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MULTIPLICATIVE JORDAN TRIPLE SEMI-DERIVATIONS ON RINGS AND STANDARD OPERATOR ALGEBRAS

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Abstract. In this paper we study the additivity of multiplicative Jordan triple semi-derivations on rings and standard operator algebras.

1. Introduction

The study of the additivity of maps on rings has been an active area of research. The first result about the additivity of multiplicative isomorphisms on rings was given by Martindale III [7]. Daif [5] introduced the definition of a multiplicative derivation of a ring and studied the additivity of such map. In the context of derivations, Brešar [3] studied the Jordan triple derivations on rings and Lu [10] studied the additivity of multiplicative Jordan triple derivations on rings. Posteriorly, Jing and Lu [6] generalized the results of Lu to a larger class of rings.

Bergen [1] introduced the notion of a semi-derivation of a ring, based on the concept of derivation, and Siddeeque and Khan [11] studied the additivity of multiplicative semi-derivation of a ring (a concept based on the notions of a semi-derivation and multiplicative derivation). Motivated by the facts mentioned above, in this paper we introduce the notions of a Jordan triple semi-derivation of a ring and multiplicative Jordan triple semi-derivation of a ring, based on the notions of a Jordan triple derivation and multiplicative Jordan triple derivation, and present a study on its additivity.

2. Multiplicative Jordan triple semi-derivations on rings

Let $\mathscr{R}$ be an associative ring, $g : \mathscr{R} \to \mathscr{R}$ an arbitrary map. A map $d : \mathscr{R} \to \mathscr{R}$ is called a *multiplicative semi-derivation of $\mathscr{R}$ (associated with g)* if the following conditions are satisfied:

$$d(ab) = d(a)b + g(a)d(b) = d(a)g(b) + ad(b),$$

for all elements $a, b \in \mathscr{R}$, and $d(g(a)) = g(d(a))$, for all element $a \in \mathscr{R}$. An additive multiplicative semi-derivation of $\mathscr{R}$ (associated with g) is called a *semi-derivation of $\mathscr{R}$ (associated with g)*.

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A map $d : \mathscr{R} \to \mathscr{R}$ is called a *multiplicative Jordan triple semi-derivation of* $\mathscr{R}$ *(associated with* g*)* if the following conditions are satisfied:

$$\begin{aligned} d(aba) &= d(a)ba + g(a)d(b)a + g(a)g(b)d(a) \\ &= d(a)g(b)g(a) + ad(b)g(a) + abd(a), \end{aligned}$$

for all elements $a, b \in \mathscr{R}$, and

$$d(g(a)) = g(d(a)),$$

for all element $a \in \mathscr{R}$. An additive multiplicative Jordan triple semi-derivation of $\mathscr{R}$ (associated with g) is called a *Jordan triple semi-derivation of* $\mathscr{R}$ *(associated with* g*)*.

Our main result reads as follows.

Theorem 2.1. *Let* $\mathscr{R}$ *be an associative ring containing a non-trivial idempotent* e_1, $\mathscr{R} = \oplus_{i,j=1,2}\mathscr{R}_{ij}$ *the Peirce decomposition of* $\mathscr{R}$, *relative to* e_1, $g : \mathscr{R} \to \mathscr{R}$ *an arbitrary endomorphism of* $\mathscr{R}$ *and* t *an element of the ring* $\mathscr{R}$. *Assume that the following properties hold:*

- (♧) $g(x_{ij})tx_{ij} = 0$ *and* $x_{ij}tg(x_{ij}) = 0$, *for all elements* $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$, *implies* $t = 0$,
- (♤) $g(x_{ij})tx_{ij} = 0$ *and* $x_{ij}tg(x_{ij}) = 0$, *for all elements* $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$, *implies* $t = 0$, *provided that the conditions are satisfied:*

$$g(x_{ji})tx_{ii} = 0 \text{ and } x_{ii}tg(x_{ij}) = 0,$$

for all elements $x_{ii} \in \mathscr{R}_{ii}$, $x_{ij} \in \mathscr{R}_{ij}$ *and* $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i, j = 1, 2)$.

Then every multiplicative Jordan triple semi-derivation of $\mathscr{R}$ *(associated with* g*) is additive.*

In addition, if $\mathscr{R}$ *is a* 2*-torsion free semiprime ring and* $g : \mathscr{R} \to \mathscr{R}$ *is a ring isomorphism of* $\mathscr{R}$, *then every multiplicative Jordan triple semi-derivation of* $\mathscr{R}$ *(associative with* g*) is a semi-derivation of* $\mathscr{R}$ *(associative with* g*).*

To prove the Theorem 2.1, let $d : \mathscr{R} \to \mathscr{R}$ be a multiplicative Jordan triple semi-derivation of $\mathscr{R}$ (associated with g).

Based on the techniques used by Jing and Lu [6], we shall organize the proof of Theorem 2.1 in a series of Steps. The following Step will be used throughout this paper, whose proof is elementary and therefore omitted.

Step 2.2. $d(0) = 0$.

Step 2.3. For arbitrary elements $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following holds: $d(a_{11} + a_{12} + a_{21} + a_{22}) = d(a_{11}) + d(a_{12}) + d(a_{21}) + d(a_{22})$.

Proof. First, write $t = d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22})$. For all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ we have

$$\begin{aligned} &d(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) \\ &= d(x_{ij})(a_{11} + a_{12} + a_{21} + a_{22})x_{ij} + g(x_{ij})d(a_{11} + a_{12} + a_{21} + a_{22})x_{ij} \\ &\qquad + g(x_{ij})g(a_{11} + a_{12} + a_{21} + a_{22})d(x_{ij}), \end{aligned} \tag{2.1}$$

$$d(x_{ij}a_{11}x_{ij}) = d(x_{ij})a_{11}x_{ij} + g(x_{ij})d(a_{11})x_{ij} + g(x_{ij})g(a_{11})d(x_{ij}), \tag{2.2}$$

$$d(x_{ij}a_{12}x_{ij}) = d(x_{ij})a_{12}x_{ij} + g(x_{ij})d(a_{12})x_{ij} + g(x_{ij})g(a_{12})d(x_{ij}), \tag{2.3}$$

$$d(x_{ij}a_{21}x_{ij}) = d(x_{ij})a_{21}x_{ij} + g(x_{ij})d(a_{21})x_{ij} + g(x_{ij})g(a_{21})d(x_{ij}) \tag{2.4}$$

and

$$d(x_{ij}a_{22}x_{ij}) = d(x_{ij})a_{22}x_{ij} + g(x_{ij})d(a_{22})x_{ij} + g(x_{ij})g(a_{22})d(x_{ij}). \tag{2.5}$$

Adding (2.2), (2.3), (2.4) and (2.5) and subtracting this sum from (2.1), we get

$$\begin{aligned} d(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) - d(x_{ij}a_{11}x_{ij}) - d(x_{ij}a_{12}x_{ij}) \\ - d(x_{ij}a_{21}x_{ij}) - d(x_{ij}a_{22}x_{ij}) \\ = g(x_{ij})(d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22}))x_{ij} \\ + g(x_{ij})(g(a_{11} + a_{12} + a_{21} + a_{22}) - g(a_{11}) - g(a_{12}) - g(a_{21}) - g(a_{22}))d(x_{ij}) \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) - d(x_{ij}a_{11}x_{ij}) - d(x_{ij}a_{12}x_{ij}) \\ - d(x_{ij}a_{21}x_{ij}) - d(x_{ij}a_{22}x_{ij}) \\ = g(x_{ij})(d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) \\ - d(a_{12}) - d(a_{21}) - d(a_{22}))x_{ij}. \end{aligned} \tag{2.6}$$

As a consequence of (2.6), we conclude that

$$g(x_{ij})tx_{ij} = 0. \tag{2.7}$$

Next, for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ we have

$$\begin{aligned} d(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) \\ = d(x_{ij})g(a_{11} + a_{12} + a_{21} + a_{22})g(x_{ij}) + x_{ij}d(a_{11} + a_{12} + a_{21} + a_{22})g(x_{ij}) \\ + x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})d(x_{ij}), \end{aligned} \tag{2.8}$$

$$d(x_{ij}a_{11}x_{ij}) = d(x_{ij})g(a_{11})g(x_{ij}) + x_{ij}d(a_{11})g(x_{ij}) + x_{ij}a_{11}d(x_{ij}), \tag{2.9}$$

$$d(x_{ij}a_{12}x_{ij}) = d(x_{ij})g(a_{12})g(x_{ij}) + x_{ij}d(a_{12})g(x_{ij}) + x_{ij}a_{12}d(x_{ij}), \tag{2.10}$$

$$d(x_{ij}a_{21}x_{ij}) = d(x_{ij})g(a_{21})g(x_{ij}) + x_{ij}d(a_{21})g(x_{ij}) + x_{ij}a_{21}d(x_{ij}) \tag{2.11}$$

and

$$d(x_{ij}a_{22}x_{ij}) = d(x_{ij})g(a_{22})g(x_{ij}) + x_{ij}d(a_{22})g(x_{ij}) + x_{ij}a_{22}d(x_{ij}). \tag{2.12}$$

Adding (2.9), (2.10), (2.11) and (2.12) and subtracting this sum from (2.8), we get

$$\begin{aligned} d(x_{ij}(a_{11} + a_{12} + a_{21} + a_{22})x_{ij}) - d(x_{ij}a_{11}x_{ij}) - d(x_{ij}a_{12}x_{ij}) \\ - d(x_{ij}a_{21}x_{ij}) - d(x_{ij}a_{22}x_{ij}) \\ = d(x_{ij})(g(a_{11} + a_{12} + a_{21} + a_{22}) - g(a_{11}) - g(a_{12}) - g(a_{21}) - g(a_{22}))g(x_{ij}) \\ + x_{ij}(d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22}))g(x_{ij}) \end{aligned}$$

which results that

$$\begin{aligned} &d(x_{ij}(a_{11}+a_{12}+a_{21}+a_{22})x_{ij}) - d(x_{ij}a_{11}x_{ij}) - d(x_{ij}a_{12}x_{ij}) \\ &\qquad - d(x_{ij}a_{21}x_{ij}) - d(x_{ij}a_{22}x_{ij}) \\ &= x_{ij}(d(a_{11}+a_{12}+a_{21}+a_{22}) - d(a_{11}) - d(a_{12}) \\ &\qquad - d(a_{21}) - d(a_{22}))g(x_{ij}). \end{aligned} \tag{2.13}$$

It therefore follows from (2.13) that

$$x_{ij}tg(x_{ij}) = 0. \tag{2.14}$$

The identities (2.7) and (2.14) and the property (♧) show that $t = 0$. Consequently, $d(a_{11}+a_{12}+a_{21}+a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22}) = 0$. □

Step 2.4. For arbitrary elements $a_{12}, b_{12} \in \mathscr{R}_{12}$, $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d(a_{12}+b_{12}t_{22}) = d(a_{12}) + d(b_{12}t_{22})$ and (ii) $d(a_{21}+t_{22}b_{21}) = d(a_{21}) + d(t_{22}b_{21})$.

Proof. First, we observe that the following identity holds

$$e_1 + a_{12} + b_{12}t_{22} = (e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22}),$$

for all elements $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $t_{22} \in \mathscr{R}_{22}$. By Step 2.3, it follows that

$$\begin{aligned} &d(e_1) + d(a_{12} + a_{12}t_{22}) \\ &= d(e_1 + a_{12} + a_{12}t_{22}) \\ &= d((e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22})) \\ &= d(e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})d(e_1 + b_{12})(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})g(e_1 + b_{12})d(e_1 + a_{12} + t_{22}) \\ &= d(e_1 + a_{12} + t_{22})(e_1 + b_{12})(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})(d(e_1) + d(b_{12}))(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})(g(e_1) + g(b_{12}))d(e_1 + a_{12} + t_{22}) \\ &= d(e_1 + a_{12} + t_{22})e_1(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})d(e_1)(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})g(e_1)d(e_1 + a_{12} + t_{22}) \\ &+ d(e_1 + a_{12} + t_{22})b_{12}(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})d(b_{12})(e_1 + a_{12} + t_{22}) \\ &+ g(e_1 + a_{12} + t_{22})g(b_{12})d(e_1 + a_{12} + t_{22}) \\ &= d((e_1 + a_{12} + t_{22})e_1(e_1 + a_{12} + t_{22})) \\ &+ d((e_1 + a_{12} + t_{22})b_{12}(e_1 + a_{12} + t_{22})) \\ &= d(e_1 + a_{12}) + d(b_{12}t_{22}) \\ &= d(e_1) + d(a_{12}) + d(b_{12}t_{22}). \end{aligned}$$

This shows that $d(a_{12}+a_{12}t_{22}) = d(a_{12}) + d(a_{12}t_{22})$. Using a similar argument to the previous case, we obtain that $d(a_{21}+t_{22}a_{21}) = d(a_{21}) + d(t_{22}a_{21})$, from the identity

$$e_1 + a_{21} + t_{22}b_{21} = (e_1 + a_{21} + t_{22})(e_1 + b_{21})(e_1 + a_{21} + t_{22}),$$

for all elements $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$. □

Step 2.5. For arbitrary elements $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $a_{21}, b_{21} \in \mathscr{R}_{21}$ the following hold: (i) $d(a_{12}+b_{12}) = d(a_{12}) + d(b_{12})$ and (ii) $d(a_{21}+b_{21}) = d(a_{21}) + d(b_{21})$.

Proof. Write $t = d(a_{12}+b_{12}) - d(a_{12}) - d(b_{12})$. For all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ we compute

$$\begin{aligned} d(x_{ij}(a_{12}+b_{12})x_{ij}) = {} & d(x_{ij})(a_{12}+b_{12})x_{ij} \\ & + g(x_{ij})d(a_{12}+b_{12})x_{ij} + g(x_{ij})g(a_{12}+b_{12})d(x_{ij}), \end{aligned} \tag{2.15}$$

$$d(x_{ij}a_{12}x_{ij}) = d(x_{ij})a_{12}x_{ij} + g(x_{ij})d(a_{12})x_{ij} + g(x_{ij})g(a_{12})d(x_{ij}) \tag{2.16}$$

and

$$d(x_{ij}b_{12}x_{ij}) = d(x_{ij})b_{12}x_{ij} + g(x_{ij})d(b_{12})x_{ij} + g(x_{ij})g(b_{12})d(x_{ij}). \tag{2.17}$$

Adding (2.16) and (2.17) and subtracting this sum from (2.15), we arrive at

$$\begin{aligned} & d(x_{ij}(a_{12}+b_{12})x_{ij}) - d(x_{ij}a_{12}x_{ij}) - d(x_{ij}b_{12}x_{ij}) \\ & = g(x_{ij})(d(a_{12}+b_{12}) - d(a_{12}) - d(b_{12}))x_{ij} \\ & + g(x_{ij})(g(a_{12}+b_{12}) - g(a_{12}) - g(b_{12}))d(x_{ij}) \end{aligned}$$

which implies that

$$\begin{aligned} & d(x_{ij}(a_{12}+b_{12})x_{ij}) - d(x_{ij}a_{12}x_{ij}) - d(x_{ij}b_{12}x_{ij}) \\ & = g(x_{ij})(d(a_{12}+b_{12}) - d(a_{12}) - d(b_{12}))x_{ij}. \end{aligned} \tag{2.18}$$

By (2.18) and Step 2.4(ii) we obtain

$$g(x_{ij})tx_{ij} = 0. \tag{2.19}$$

Using a similar argument to the previous case, we obtain that

$$x_{ij}tg(x_{ij}) = 0, \tag{2.20}$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$. The identities (2.19) and (2.20) and the property (♧) show that $t = 0$. Therefore, $d(a_{12}+b_{12}) - d(a_{12}) - d(b_{12}) = 0$.

The other case is proved in a similar way. □

Step 2.6. For arbitrary elements $a_{11}, b_{11} \in \mathscr{R}_{11}$ and $a_{22}, b_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d(a_{11}+b_{11}) = d(a_{11}) + d(b_{11})$ and (ii) $d(a_{22}+b_{22}) = d(a_{22}) + d(b_{22})$.

Proof. Write $t = d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11})$. For all $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$ we compute

$$\begin{aligned} d(x_{ij}(a_{11}+b_{11})x_{ij}) = {} & d(x_{ij})(a_{11}+b_{11})x_{ij} \\ & + g(x_{ij})d(a_{11}+b_{11})x_{ij} + g(x_{ij})g(a_{11}+b_{11})d(x_{ij}), \end{aligned} \tag{2.21}$$

$$d(x_{ij}a_{11}x_{ij}) = d(x_{ij})a_{11}x_{ij} + g(x_{ij})d(a_{11})x_{ij} + g(x_{ij})g(a_{11})d(x_{ij}) \tag{2.22}$$

and

$$d(x_{ij}b_{11}x_{ij}) = d(x_{ij})b_{11}x_{ij} + g(x_{ij})d(b_{11})x_{ij} + g(x_{ij})g(b_{11})d(x_{ij}). \tag{2.23}$$

Adding (2.22) and (2.23) and subtracting this sum from (2.21), we arrive at

$$\begin{aligned}&d(x_{ij}(a_{11}+b_{11})x_{ij}) - d(x_{ij}a_{11}x_{ij}) - d(x_{ij}b_{11}x_{ij})\\ &= g(x_{ij})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ij}\\ &+ g(x_{ij})(g(a_{11}+b_{11}) - g(a_{11}) - g(b_{11}))d(x_{ij})\end{aligned}$$

which leads at

$$\begin{aligned}&d(x_{ij}(a_{11}+b_{11})x_{ij}) - d(x_{ij}a_{11}x_{ij}) - d(x_{ij}b_{11}x_{ij})\\ &= g(x_{ij})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ij}.\end{aligned} \tag{2.24}$$

From (2.24) we get

$$g(x_{ij})tx_{ij} = 0. \tag{2.25}$$

Using a similar argument to the previous case, we can prove that

$$x_{ij}tg(x_{ij}) = 0, \tag{2.26}$$

for all $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$. Also, for all $x_{ii} \in \mathscr{R}_{ii}$ and $x_{ji} \in \mathscr{R}_{ji}$ $(i \neq j; i, j = 1, 2)$ we compute

$$\begin{aligned}&d(x_{ii}(a_{11}+b_{11})x_{ii}) + d(x_{ji}(a_{11}+b_{11})x_{ii}) = d((x_{ii}+x_{ji})(a_{11}+b_{11})(x_{ii}+x_{ji}))\\ &= d(x_{ii}+x_{ji})(a_{11}+b_{11})(x_{ii}+x_{ji}) + g(x_{ii}+x_{ji})d(a_{11}+b_{11})(x_{ii}+x_{ji})\\ &+ g(x_{ii}+x_{ji})g(a_{11}+b_{11})d(x_{ii}+x_{ji}),\end{aligned} \tag{2.27}$$

$$\begin{aligned}&d(x_{ii}a_{11}x_{ii}) + d(x_{ji}a_{11}x_{ii}) = d((x_{ii}+x_{ji})a_{11}(x_{ii}+x_{ji}))\\ &= d(x_{ii}+x_{ji})a_{11}(x_{ii}+x_{ji}) + g(x_{ii}+x_{ji})d(a_{11})(x_{ii}+x_{ji})\\ &+ g(x_{ii}+x_{ji})g(a_{11})d(x_{ii}+x_{ji})\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}&d(x_{ii}b_{11}x_{ii}) + d(x_{ji}b_{11}x_{ii}) = d((x_{ii}+x_{ji})b_{11}(x_{ii}+x_{ji}))\\ &= d(x_{ii}+x_{ji})b_{11}(x_{ii}+x_{ji}) + g(x_{ii}+x_{ji})d(b_{11})(x_{ii}+x_{ji})\\ &+ g(x_{ii}+x_{ji})g(b_{11})d(x_{ii}+x_{ji}).\end{aligned} \tag{2.29}$$

Adding (2.28) and (2.29) and subtracting this sum from (2.27), we arrive at

$$\begin{aligned}&d(x_{ii}(a_{11}+b_{11})x_{ii}) - d(x_{ii}a_{11}x_{ii}) - d(x_{ii}b_{11}x_{ii})\\ &+ d(x_{ji}(a_{11}+b_{11})x_{ii}) - d(x_{ji}a_{11}x_{ii}) - d(x_{ji}b_{11}x_{ii})\\ &= g(x_{ii}+x_{ji})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))(x_{ii}+x_{ji})\\ &+ g(x_{ii}+x_{ji})(g(a_{11}+b_{11}) - g(a_{11}) - g(b_{11}))d(x_{ii}+x_{ji})\end{aligned}$$

which leads to

$$\begin{aligned}&d(x_{ii}(a_{11}+b_{11})x_{ii}) - d(x_{ii}a_{11}x_{ii}) - d(x_{ii}b_{11}x_{ii})\\ &= g(x_{ii}+x_{ji})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))(x_{ii}+x_{ji}),\end{aligned} \tag{2.30}$$

by Step 2.24. Now, note that

$$\begin{aligned} & d(x_{ii}(a_{11}+b_{11})x_{ii}) - d(x_{ii}a_{11}x_{ii}) - d(x_{ii}b_{11}x_{ii}) \\ & = g(x_{ii})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ii}. \end{aligned} \tag{2.31}$$

Thus from (2.30) and (2.31) it follows that

$$\begin{aligned} & g(x_{ii}+x_{ji})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))(x_{ii}+x_{ji}) \\ & = g(x_{ii})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ii} \end{aligned}$$

and by (2.25) we arrive at

$$\begin{aligned} & g(x_{ii})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ji} \\ & + g(x_{ji})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ii} = 0. \end{aligned} \tag{2.32}$$

Since $i \neq j$ and g is an endomorphism of $\mathscr{R}$, then multiplying the identity (2.32) on the left by $g(e_1)$, we can then conclude that

$$g(x_{ji})(d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}))x_{ii} = 0.$$

It follows directly from this that

$$g(x_{ji})tx_{ii} = 0. \tag{2.33}$$

Using a similar argument to the previous case, we can prove that, for all $x_{ii} \in \mathscr{R}_{ii}$ and $x_{ij} \in \mathscr{R}_{ij}$ $(i \neq j; i, j = 1, 2)$ we have

$$x_{ii}tg(x_{ij}) = 0. \tag{2.34}$$

The identities (2.25), (2.26), (2.33) and (2.34) and the property ($\spadesuit$) show that $t = 0$. Therefore, $d(a_{11}+b_{11}) - d(a_{11}) - d(b_{11}) = 0$.

The other case is proved in a similar way. □

Step 2.7. d is an additive map.

Proof. The result is direc consequence of Steps 2.3, 2.5 and 2.6. □

To complete the proof of the Theorem 2.1 we will assume in addition that $\mathscr{R}$ is a 2-torsion free semiprime ring and $g : \mathscr{R} \to \mathscr{R}$ is a ring isomorphism of $\mathscr{R}$.

Step 2.8. d is a semi-derivation of $\mathscr{R}$ (associated with g).

Proof. First, note that every Jordan triple semi-derivation of $\mathscr{R}$ (associative with g) is also a Jordan triple $(id_{\mathscr{R}}, g)$-derivation and a Jordan triple $(g, id_{\mathscr{R}})$-derivation, by [9, identity (†), pp. 1397]. Thus, by [9, Theorem 1], we conclude that d is a $(id_{\mathscr{R}}, g)$-derivation and a $(g, id_{\mathscr{R}})$-derivation, respectively, which allows to conclude that d is a semi-derivation of $\mathscr{R}$ (associative with g). □

3. Multiplicative Jordan triple semi-derivations on standard operator algebras

In order to present the main result of this section, we will take into account the results of Chang [4, Theorem 1], that proved that all maps which are associated with arbitrary semi-derivations of a prime ring are always homomorphisms, and of Brešar and Sěmrl [2, Corollary 7.3.], that characterized the class of isomorphisms between standard operator algebras. Thus, in view of these facts and the main result of the previous section, we can state the following:

Theorem 3.1. *Let $\mathscr{X}$ be a Hilbert space with* $\dim \mathscr{X} \geq 2$, *$\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$, $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ a standard operator algebra and $g : \mathscr{A} \to \mathscr{A}$ an isomorphism of $\mathscr{A}$. Then the following statement holds: every multiplicative Jordan triple semi-derivation of $\mathscr{A}$ (associated with g) is a semi-derivation of $\mathscr{A}$ (associated with g).*

Proof. First of all, it is well known that $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$ are prime rings and $\mathscr{A}$ contains a non-trivial idempotent e_1. Also, $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$ under the strong operator topology and, by [2, Corollary 7.3.], there exist a linear invertible operator of $\mathscr{B}(\mathscr{X})$, $y : \mathscr{X} \to \mathscr{X}$, such that $g(a) = yay^{-1}$, for all element $a \in \mathscr{A}$.

Let $\mathscr{B}(\mathscr{X}) = \oplus_{i,j=1,2}\mathscr{B}(\mathscr{X})_{ij}$ and $\mathscr{A} = \oplus_{i,j=1,2}\mathscr{A}_{ij}$ be the Peirce decompositions of $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$, relative to e_1, respectively, and for an arbitrary element $t \in \mathscr{A}$, consider $y^{-1}t = (y^{-1}t)_{11} + (y^{-1}t)_{12} + (y^{-1}t)_{21} + (y^{-1}t)_{22}$ and $ty = (ty)_{11} + (ty)_{12} + (ty)_{21} + (ty)_{22}$ be the Peirce decompositions of $y^{-1}t$ and ty, relative to e_1. Write $e_2 = 1_{\mathscr{B}(\mathscr{X})} - e_1$, where $1_{\mathscr{B}(\mathscr{X})}$ is a multiplicative identity of $\mathscr{B}(\mathscr{X})$.

Assume that the following conditions hold: $g(x_{ij})tx_{ij} = 0$ and $x_{ij}tg(x_{ij}) = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$. Then $yx_{ij}y^{-1}tx_{ij} = 0$ and $x_{ij}tyx_{ij}y^{-1} = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$, which leads to $x_{ij}y^{-1}tx_{ij} = 0$ and $x_{ij}tyx_{ij} = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i, j = 1, 2)$. As $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, then we can conclude that $x_{ij}y^{-1}tx_{ij} = 0$ and $x_{ij}tyx_{ij} = 0$, for all elements $x_{ij} \in \mathscr{B}(\mathscr{X})_{ij}$ $(i, j = 1, 2)$. Hence, by [8, Lemma 2(i)], we get $(y^{-1}t)_{ji} = 0$ and $(ty)_{ji} = 0$ $(i, j = 1, 2)$, respectively. This shows that $y^{-1}t = 0$ and $ty = 0$. From either of the two identities, we obtain $t = 0$. Now, assume that the following conditions hold: $g(x_{ij})tx_{ij} = 0$ and $x_{ij}tg(x_{ij}) = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$ $(i \neq j; i, j = 1, 2)$, provided that the conditions are met: $g(x_{ji})tx_{ii} = 0$ and $x_{ii}tg(x_{ij}) = 0$, for all elements $x_{ii} \in \mathscr{A}_{ii}$, $x_{ij} \in \mathscr{A}_{ij}$ and $x_{ji} \in \mathscr{A}_{ji}$ $(i \neq j; i, j = 1, 2)$. First, note that using a similar reasoning to the previous case we can deduce that $(y^{-1}t)_{ji} = 0$ and $(ty)_{ji} = 0$ $(i \neq j; i, j = 1, 2)$. Also, $yx_{ji}y^{-1}tx_{ii} = 0$ and $x_{ii}tyx_{ij}y^{-1} = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$, $x_{ij} \in \mathscr{A}_{ij}$ and $x_{ji} \in \mathscr{A}_{ji}$ $(i \neq j; i, j = 1, 2)$ results in $x_{ji}y^{-1}tx_{ii} = 0$ and $x_{ii}tyx_{ij} = 0$, for all elements $x_{ij} \in \mathscr{A}_{ij}$, $x_{ij} \in \mathscr{A}_{ij}$ and $x_{ji} \in \mathscr{A}_{ji}$. As $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, then we can conclude that $x_{ji}y^{-1}tx_{ii} = 0$ and $x_{ii}tyx_{ij} = 0$, for all elements $x_{ii} \in \mathscr{B}(\mathscr{X})_{ii}$, $x_{ij} \in \mathscr{B}(\mathscr{X})_{ij}$ and $x_{ji} \in \mathscr{B}(\mathscr{X})_{ji}$ $(i \neq j; i, j = 1, 2)$. This implies that $(y^{-1}t)_{ii} = 0$ and $(ty)_{ii} = 0$, in view of the primeness of $\mathscr{B}(\mathscr{X})$. It therefore follows that $y^{-1}t = 0$ and $ty = 0$, respectively. From either of the two identities, we obtain $t = 0$. Therefore, by Theorem 2.1, d is an additive map and so we obtain the desired result. □

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