banach space. Berzig and Samet [8], extended coupled fixed point concept in higher dimensions and applications. Kamalov and Leung [9] reviewed fixed point theory. Gu and Yin [10], deduced a new common coupled fixed point theorem in generalized metric space and applications to integral equations. Sahin *et al.* [11], obtained some fixed-point results for the KF-iteration process in hyperbolic metric spaces. Sharma *et al.*[12], constructed a new three-step fixed point iteration scheme with strong convergence and application.

2. Preliminaries

Definition 2.1 (Coupled Fixed Point)
Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $f : X \times X \to X$ if it is a solution of the system.

$$F(x, y) = x \text{ and } f(x, y) = y.$$

Definition 2.2 (Partially Ordered Normed Space)
The Triple $(x, ||\cdot||, \leq)$ is called Partially Ordered Normed Spaces, if $(x, \leq)$ is a Partially Ordered Set and $(x, ||\cdot||)$ is a Normed Space.
Definition 2.3 (Weakly non-expansive)
Let Y be a normed linear space and C be a subset of Y. A mapping $F : C\times C \to Y$ is called Weakly non-expansive if

$$||F(x, y) - F(u, v)|| \leq a||x - u|| + b||y - v||, \tag{2.1}$$

for all $x, y, u, v \in C$, where $a, b \geq 0$ and $a + b \leq 1$.
Definition 2.4 (Non-expansive)
Let X be a normed linear space and C be a subset of X. A mapping $F : C\times C \to X$ is called non-expansive if

$$||F(x, y) - F(u, v)|| \leq \frac{1}{2}\left(||x - u|| + ||y - v||\right),$$

for all $x, y, u, v \in C$. Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called a Coupled fixed point of the mapping

$$E(x, y) = x \text{ and } E(y, x) = y.$$

Consider a coupled fixed point problem in a partially ordered normed space for mixed monotone mapping $E : X\times X \to X$ in conjunction with a contraction-type condition of the form

$$||E(x, y) - E(u, v)|| \leq \frac{h}{2}max\left[\left(||x - u|| + ||y - v||\right)||x - u||, ||y - v||||x - y||\right], \tag{2.2}$$

where $h \in [0, 1]$.

Theorem 2.1. (Bhasker and Lakshmikantham, [4])
Let $(X, \leq)$ be a partially ordered set and suppose there is a normed $||\cdot||$ on X such that $(X, ||\cdot||)$ is a complete normed space. Let $: X \times X \to$ be a continuous mapping having mixed monotone property on X.
If E satisfies (2.2) and there exist $x_0, y_0 \in X$ such that

$$x_0 \leq E(x_0, y_0) \text{ and } y_0 \leq E(x_0, y_0),$$

then there exist

$$\overline{x} = E(\overline{x}, \overline{y}) \text{ and } \overline{y} = E(\overline{y}, \overline{x}).$$

Suppose additionally, that $x_0, y_0 \in X$ are comparable. Then for the coupled fixed point $(\overline{x}, \overline{y})$, we have $\overline{x} = \overline{y}$, that is E has a fixed point;

$$F(\overline{x}, \overline{x}) = \overline{x}.$$

Theorem 2.2. (Berinde [8])
Let F be a bounded, close and convex subset of a Banach space X and let $E : F \times F \to F$ be a (weakly) non expansive operator. Then E has at least one coupled fixed point in F.

Proof. Let $T : F - F$ be begin by $T(x) = E(x, x), x \in F$. By the (weakly) non-expansiveness property of E, we obtain the non expansiveness of T and hence, by the Browder-Gohde-Kirk fixed point theorem, it follows that $FiX(T) \neq \phi$. □

Remark 2.1
Theorem 2.2 shows that E has at least one (coupled) fixed point of the form $(\overline{x}, \overline{x}) \in F \times F$, but, in general, for a bi-variate mapping E, it is also possible to have coupled fixed points $(\overline{x}, \overline{y})$ with unequal components, i.e., such that $\overline{x} \neq \overline{y}$.

Lemma 2.3. (Berinde and pacurar [7])
If $\{x_n\}_n \geq 0$ is a sequence of non negative real numbers satisfying

$$x_{n+1} \leq \beta_1 x_n + \beta_2 x_{n-1}, n \geq 1, \tag{2.3}$$

where $\beta_1, \beta_2 \in (0, 1)$ are such that $\beta_1 + \beta_2 \leq 1$, then:

(a) *$\{x_n\}_{n\geq 0}$ is convergent;*
(b) *There exist $L > 0$ and $\theta \in [0, 1]$ such that*

$$x_n \leq L \cdot \theta^n, \text{ for all } n \geq 1. \tag{2.4}$$

(c) *If $\beta_1 + \beta_2 < 1$, then $x_n \to 0$ as $\to \infty$.*

Proof. We consider the auxiliary sequence $\{y_n\}_{n\geq 0}$, defined by

$$y_{n+1} = x_{n+1} + (1 - \beta_1)x_n, \ n \geq 0,$$

and prove fast that if $\{x_n\}_{n\geq 0}$ satisfies (2.3), then $\{y_n\}_{n\geq 0}$ is convergent. Indeed, by the definition of y_n and (2.3) we have

$$\begin{aligned} y_{n+1} &= x_{n+1} + (1 - \beta_1)x_n \leq \beta_1 x_n + \beta_2 x_{n-1} + (1 - \beta_1)x_n, \\ &= x_n + \beta_2 x_{n-1} \leq x_n + (1 - \beta_1)x_{n-1} = y_n, \end{aligned}$$

which shows that $\{y_n\}_{n\geq 0}$ is decreasing sequence of non negative real numbers, hence it is convergent. Denote

$$\lim_{n\to\infty} y_n = l$$

and let $0<\epsilon<1$ be arbitrary. Then there exist a rank $N=N(\epsilon)$ such that for all $n\geq N$, we have

$$l-\epsilon-(1-\beta_1)x_n \leq x_{n+1} \leq l+\epsilon-(1-\beta_1)x_n \tag{2.5}$$

Let now $m\in N$ and take $n=m+N$ in (2.5) to get

$$\begin{aligned} x_{N+m+1} &\leq l+\epsilon-(1-\beta_1)x_{N+m} \\ &\leq l+\epsilon-(1-\beta_1)\left[l-\epsilon-(1-\beta_1)x_{N+m-1}\right] \\ &= l\left[1-(1-\beta_1)\right]+\epsilon\left[1-(1-\beta_1)\right]+(1-\beta_1)^2 x_{N+m-1}. \end{aligned}$$

Inductively, we obtain

$$x_{N+m+1} \leq l\left[1-(1-\beta_1)+(1-\beta_1)^2+\cdots+(-1)^m(1-\beta_1)^m\right]+\epsilon\left[1+(1-\beta_1)+\cdots+(1-\beta_1)^m\right]+(1- \tag{2.6}$$

Similarly, one obtains

$$\begin{aligned} x_{N+m+1} &\geq l-\epsilon-(1-\beta_1)x_{N+m} \\ &\geq l-\epsilon-(1-\beta_1)\left[l+\epsilon-(1-\beta_1)x_{N+m-1}\right] \\ &= l\left[1-(1-\alpha_1)\right]-\epsilon\left[1+(1-\beta_1)\right]+(1-\beta_1)^2 x_{N+m-1}, \end{aligned}$$

and hence, inductively, we obtain

$$x_{N+m+1} \geq l\left[1-(1-\beta_1)+(1-\beta_1)^2+\cdots+(-1)^m(1-\beta_1)^m\right]-\epsilon\left[1+(1-\beta_1)+\cdots+(1-\beta_1)^m\right]+(1- \tag{2.7}$$

Therefore by (2.6) and (2.7) we get the following double inequality

$$\begin{aligned} l\cdot a_m-\epsilon\cdot b_m+(1-\beta_1)^{m+1}x_N &\leq x_{N+m+1} \\ &\leq l\cdot a_m+\epsilon\cdot b_m+(1-\beta_1)^{m+1}x_N \end{aligned} \tag{2.8}$$

where

$$a_m=\frac{1-(-1)^{m+1}(1-\beta_1)^{m+1}}{2-\beta_1}\to\frac{1}{2-\beta_1} \text{ as } n\to\infty,$$

and

$$b_m=\frac{1-(1-\beta_1)^{m+1}}{\beta_1}\to\frac{1}{\beta_1} \text{ as } n\to\infty,$$

since $1-\beta_1<1$.
Now we can choose $m_0=m_0(\epsilon)$ such that for $m\geq m_0$ one has

$$\left|a_m-\frac{1}{2-\beta_1}\right|<\frac{\epsilon}{3l},\ \left|b_m-\frac{1}{\beta_1}\right|<\frac{1}{3}$$

and

$$\left|(1-\beta_1)^{m+1}x_N\right|<\frac{\epsilon}{3}.$$

Let us denote $n_0 = max\{N, m_0\}$. Then, in view of the previous three inequalities, for all $m \geq n_0 - N$, by (2.8) we have

$$\left| x_{m+N+1} - \frac{1}{2-\beta_1} \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{\beta_1} + \frac{\epsilon}{3} = \epsilon + \frac{\epsilon}{\beta_1} := \epsilon_1,$$

which shows that

$$\lim_{n\to\infty} x_n = \frac{1}{2-\beta_1},$$

and this proves part (a) of the Lemma.
(b) Let $f(t) = t^2 - \beta_1 t - \beta_2, t \in IR$. Since $f(0) = -\beta_2 < 0, f(1) = 1 - \beta_1 - \beta_2 \geq 0$ and f is continuous, it follows that there exists $\theta \in [0,1]$ such that $f(\theta) = 0$, that is,

$$\theta^2 = \beta\theta + \beta_2 \tag{2.9}$$

Assume $x_0 > 0$ or $x_1 > 0$ (otherwise, $\{x_n\}$ would be the trivial null sequence) and denote

$$L = max\left\{x_0, \frac{x_1}{\theta}\right\} > 0.$$

Then (2.4) is clearly satisfied for $n = 0$ and $n = 1$.
We proved by induction. Suppose (2.4) is true for $n = k$ and $n = k+1$ and the prove that it is also true for $n = k+2$.
Indeed, by (2.9),(2.3), and the induction hypothesis, we get

$$x_{k+2} \leq \beta_1 x_{k+1} + \beta_2 x_k \leq \beta_1 L\theta^{k+1} + \beta_2 L\theta^k = L\theta^k(\beta_1\theta + \beta_2) = L\theta^{k+2},$$

as required. (c) If $\beta_1 + \beta_2 < 1$, then by the arguments above, we have $\theta \in (0,1)$ and hence in view of (2.9) the conclusions follows. □

3. Main Results

This section present results of strong convergence theorem for a double Krasnoselskij-type algorithm associated with bi-variate weakly non-expansive operators on Banach spaces.

Theorem 3.1. *Let $(X, ||\cdot||)$ be a Banach space and let $E : X \times X \to X$ be a (b, θ, θ, L)-almost contraction. Then*

(a) *$Fix(E) \neq \phi$;*
(b) *for any $x_0, y_0 \in X$, there exists $\lambda \in (0,1)$ such that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = \lambda x_n + (1-\lambda)E(x_n, x_n),\ n \geq 0, \tag{3.1}$$

Converges to some $\overline{x} \in Fix(E)$, for any $x_0 \in X$;
(c) $||x_{n+i-1} - \overline{x}|| \leq \dfrac{\delta^i}{1-\delta}||x_n - x_{n-1}||, n = 0,1,2,\cdots;\ i = 1,2,\cdots$ *holds with* $\delta = \dfrac{\theta}{b+1}$

Proof. For any $\lambda \in (0,1)$, consider the averaged mapping E_λ, given by

$$E_\lambda = \lambda_x + (1-\lambda)E(x,x),\ \forall x \in X \tag{3.2}$$

It is well known that E_λ possesses the following important property:

$$Fix(E_\lambda) = Fix(E).$$

Now, since E is a (b,δ,L)-almost contraction, then there exist $b \in [0,\infty]$, $\theta \in (0, b+1)$ and $L \geq 0$ such that

$$||b(x-y) + E_x - E_y|| \leq \theta||x-y|| + L||b(x-y) + E_x - y|| \text{ for all } x,y \in X. \tag{3.3}$$

If $b = 0$, then we have that if $(X, ||\cdot||)$ is a complete normed space and $E : X \times X \to X$ be a (δ, L)-almost contraction for which there exist $\delta_1 \in (0,1)$ and $L_1 \geq 0$ such that

$$||E_x - E_y|| \leq \delta_1||x-y|| + L_1||x - E_x||, \text{ for all } x,y \in X. \tag{3.4}$$

then we have that E has a unique fixed points that is $Fix(E) = \{\overline{x}\}$, the picard iteration $\{x_n\}_{n=0}^{\infty}$ associated to E converges to $\overline{x}$ for any $x \in X$. Then

$$||x_{n+i-1} - \overline{x}|| \leq \frac{\delta^i}{1-\delta}||x_n - x_{n-1}|| \text{ holds for } n = 0,1,2,\cdots; \text{ and } i = 1,2,\cdots$$

Thus, the rate of convergence of the Picard iteration is given by

$$||x_n - \overline{x}|| \leq \delta||x_{n-1} - \overline{x}||,\ n = 0,1,2,\cdots.$$

If $b > 0$ in (3.3), then let put $\lambda = \dfrac{1}{b+1}$. Obviously we have $0 < \lambda < 1$ and thus the contractive condition (3.3) becomes

$$\left|\left|\left(\frac{1}{\lambda} - 1\right)(x-y) + E_x - E_y\right|\right| \leq \theta||x-y|| + L\left|\left|\left(\frac{1}{\lambda} - 1\right)(x-y) + E_x - E_y\right|\right| \ \forall\, x,y \in X \tag{3.5}$$

where $\delta = \frac{\theta}{b+1} \in (0,1)$.
The above inequality shows that E_λ is an almost contraction in the Banach space X. We shall write in the following the norm in X as induced distance, that is

$$||x-y|| = d(x,y),$$

in order to emphasize the fact that the rest of the proof works in the general case of a complete metric space. We shall prove that E_λ satisfying (3.5) has at least a fixed point in X. To this end, let $x_0 \in X$ be an arbitrary but fixed and let $\{x_n\}_{n=0}^{\infty}$ be picard iteration defined by

$$x_{n+1} = E_\lambda x_n\ n \geq 0 \tag{3.6}$$

Take $x := x_{n-1},\ y := x_n$ in (3.5) to obtain

$$d(E_\lambda x_{n-1}, E_\lambda x_n) \leq \delta \cdot d(x_{n-1}, x_n),$$

which shows that

$$d(x_n, x_{n+1}) \leq \delta \cdot d(x_{n-1}, x_n) \tag{3.7}$$

Using (3.7) we obtain by induction

$$d(x_n, x_{n+1}) \leq \delta^n \cdot d(x_0, x_1),\ n = 0,1,2,\cdots$$

and then

$$\begin{aligned} d(x_n, x_{n+p}) &\le \delta^n(1+\delta+\cdots+\delta^{p-1})d(x_0,x_1) \\ &= \frac{\delta^n}{1-\delta}(1-\delta^p)\cdot d(x_0,x_1),\ n,p\in N,\ p\ne 0 \end{aligned} \tag{3.8}$$

Since $0<\delta<1$, then (3.8) shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and hence it is convergent. Let us denote

$$\overline{x} = \lim_{n\to\infty} x_n \tag{3.9}$$

Then

$$\begin{aligned} d(\overline{x}, E_\lambda\overline{x}) &\le d(\overline{x}, x_{n+1}) + d(x_{n+1}, E_\lambda\overline{x}) \\ &= d(x_{n+1}, \overline{x}) + d(E_\lambda x_n, E_\lambda\overline{x}) \end{aligned}$$

By $d(E_x - E_y) \le \delta\cdot d(x,y) + Ld(y, E_x)$ for all $x,y\in X$, then we have

$$d(E_\lambda x_n, E_\lambda\overline{x}) \le \delta d(x_n,\overline{x}) + Ld(\overline{x}, E_\lambda x_n)$$

and hence, by combining the previous two inequalities, we obtains

$$d(\overline{x}, E_\lambda\overline{x}) \le (1+L)d(\overline{x}, x_{n+1}) + \delta\cdot d(x_n,\overline{x}) \tag{3.10}$$

which is valid for all $n\ge 0$. Letting $n\to\infty$ in (3.10) we obtain

$$d(\overline{x}, E_\lambda\overline{x}) = 0$$

which shows that $\overline{x}$ is a fixed point of E_λ. As

$$Fix(E_\lambda) = Fix(E),$$

the first part of the theorem follows. Now by letting $P\to\infty$ in (3.8), we obtains the prior estimate

$$d(x_n,\overline{x}) \le \frac{\delta^n}{1-\delta}d(x_0,x_1),\ n=0,1,2,\cdots \tag{3.11}$$

where $\delta = \frac{\theta}{b+1}\in(0,1)$.
On the other hand, let us observe that by (3.7) we inductively obtain

$$d(x_{n+k}, x_{n+k+1}) \le \delta^{k+1}\cdot d(x_{n-1},x_n),\ k,n\in N$$

and hence, similarly to deriving (3.8), we get

$$d(x_n,x_{n+p}) \le \frac{\delta(1-\delta^p)}{1-\delta}d(x_{n-1},x_n),\ n\ge 1, p\in N^* \tag{3.12}$$

Now, by letting $p\to\infty$ in (3.12), the following a posterior estimate follows.

$$d(x_n,\overline{x}) \le \frac{\delta}{1-\delta}d(x_{n-1},x_n),\ n=1,2,\cdots \tag{3.13}$$

by merging (3.11) and (3.13), we obtain

$$d(x_{n+1},\overline{x}) \le \frac{\delta^n}{1-\delta}d(x_n,x_n),\ n=0,1,2,\cdots$$

Thus, we have

$$d(x_{n+i-1},\overline{x}) \le \frac{\delta^i}{1-\delta}d(x_n,x_{n-1}),\ n=0,1,2,\cdots\ i=1,2,\cdots$$

as required. □

Theorem 3.2. *Let F be a bounded, closed and convex subset of a Banach space X and let $E : F \times F \to$ be weakly non-expansive and demicompact operator. Then the set of coupled fixed points of E is nonempty and the double iterative algorithm $\{(x_n, x_n)\}_{n=0}^{\infty}$ given by x_0 in F and*

$$x_{n+1} = \lambda x_n + (1-\lambda)E(x_n, x_n), \ n \geq 0 \tag{3.14}$$

where $\lambda \in (0,1)$, converges (strongly) to a coupled fixed point of E.

Proof. By Theorem 2.2, E has at least one coupled fixed point with equal components, $(\overline{x}, \overline{x}) \in F \times F$.
We first show that the sequence $\{x_n - E(x_n, x_n)\}_{n \in N}$ converges strongly to zero. We have

$$\begin{aligned} ||x_{n+1} - \overline{x}||^2 &= ||\lambda x_n + (1-\lambda)E(x_n, x_n) - \overline{x}||^2 \\ &= ||\lambda(x_n - \overline{x}) + (1-\lambda)E(x_n, x_n) - \overline{x}||^2 \\ &= \lambda^2 ||x_n - \overline{x}||^2 + (1-\lambda)^2 \cdot ||E(x_n, x_n) - \overline{x}||^2 \\ &+ 2\lambda(1-\lambda) \left|(E(x_n, x_n) - \overline{x}, x_n - \overline{x})\right| . \end{aligned} \tag{3.15}$$

Similarly

$$||x_n - E(x_n, x_n)||^2 = ||x_n - \overline{x}||^2 + ||E(x_n, x_n) - \overline{x}||^2 - 2\left|(E(x_n, x_n) - \overline{x}, x_n - \overline{x})\right| . \tag{3.16}$$

On the other hand by weak non-expansiveness condition (2.1) and $E(\overline{x}, \overline{x}) = \overline{x}$ we obtain

$$||E(x_n, x_n) - \overline{x}|| = ||E(x_n, x_n) - E(\overline{x}, \overline{x})|| \leq ||x_n - \overline{x}||$$

Now by (3.15), (3.16) and the inequality above, it follows that for any real number q we have

$$\begin{aligned} ||x_{n+1} - \overline{x}||^2 + q^2 ||x_n - E(x_n, x_n)||^2 &\leq [2q^2 + \lambda^2 + (1-\lambda)^2] \cdot ||x_n - \overline{x}||^2 \\ &+ 2[\lambda(1-\lambda) - q^2] \cdot \left|(E(x_n, x_n) - \overline{x}, x_n - \overline{x})\right| . \end{aligned} \tag{3.17}$$

If we choose now a nonzero q such that $q^2 \leq \lambda(1-\lambda)$, then from the last inequality, we obtain

$$\begin{aligned} ||x_{n+1} - \overline{x}||^2 + q^2 ||x_n - E(x_n, x_n)||^2 \leq &(2q^2 + \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) \\ &- 2q^2)||x_n - \overline{x}||^2 = ||x_n - \overline{x}||^2 \end{aligned} \tag{3.18}$$

We used the Cauchy-Scharz inequality,

$$\begin{aligned} (E(x_n, x_n) - \overline{x}, x_n - \overline{x}) &\leq ||E(x_n, x_n) - \overline{x}|| \cdot ||x_n - \overline{x}|| \\ &\leq ||x_n - \overline{x}||^2 \end{aligned}$$

So by (3.18), we get

$$q^2 ||x - E(x_n, x_n)||^2 \leq ||x_n - \overline{x}||^2 - ||x_{n+1} - \overline{x}||^2, \ n \geq 0 \tag{3.19}$$

By (3.18) we deduce that $\{||x_n - \overline{x}||\}$ is decreasing sequence of non negative real numbers, hence it is convergent. By (3.18), we also have

$$0 \leq ||x_n - E(x_n, x_n)||^2 \leq \frac{1}{q^2}(||x_n - \overline{x}||^2 - ||x_{n+1} - \overline{x}||^2),\ n \geq 0,$$

from which, by letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} ||x_n - E(x_n, x_n)|| = 0 \tag{3.20}$$

This shows that $x_n - E(x_n, x_n) \to 0$ (strongly) and so it follows by demicompactness of E that there exist a subsequence $\{x_{nk}\} \subset F$ and a point $p \in F$ such that

$$\lim_{k\to\infty} x_{nk} = p$$

As E is non-expansive, it is continuous.
This implies

$$\lim_{k\to\infty} E(x_{nk}, x_{nk}) = E(p, p).$$

By (3.20),

$$0 = \lim_{k\to\infty} (x_{nk} - E(x_{nk}, x_{nk})) = p - E(p, p).$$

which shows that (p, p) is coupled fixed point of E.
Using now (3.19), with $\overline{x} = p$, we have deduce that the sequence of non negative real numbers $\{||x_{nk} - p||\}_{k\geq 0}$ converges to 0, it follows that the sequence $\{||x_n - p||\}_{n\geq 0}$ itself converges to 0, that is, the sequence converges strongly to (p, p), as $n \to 0$ and this achieved the proof. □

Theorem 3.3. *Suppose X is a Banach space, F is a closed, bounded and convex subset of X, and $E : F \times F \to F$ a weakly non expansive mapping such that $Fix(E) = \{m\}$. Then the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by x_0 in F and*

$$x_{n+1} = (1 - \lambda)x_n + \lambda E(x_n, x_n),\ n \geq 0 \tag{3.21}$$

Converges weakly to m, for any $\lambda \in (0, 1)$.

Proof. It suffices to show that if $\{x_{nj}\}_{j=0}^{\infty}, x_{nj} = T^{nj}x$, where $Tx = \lambda x + (1 - \lambda)E(x, x)$, converges weakly to a certain m_0, then m_0 is a fixed point of T (and hence $m_0 \in Fix(E)$) and therefore $m_0 = m$. Assume $\{x_{nj}\}_{j=0}^{\infty}$ does not converge weakly to m. As E is weakly non-expansive, we have

$$\begin{aligned}||T_x - T_y|| &\leq \lambda||x - y|| + (1 - \lambda)||E(x, x) - E(y, y)|| \\ &\leq \lambda||x - y|| + \frac{(1 - \lambda)}{2} max[(||x - y|| - ||x - y||)||x - y||, \\ &||x - y||||x - x||] = ||x - y||\end{aligned} \tag{3.22}$$

which shows that T is non-expansive and hence we get

$$\begin{aligned}||x_{nj} - T_{mo}|| &\leq ||Tx_{nj} - T_{mo}|| + ||x_{nj} - Tx_{nj}|| \\ &\leq ||x_{nj} - m_o|| + ||x_{nj} - Tx_{nj}||.\end{aligned} \tag{3.23}$$

Using the arguments in the proof of Theorem 2.2, it follows

$$||x_{nj} - Tx_{nj}|| \to 0,\ \text{as } n \to \infty$$

and so that (3.23) implies

$$\limsup(||x_{nj} - T_{mo}|| - ||x_{nj} - m_o||) \leq 0 \tag{3.24}$$

As in proof of Theorem 2.2, we have

$$\begin{aligned} ||x_{nj} - T_{mo}||^2 &= ||(x_{nj} - m_o) + (m_o - T_{mo})||^2 \\ &= ||x_{nj} - m_o||^2 + ||m_o - T_{mo}||^2 + 2|(x_{nj} - m_o, m_o - T_{mo})|, \end{aligned}$$

which shows, together with $x_{nj} \to m_o$ (as $j \to \infty$), that

$$\lim_{n\to\infty}[||x_{nj} - T_{mo}||^2 - ||x_{nj} - m_o||^2] = ||m_o - T_{mo}||^2. \tag{3.25}$$

On the other hand, we have

$$\begin{aligned} ||x_{nj} - T_{mo}||^2 - ||x_{nj} - m_o||^2 = (||x_{nj} - T_{mo}|| - ||x_{nj} - m_o||) \cdot (||x_{nj} - T_{mo}|| \\ + ||x_{nj} - mo||) \end{aligned} \tag{3.26}$$

Since F is bounded, the sequence

$$\{||x_{nj} - T_{mo}|| + ||x_{nj} - m_o||\}$$

is bounded too, and by (3.24) - (3.26) we get

$$||m_o - T_{mo}|| \leq 0, \text{ that is}$$

$$T_{mo} = m_o \Leftrightarrow m_o = E(m_o, m_o) = m.$$

Hence, m is the fixed point of E and its coupled. □

Theorem 3.4. *Assume that F is a bounded, closed and convex subset of a Banach space and $E : F \times F \to F$ is a weakly non-expansive operator. Then the Krasnoselskij algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in F and defined by*

$$x_{n+1}(1-\lambda)x_n + \lambda E(x_n, x_n), \; n \geq 0 \tag{3.27}$$

Converges weakly to a coupled fixed point of E.

Proof. For each $m \in Fix(E)w$ and each n, we have, as in proof of Theorem 3.2, so that

$$||x_{n+1} - m|| \geq ||x_n - m||,$$

which shows that the function $g(m) = \lim_{n\to\infty} ||x_n - m||$ is well defined and is a lower semi-continuous convex function of the nonempty convex set $Fix(E)$. Let

$$d_0 = inf\{g(m) : Fix(E)\}.$$

For each $\epsilon > 0$, the set

$$G_\epsilon = \{y : g(y) \leq d_0 + \epsilon\}$$

is closed, convex and hence, weakly compact.
Therefore $\bigcap_{\epsilon>0} G_\epsilon \neq \phi$ $\big($infact $\bigcap_{\epsilon>0} G_\epsilon = \{y : g(y) = d_0\} \equiv E_0\big)$.

Moreover, E_0 contains exactly one point. Indeed, since E_0 is convex and closed, for $m_0 m_1 \in E_0$ and $m_\lambda = (1-\lambda)m_0 + \lambda m_1$,

$$\begin{aligned}
&g^2(m\lambda) = \lim_{n\to\infty} ||m_\lambda - x_n||^2 = \lim_{n\to\infty}(||\lambda(m_1 - x_n) + (1-\lambda)(m_0 - x_n)||^2) \\
&= \lim_{n\to\infty}(\lambda^2||m_1 - x_n||^2 + (1-\lambda)^2||m_0 - x_n||^2 + 2\lambda(1-\lambda)|(m_1 - x_n, m_0 - x_n)|) \\
&= \lim_{n\to\infty}(\lambda^2||m_1 - x_n||^2 + (1-\lambda)^2||m_0 - x_n||^2 + 2\lambda(1-\lambda)||m_1 - x_n|| \cdot ||m_0 - x_n||) \\
&\quad + \lim_{n\to\infty}\{2\lambda(1-\lambda)[(m_1 - x_n, m_0 - x_n) - ||m_1 - x_n|| \cdot ||m_0 - x_n||]\}
\end{aligned} \tag{3.28}$$

Since $||m_1 - x_n|| \to d_0$ and $||m_0 - x_n|| \to d_0$, the latter relation implies that]

$$\begin{aligned}
||m_1 - m_0||^2 &= ||(m_1 - x_n) + (x_n - m_0)||^2 \\
&= ||m_1 - x_n||^2 + ||x_n - m_0||^2 - 2|(m_1 - x_n, m_0 - x_n)| \\
&\to d_0^2 + d_0^2 - 2d_0^2 = 0,
\end{aligned}$$

giving a contraction.

Now, in order to show that $x_n = E^n(x_0, x_0) \to m_0$, it suffices to assume that $x_{nj} \to m$ for an infinite subsequence and then prove that $m = m_0$. By the arguments in the proof of Theorem 3.2, $m \in Fix(E)$. Considering the definition of g and the fact that $x_{nj} \to m$, then we have

$$\begin{aligned}
||x_{nj} - m_0||^2 &= ||x_{nj} - m + m - m_0)||^2 \\
&= ||x_{nj} - m||^2 + ||m - m_0||^2 - 2|(x_{nj} - m, m - m_0)| \\
&\to g^2(m) + ||m - m_0||^2 = g^2(m_0) = d_0^2.
\end{aligned} \tag{3.29}$$

Since $g^2(m) \geq d_0^2$, the last inequality implies that

$$||m - m_0|| \leq 0,$$

which implies that $m = m_0$ as required. □

Theorem 3.5. *Let $(X, ||\cdot||)$ be a complete normed space. Suppose $E : X \times X$ satisfies the following contractive condition for all $x, y, u, v \in X$ in (2.1). Then*

(a) *E has a unique coupled fixed point with identical components, that is, there exists $\overline{x} \in X$, such that $E(\overline{x}, \overline{x}) = \overline{x}$;*

(b) *The sequence $\{x_n\}_{n\geq 0}$, defined by*

$$x_{n+1} = E(x_n, x_n),\ n \geq 0, \tag{3.30}$$

converges to $\overline{x}$, for any initial values $x_0 \in X$;

(c) *The sequence $\{y_n\}_{n\geq 0}$, defined by*

$$y_{n+1} = E(y_{n-1}, y_n),\ n \geq 0, \tag{3.31}$$

converges to $\overline{x}$, for any initial values $y_0, y_1 \in X$;

(d) *The following estimates hold*

$$||x_{n+i-1} - \overline{x}|| \leq \frac{\delta^i}{1-\delta}||x_n - x_{n-1}||,\ n = 1, 2, \cdots,\ i = 1, 2, \cdots \tag{3.32}$$

$$||y_n - \overline{x}|| \leq \theta^n \max\left\{y_0, \frac{y_1}{\theta}\right\},\ n = 1, 2, \cdots \tag{3.33}$$

for some $\theta \in (0,1)$ and $\delta = h < 1$.

Proof. We first prove claims a and b. Assume $T : X \to X$ defined by $T(x) = E(x, x)$, for any $x \in X$. Then

$$||Tx - Ty|| = ||E(x,x) - E(y,y)|| \leq ||E(x,x) - E(x,y)|| + ||E(x,y) - E(y,y)||$$

and by using the contractive condition in (2.2), we obtain

$$||Tx - Ty|| \leq \frac{h}{2} \max ||x - y||,\ \forall x, y \in X, \tag{3.34}$$

which by assumption $h < 1$, shows that T is a contraction in the complete normed space $(X, ||\cdot||)$.
So, by the contraction mapping principle, claims 1,2 and the estimates (3.32) easily follow by Theorem 3.1.
Now, to prove C, we first prove the sequence $\{||y_n - y_{n+1}||\}_{n\geq 0}$ converges to zero. Indeed, by the contractive condition (2.2), we get

$$\begin{aligned} ||y_n - y_{n+1}|| =& ||E(y_{n-2}, y_{n-1}) - E(y_{n-1}, y_n)|| \leq \frac{h}{2} \max[(||y_{n-2} \\ & - y_{n-1}|| + ||y_{n-1} - y_n||)||y_{n-2} - y_{n-1}||, ||y_{n-1} \\ & - y_n||||y_{n-2} - y_{n-1}||],\ n \geq 2 \end{aligned} \tag{3.35}$$

and the claims easily follows by Lemma 2.3 with $\beta_1 - \beta_2 = h$ and $x_n = ||y_{n-1} - y_n||$. In order to prove that $\{y_n\}_{n\geq 0}$ converges to $\overline{x}$, the unique fixed point of T(and unique coupled fixed point of E), we denote.

$$\Delta_n = ||y_n - \overline{x}||,\ n \geq 0. \tag{3.36}$$

We have

$$\begin{aligned} ||y_n - \overline{x}|| =& ||E(y_{n-1}, y_n) - E(\overline{x}, \overline{x})|| \\ & \leq ||E(y_{n-1}, y_n) - E(y_n, \overline{x})|| + ||E(y_n, \overline{x}) - E(\overline{x}, \overline{x})|| \end{aligned}$$

and so by the contractive condition (2.2), this yields

$$\Delta_{n+1} \leq \frac{h}{2} \max\{2||y_n - \overline{x}|| + ||y_{n-1} - y_n||,\ n \geq 1, \tag{3.37}$$

that is,

$$\Delta_{n+1} \leq h \cdot \Delta_n + \frac{h}{2} \max ||y_{n-1} - y_n||,\ n \geq 1.$$

Now let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences of positive real numbers and $q \in (0,1)$ such that $a_{n+1} \leq qa_n + b_n$, whenever $n \geq n_0$ and $b_n \to 0$ as $n \to \infty$, so that

$$\lim_{n\to\infty} a_n = 0.$$

With $a_n = \Delta_n$, $b_n = \dfrac{h}{2} \max ||y_{n-1} - y_n||$ and $q = h$, to get the required conclusion, we use Lemma 2.3 to obtained the estimate 3.33, and this achieved the proof. $\square$

4. Conclusion

In this paper, results of coupled fixed point theorem of weakly non-expansive operator has been established by showing that the subset F of a Banach space X is bounded, closed and convex.

Acknowledgment
The authors acknowledge the management of the University of Ilorin for providing us with a suitable research laboratory and library to enable us carried out this research.

References

1. A. Aghajani, R. Arab, *Fixed points of (ψ, φ, θ)-contractive mappings in partially ordered $b-$metric spaces and application to quadratic integral equations,* Fixed Point Theory Applications. **245** (2013). https://doi.org/10.1186/1687-1812-2013-245
2. A. Amini-Harandi, *Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem,* Mathematics Computation Model. **57** (2013), 2343-2348. https://doi.org/10.1016/j.mcm.2011.12.006
3. H. Aydi, B. Samet, and C. Vetro *Coupled fixed point results in cone metric spaces for $\overline{\omega}-$compatible,* Fixed point Theory Application. **27** (2011), 1687-1812. https://doi.org/10.1186/1687-1812-2011-27
4. T. G. Bhaskar, and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications* Nonlinear Analysis: Theory, Methods and Application **65(7)** (2006), 1379-1393. https://doi.org/10.1016/j.na.2005.10.017
5. V. Berinde, *Iterative approximation of fixed points,* Second edition, Lecture Notes in Mathematics Springer, Berlin. (2007). https://doi.org/10.1007/978-3-540-72234-2
6. V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Analysis **74** (2011), 7347-7355. https://doi.org/10.1016/j.na.2011.07.053
7. V. Berinde, and M. Pacurar, *Krasnoselskij-type algorithm for variational inequality problems and fixed point problems in Banach space*, arxiv **10(2)** (2021), 1-17. https://doi.org/10.48550/arXiv.2103.10289
8. M. Berzig, and B. Samet *An extension of coupled fixed point's concept in in higher dimension analysis*, Computer Mathematics Applications **63(8)** (2012), 1319-1334. https://doi.org/10.1016/j.camwa.2012.01.018
9. F. Kamalov and H. H. Leung *Fixed point theory: A review*, arXiv:2309.03226 [math.HO] (2023). https://doi.org/10.48550/arXiv.2309.03226
10. F. Gu, and Y. Yin, *A new common coupled fixed point theorem in generalized metric space and application to quadratic integral equations*, Fixed Point Theory Application**266** (2013). https://doi.org/10.1186/1687-1812-2013-266
11. P. Sharma, H. Ramos, R. Behl, and V. Kanwar, *A new three-step fixed iteration scheme with strong convergence and applications*, Journal of Computational and Applied Mathematics. **430** (2023), 1-15. https://doi.org/10.1016/j.cam.2023.115242
12. A. Sahin, E. Ozturk, and A. Aggarwal, *Some fixed-point results for KF-iteration process in hyperbolic metric space*, Symmetry **15(7)** (2023) 1360. https://doi.org/10.3390/sym15071360

[1] Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
Email address: krauf@unilorin.edu.ng, olizot.aa@gmail.com