Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 20 (2024) 82-97 https://doi.org/10.56947/amcs.v20.232

FIXED POINT THEOREMS FOR MODIFIED F-WEAK CONTRACTIONS VIA α-ADMISSIBLE MAPPING WITH APPLICATION TO PERIODIC POINTS

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Abstract. In (Fixed Point Theory Appl. 94:6, 2012), Wardowski introduced a new type of contraction called F-contraction and proved a fixed point result in complete metric spaces, which in turn generalizes the Banach contraction principle. The aim of this paper is to introduce a modified α-F-weak contractions with respect to a self-mapping on a metric space and to obtain fixed point results. Examples are provided to support results and concepts presented herein. As an application of our results, periodic point results for the F-contractions in metric spaces are proved.

1. Introduction and preliminaries

It is well known that Banach's fixed point theorems for contraction mappings is one of the pivotal results of mathematical analysis. Banach contraction principle [4] of is one of the most notable results which has played a vital role in the development of a metric fixed point theory. This principle and its variants provide a useful apparatus in guaranteeing the existence and uniqueness of solutions of various nonlinear problems: differential equations, integral equations, optimization problems, variational inequalities. A host of this principle has been made modified and extended by several mathematicians in different perspectives, some of them are as follows:

In 2012, Wardowski [15] introduced the concept of F-contractive mapping on a metric space and proved a fixed point theorem for such a map on a complete metric space.

Later, Wardowski [16] introduced the concepts of F-contraction and F-weak contraction to generalize the Banach's contraction in many ways. On the other hand, Hussain et al. [9] introduced the concept of α-GF-contraction as a generalization of F-contraction and obtained some interesting fixed point results.

Gopal et al. [7] introduced the concept of α-type F-contractions with respect to a self-mapping in a complete metric space and proffer sufficient conditions for the existence and uniqueness of fixed point results. For other generalizations, we refer to [2, 3, 5, 8, 10, 11, 12, 14]. Following these directions of research, we

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Date: Received: Nov 1, 2023; Accepted: Dec 3, 2023.

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2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.

Key words and phrases. Fixed point, F-contraction , Periodic points, α-admissible mappings.

introduce a modified α-F-weak contractions with respect to a self-mapping on a metric space and to obtain fixed point results. Examples are provided to support results and concepts presented herein. As an application of our results, periodic point results for the F-contractions in metric spaces are proved.

Throughout this article, we denot R as the set of all real numbers, R^+ the set of all positive real numbers and N the set of natural numbers.

In the sequel, we shall present the notion of F-contraction and some relevant definitions.

Definition 1.1. [7] Let $F : (0, +\infty) \to R$ be a mapping satisfying :
$(F_1)F$ is strictly increasing, i.e $\forall\ \alpha,\beta \in (0, +\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;
(F_2) For each sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(F_3)\exists k \in (0, 1)$ such that $\lim_{n\to 0^+} \alpha^k F(\alpha) = 0$
We denote by F, the family of all functions F satisfying the condition (i)-(iii)

Example 1.2. [16] The following functions $F : (0, +\infty) \to R$ belongs to F are:

i. $F(\alpha) = ln\alpha, \alpha > 0$;
ii. $F(\alpha) = ln\alpha + \alpha, \alpha > 0$;
iii. $F(\alpha) = \frac{-1}{\sqrt{\alpha}}, \alpha > 0$;
iv. $F(\alpha) = ln(\alpha^2 + \alpha), \alpha > 0$.

Definition 1.3. [16] Let (X, d) be a metric space. A mapping $T : X \to X$ is called an F-contraction on X if $\exists F \in F$ and $\tau > 0$ such that, $\forall x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y) \tag{1.1}$$

Definition 1.4. Let (X, d) be a metric space. A mapping $T : X \to X$ is called an F-weak contraction on X if $\exists F \in F$ and $\tau > 0$ such that,$\forall x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y) \tag{1.2}$$

where,

$$M(x, y) = \max \left\{ \begin{array}{r} d(x, y), d(x, Tx), d(y, Ty), \\ \dfrac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}, \\ \dfrac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} \end{array} \right\}$$

Remark 1.5. [16] Every F-contraction is an F-weak contraction but converse is not necessary true.

Further details on these examples, we refer to [16].

Definition 1.6. [13] A mapping $T : X \to X$ is α-admissible if there exist a function $\alpha : X \times X \to R^+$ such that

$$x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1 \tag{1.3}$$

Definition 1.7. [7] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-F-contraction on X if $\exists\ \tau > 0$ and two functions $F \in F$ and a function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\forall\ x, y \in X$ satisfying $d(Tx, Ty) > 0$, then

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1.4}$$

Definition 1.8. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a modified α-F-contraction on X if $\exists\ \tau > 0$ and two functions $F \in F$ and a function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\forall\ x, y \in X$ satisfying $d(Tx, Ty) > 0$, then

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(M(x, y)) \tag{1.5}$$

where

$$M(x, y) = \max \left\{ \begin{array}{r} d(x, y), d(x, Tx), d(y, Ty), \\ \dfrac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}, \\ \dfrac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} \end{array} \right\}$$

Remark 1.9. Every α-F-contraction is a modified α-F-weak contraction but converse is not necessary true.

2. Main results

In this section, we begin with the following Theorem.

Theorem 2.1. *Let (X, d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:*

i. *T is α-admissible,*
ii. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
iii. *T is continuous.*

Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in N}$ is convergent to X^*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n\ \forall n \in N$. If $\exists n_0 \in N$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$. This proves that x_{n_0} is a fixed point of T .
Now, suppose that $x_{n+1} \neq x_n$ for every $n \in N$. It follows from condition (i) and (ii), that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \tag{2.1}$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \forall n \in N \tag{2.2}$$

Since T is a modified α-F- week contraction, then $\forall n \in N$, we have

$$F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \leq \alpha(x_n, x_{n-1})F(d(Tx_n, Tx_{n-1}))$$

Consequently, we have

$$\tau + F(d(x_{n+1}, x_n)) \leq \tau + \alpha(x_n, x_{n-1})F(d(Tx_n, Tx_{n-1})) \leq F(M(x_n, x_{n-1}))$$

where,

$$M(x_n, x_{n-1}) = max \left\{ \begin{array}{r} d(x_n, x_{n-1}), d((x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \\ \dfrac{d(x_n, Tx_n)d(x_n, Tx_{n-1} + d(x_{n-1}, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}, \\ \dfrac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)} \end{array} \right\}$$

$$= max \left\{ \begin{array}{r} d(x_n, x_{n-1}), d((x_n, x_{n+1}), d(x_{n-1}, x_n), \\ \dfrac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n}, \\ \dfrac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1}) + d(x_n, x_n) + d(x_{n-1}, x_{n+1})} \end{array} \right\}$$

$$\leq max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \tag{2.3}$$

If $\exists n \in N$ such that $max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then(2.3) becomes

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1})) - \tau \tag{2.4}$$

which is a contradiction. Therefore,

$$max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n) \forall n \in N.$$

Hence, from (2.3), we get

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau \ \forall n \in N. \tag{2.5}$$

By induction, we have

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \ \forall n \in N. \tag{2.6}$$

On taking limit as $n \to \infty$ in (2.6), we get

$$F(d(x_n, x_{n+1})) = -\infty \tag{2.7}$$

Consider (2.7) and (F_2), we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.8}$$

From (2.6), $\exists\ k \in (0, 1)$ such that

$$\lim_{n\to\infty} ((d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) = 0 \tag{2.9}$$

From (2.6), $\forall n \in N$, we have

$$(d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) - F(d(x_0, x_1)) \leq -(d(x_n, x_{n+1}))^k n\tau \leq 0 \tag{2.10}$$

Using (2.8), (2.9) and taking the limit as $n \to \infty$ in (2.10), we get

$$\lim_{n\to\infty}(n(d(x_n, x_{n+1}))^k) = 0 \tag{2.11}$$

Then, $\exists\ n_1 \in N$ such that $n(d(x_n, x_{n+1}))^k \leq 1 \forall\ n \geq n_1$, that is,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \forall\ n \geq n_1 \tag{2.12}$$

For all $m > n > n_1$, by using (2.12) and the triangular inequality, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \ldots + d(x_m, x_{m+1}) \\ &< \sum_{i=1}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}} \end{aligned} \tag{2.13}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, taking the limit as $n \to +\infty$ we get

$$\lim_{m,n\to+\infty} d(x_n, x_m) = 0 \tag{2.14}$$

This proves that $\{x_n\}$ is Cauchy sequence in X.
Since X is complete, $\exists x^* \in X$ such that

$$\lim_{n\to+\infty} x_n = X^*.$$

Then, by the continuity of T, we have

$$d(x^*, Tx^*) = \lim_{n\to+\infty} d(x_n, Tx_n) = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0 \tag{2.15}$$

Thus, $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$. Hence, x^* is the fixed point of T. □

We now consider the following Theorem by omitting the continuity of hypothesis of T.

Theorem 2.2. *Let (X, d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:*

i. *T is α-admissible,*
ii. *$\exists\ x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
iii. *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then, $\alpha(x_n, x) \geq 1$ for all $n \in N$,*
iv. *F is continuous.*

Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in N}$ is convergent to X^*.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in N$. In view of Theorem 2.1, we obtain that $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d). Then, there exist $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. On the other hand, from (2.2) and condition (iii), we have

$$\alpha(x_n, x^*) \geq 1\ \forall n \in N. \tag{2.16}$$

We shall now prove that x^* is a fixed point of T by the following cases

Case 1 Suppose for each $n \in N$ $\exists i_n \in N$ such that $x_{i_n+1} = Tx^*$ and $i_n > i_{n-1^*}$ Then, we have

$$x^* = \lim_{n\to+\infty} x_{i_n+1} = \lim_{n\to+\infty} Tx^* = Tx^*. \tag{2.17}$$

That is, $Tx^* = x^*$. Hence, x^* is the fixed point of T.

Case 2 Suppose $\exists$ $n_0 \in N$ such that $x_{n+1} \neq Tx$ for all $n > n_0$, that is, $d(Tx_n, Tx^*) > 0$ for all $n > n_0$. Then from (1.5) and (F_1), we have

$$\begin{aligned}\tau + F(d(x_{n+1}, Tx^*)) &= \tau + F(d(Tx_n, Tx^*))\\ &\leq \tau + \alpha(x_n, x^*)F(d(Tx_n, Tx^*))\\ &\leq F(M(x_n, x^*))\end{aligned}$$

where

$$M(x_n, x^*) = max\left\{\begin{array}{r} d(x_n, x^*), d((x_n, Tx_n), d(x^*, Tx^*),\\ \dfrac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx_n)d(x^*, Tx^*)}{d(x^*, Tx_n) + d(x_n, Tx^*)},\\ \dfrac{d(x_n, Tx_n)d(x^*, Tx^*)}{d(x_n, x^*) + d(x_n, Tx^*) + d(x^*, Tx_n)} \end{array}\right\}$$

$$= max\left\{\begin{array}{r} d(x_n, x^*), d((x_n, x_{n+1}), d(x^*, Tx^*),\\ \dfrac{d(x_n, x_{n+1})d(x_n, Tx^*) + d(x^*, x_{n+1})d(x^*, Tx^*)}{d(x^*, x_{n+1}) + d(x_n, Tx^*)},\\ \dfrac{d(x_n, x_{n+1})d(x^*, Tx^*)}{d(x_n, x^*) + d(x_n, Tx^*) + d(x^*, x_{n+1})} \end{array}\right\} \tag{2.18}$$

If $d(x^*, Tx^*) > 0$ and the fact that

$$\lim_{n\to+\infty} d(x_n, x^*) = \lim_{n\to+\infty} d(x_{n+1}, x^*) = 0, \tag{2.19}$$

$\exists$ $n_1 \in N$ such that $\forall n > n_1$, we have

$$= max\left\{\begin{array}{r} d(x_n, x^*), d((x_n, x_{n+1}), d(x^*, Tx^*),\\ \dfrac{d(x_n, x_{n+1})d(x_n, Tx^*) + d(x^*, x_{n+1})d(x^*, Tx^*)}{d(x^*, x_{n+1}) + d(x_n, x^*) + d(x^*, Tx^*)},\\ \dfrac{d(x_n, x_{n+1})d(x^*, Tx^*)}{d(x_n, x^*) + d(x_n, x^*) + d(x^*, Tx^*) + d(x^*, x_{n+1})} \end{array}\right\}$$

$$M(x_n, x^*) = d(x^*, Tx^*) \tag{2.20}$$

By (2.18) and (2.20), we get

$$\tau + F(d(x_{n+1}, Tx^*)) \leq F(d(x^*, Tx^*)) \tag{2.21}$$

for all $n \geq max\{n_0, n_1\}$. Since F is continuous, taking limit as $n \to +\infty$ in (2.21), we get

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)) \tag{2.22}$$

which is a contradiction. Thus, $d(x^*, Tx*) = 0$, that is $Tx^* = x^*$. Hence, x^* is the fixed point of T. $\square$

Example 2.3. Let $X = [0, \frac{5}{2}]$ and d be the usual metric on X. Define $T : X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{5}{2}] \\ \frac{5}{2}, & \text{otherwise} \end{cases}$$

Proof. Let $\alpha(x, y) = 1 \;\forall\; x, y \in X, \tau > 0$ such that $e^{\tau} = \frac{8}{9}$ and $x = 0$ and $y = 1$, by putting $F(t) = \ln t$ with $t > 0$. Clearly, the above satisfies the hypothesis of Theorem 2.2. Consequently, T has atleast a fixed point. Thus, $x = 0$ and $x = \frac{5}{2}$ are two fixed point of T. □

Next, to ensure the existence and uniqueness of fixed point, we will consider the following hypothesis:

$$(U) \forall x, y \in Fix(T), d(x, y) \geq 1.$$

Theorem 2.4. *Adding condition (U) to the hypothesis of Theorem 2.1 (respectively Theorem 2.2), then the uniqueness of fixed point is obtained.*

Proof. Suppose that y^* is another fixed point of T, so that $d(x^*, Ty^*) = d(x^*, y^*) > 0$. Then, we get

$$\begin{aligned} \tau + F(d(x^*, y^*)) &= \tau + F(d(Tx^*, Ty^*)) \\ &\leq \tau + \alpha(x^*, y^*) F(d(Tx^*, Ty^*)) \\ &\leq F(M(x^*, y^*)) \end{aligned}$$

where

$$M(x^*, y^*) = max \left\{ \begin{array}{r} d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \\ \dfrac{d(x^*, Tx^*) d(x^*, Ty^*) + d(y^*, Tx^*) d(y^*, Ty^*)}{d(y^*, Tx^*) + d(x^*, Ty^*)}, \\ \dfrac{d(x^*, Tx^*) d(y^*, Ty^*)}{d(x^*, y^*) + d(x^*, Ty^*) + d(y^*, Tx^*)} \end{array} \right\}$$

$$\begin{aligned} &= max \left\{ \begin{array}{r} d(x^*, y^*), d(x^*, x^*), d(y^*, y^*), \\ \dfrac{d(x^*, x^*) d(x^*, y^*) + d(y^*, x^*) d(y^*, y^*)}{d(y^*, x^*) + d(x^*, y^*)}, \\ \dfrac{d(x^*, x^*) d(y^*, y^*)}{d(x^*, y^*) + d(x^*, y^*) + d(y^*, x^*)} \end{array} \right\} \\ &= d(x^*, y^*) \end{aligned} \tag{2.23}$$

consequently, we have

$$\tau + F(d(x^*, y^*)) \leq F(d(x^*, y^*))$$

which is a contradiction. Thus, $d(x^*, y^*) = 0$, that is, $x^* = y^*$. □

Example 2.5. Let $X = [0,1]$ and d be the usual metric on X. Define $T : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0,1) \\ 0, & \text{if } x = 1. \end{cases}$$

And also define $F(t) = \ln t$ with $t > 0$.

Proof. Let $\alpha(x,y) = 1$ for all $x, y \in X$ and $\tau > 0$ such that $e^{\tau} = \frac{1}{2}$. Obviously, the above satisfied all the hypothesis of Theorem 2.4. Hence, T has a unique fixed point $x = \frac{1}{2}$ □

Corollary 2.6. Let (X,d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:

i. if there exist $\tau > 0$ a function $F \in F$ such that for all $x, y \in X$ satisfying $d(Tx,Ty) > 0$, then

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y))$$

where

$$M(x,y) = max \left\{ \begin{array}{r} d(x,y), d(x,Tx), d(y,Ty), \\ \dfrac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(y,Tx) + d(x,Ty)}, \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y) + d(x,Ty) + d(y,Tx)} \end{array} \right\} \tag{2.24}$$

ii. T is continuous.

Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in N}$ is convergent to X^*.

Proof. Let $\alpha(x,y) = 1$ for all $x, y \in X$ in Theorem 2.1, we get the result immediately. □

Corollary 2.7. Let (X,d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:

i. if there exist $\tau > 0$ a function $F \in F$ such that for all $x, y \in X$ satisfying $d(Tx,Ty) > 0$, then

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y))$$

where

$$M(x,y) = max \left\{ \begin{array}{r} d(x,y), d(x,Tx), d(y,Ty), \\ \dfrac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(y,Tx) + d(x,Ty)}, \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y) + d(x,Ty) + d(y,Tx)} \end{array} \right\} \tag{2.25}$$

ii. F is continuous.

Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in N}$ is convergent to X^*.

Proof. Let $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.2, we get the result immediately. □

Corollary 2.8. Adding the condition, for all $x, y \in Fix(T), d(x, y) \geq 1$ to the hypothesis of corollary 2.6 (respectively Corollary 2.7) then, the uniqueness of fixed point is obtained.

Proof. For all $x, y \in X$ and $\alpha(x, y) = 1$ in Theorem 2.4, we get the result immediately. □

Corollary 2.9. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:

i. if there exist $\tau > 0$ and two functions $F \in F$ and a function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, then

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(d(x, y)), \tag{2.26}$$

ii. T is α-admissible,
iii. there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
iv. T is continuous.
Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in N}$ is convergent to X^*.

Proof. For all $x, y \in X$ we get the result immediately from Theorem 2.1. □

Corollary 2.10. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:

i. if there exist $\tau > 0$ and two functions $F \in F$ and a function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, then

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(d(x, y)), \tag{2.27}$$

ii. T is α-admissible,
iii. there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
iv. F is continuous.
Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in N}$ is convergent to X^*.

Proof. For all $x, y \in X$, we get the result immediately from Theorem 2.2. □

Corollary 2.11. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:

i. if there exist $\tau > 0$ a function $F \in F$ such that for all $x, y \in X$ where $a, b, c, d, e \geq 0$ and $a + b + c + d + e < 1$ satisfying $d(Tx, Ty) > 0$, then

$$\tau + F(d(Tx,Ty)) \leq F\begin{pmatrix} ad(x,y)+bd(x,Tx)+cd(y,Ty)+ \\ d\dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}+ \\ e\dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \end{pmatrix} \tag{2.28}$$

ii. F is continuous.
Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in N}$ is convergent to X^*.

Proof. For all $x, y \in X$, we have

$$ad(x,y) + bd(x,Tx) + cd(y,Ty) + d\frac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(y,Tx) + d(x,Ty)} + e\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty) + d(y,Tx)}$$

$$\leq (a+b+c+d+e)\ max\begin{pmatrix} d(x,y), d(x,Tx), d(y,Ty) \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}, \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \end{pmatrix}$$

$$\leq\ max\begin{pmatrix} d(x,y), d(x,Tx), d(y,Ty), \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}, \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \end{pmatrix}$$

Then, by (F_1) we see that (2.24) is a consequence of (2.28). Then, the corollary is proved. □

Corollary 2.12. Let (X,d) be a complete metric space. A mapping $T : X \to X$ is said to be a modified α-F-weak contraction satisfying the following conditions:

i. if there exist $\tau > 0$ a function $F \in F$ such that for all $x, y \in X$ where $a, b, c, d, e \geq 0$ and $a+b+c+d+e < 1$ satisfying $d(Tx,Ty) > 0$, then

$$\tau + F(d(Tx,Ty)) \leq F\begin{pmatrix} ad(x,y)+bd(x,Tx)+cd(y,Ty)+ \\ d\dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}+ \\ e\dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \end{pmatrix} \tag{2.29}$$

ii. F is continuous.
Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in N}$ is convergent to X^*.

Proof. For all $x, y \in X$, we have

$$ad(x,y) \ + \ bd(x,Tx) + cd(y,Ty) \ + \ d\frac{d(x,Tx)d(x,Ty) \ + \ d(y,Tx)d(y,Ty)}{d(y,Tx) \ + \ d(x,Ty)}$$
$$+\, e\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty) \ + \ d(y,Tx)}$$
$$\leq (a+b+c+d+e)\ max\begin{pmatrix} d(x,y), d(x,Tx), d(y,Ty), \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}, \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \end{pmatrix}$$

$$\leq \ max\begin{pmatrix} d(x,y), d(x,Tx), d(y,Ty), \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}, \\ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \end{pmatrix}$$

Then, by (F_1) we see that (2.25) is a consequence of (2.29). Then, the corollary is proved. □

3. Application

In this section, as an application of our results, we establish the existence of periodic point result for self mapping on a complete metric space. In sequel, we begin with the following definition.

Definition 3.1. A mapping $T : X \to X$ is said to have property (P) if $Fix(T^n) = Fix(T)$ for every $n \in N$, where

$$Fix(T) : \{x \in X : Tx = x\}\,. \tag{3.1}$$

Further details on this property, we refer to [[1],[11]].

Theorem 3.2. *Let (X, d) be a complete metric space. $T : X \to X$ is said to be a mapping satisfying the following conditions.*

i. *there exist $\tau > 0$ and two functions $F \in F$ and a function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that,*

$$\tau + \alpha(x,Tx)F(d(Tx,T^2x)) \leq F(M(x,Tx)), \tag{3.2}$$

where

$$M(x,Tx) = \ max \left\{ \begin{array}{r} d(x,Tx), d(x,Tx), d(Tx,T^2x), \\ \dfrac{d(x,Tx)d(x,T^2x) + d(Tx,Tx)d(Tx,T^2x)}{d(Tx,Tx) + d(x,T^2x)}, \\ \dfrac{d(x,Tx)d(Tx,T^2x)}{d(x,Tx) + d(x,T^2x) + d(Tx,Tx)} \end{array} \right\}$$

holds for $x \in X$ with $F(d(Tx,T^2x)) > 0$,

ii. *there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

iii. *T is α-admissible,*

iv. *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then, $Tx_n \to Tx$ as $n \to +\infty$,*

v. *if $w \in Fix(T^n)$ and $w \notin Fix(T)$, then $\alpha(T^{n-1}w, T^n w) \geq 1$.*

Then T has a property (P).

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\}$ in X by $x_n = T^n x_n = Tx_{n-1}$. By condition (iii) , we get

$$\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1,$$

By induction, we have

$$\alpha(Tx_n, Tx_{n+1}) \geq 1 \ \forall \ n \in N \tag{3.3}$$

If there exist $n_0 \in N$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$. This proves that x_{n_0} is a fixed point of T.

Now, suppose that $x_{n+1} \neq x_n$ or $d(Tx_{n-1}, T^2x_{n-1} > 0$ for every $n \in N$. From (3.3) and (i), we have

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) &= \tau + F(d(Tx_{n-1}, T^2x_{n-1})) \\ &\leq \tau + \alpha(x_{n-1}, Tx_{n-1})F(d(Tx_{n-1}, T^2x_{n-1})) \\ &\leq F(M(x_{n-1}, Tx_{n-1})) \end{aligned}$$

or equivalently,

$$F(d(x_n, x_{n+1})) \leq F(M(x_{n-1}, x_n)) - \tau,$$

where

$M(x_{n-1}, Tx_{n-1})$

$$= \ max \left\{ \begin{array}{r} d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1}), d(Tx_{n-1}, T^2x_{n-1}), \\ \dfrac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, T^2x_{n-1}) + d(Tx_{n-1}, Tx_{n-1})d(Tx_{n-1}, T^2x_{n-1})}{d(Tx_{n-1}, Tx_{n-1}) + d(x_{n-1}, T^2x_{n-1})}, \\ \dfrac{d(x_{n-1}, Tx_{n-1})d(Tx_{n-1}, T^2x_{n-1})}{d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, T^2x_{n-1}) + d(Tx_{n-1}, Tx_{n-1})} \end{array} \right\}$$

$$= max \left\{ d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1}), d(Tx_{n-1}, T^2x_{n-1}, d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1}) \right\}$$

$$= max \left\{ d(x_{n-1}, Tx_{n-1}), d(Tx_{n-1}, T^2x_{n-1}) \right\} \tag{3.4}$$

If there exists $n \in N$ such that

$$max\left\{d(x_{n-1},Tx_{n-1}),d(Tx_{n-1},T^2x_{n-1})\right\} = d(Tx_{n-1},T^2x_{n-1}),$$

then (3.4) becomes

$$F(d(x_n,x_{n+1})) = F(d(Tx_{n-1},T^2x_{n-1})) \leq F(d(Tx_{n-1},T^2x_{n-1})) - \tau \;\; \forall\, n \in N \tag{3.5}$$

which is a contradiction. Therefore,

$$max\left\{d(x_{n-1},Tx_{n-1}),d(Tx_{n-1},T^2x_{n-1})\right\} = d(x_{n-1},Tx_{n-1})\forall\, n \in N.$$

Hence, from (3.4), we get

$$F(d(x_n,x_{n+1})) = F(d(Tx_{n-1},T^2x_{n-1})) \leq F(d(x_{n-1},Tx_{n-1})) - \tau \;\; \forall\, n \in N \tag{3.6}$$

or equivalently,

$$F(d(x_n,x_{n+1})) \leq F(d(x_{n-1},x_n)) - \tau \;\; \forall\, n \in N \tag{3.7}$$

Following the proof of Theorem 2.1, we get that the sequence $\{x_n\}$ is a Cauchy sequence and hence the completeness of (X,d) ensure that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$.

From (iv), we get $x_{n+1} = Tx_n \to x^*$ as $n \to +\infty$, that is $x^* = Tx^*$. Thus, T has a fixed point and $Fix(T^n) = Fix(T)$ is true for $n = 1$. Now, let $n > 1$ and assume, by contradiction, that $w \in Fix(T^n)$ and $w \notin Fix(T)$,then $d(w,Tw) > 0$.

Applying (v) and (i), we get

$$\begin{aligned}\tau + F(d(w,Tw)) &= \tau + F(d(T(T^{n-1}w),T^2(T^{n-1}w)))\\ &\leq \tau + \alpha(T^{n-1}w,T(T^{n-1}w))F(d(T(T^{n-1}w),T^2(T^{n-1}w)))\end{aligned}$$

and

$$\begin{aligned}\tau + F(d(w,Tw)) &\leq \tau + \alpha(T^{n-1}w,T^nw)F(d(T(T^{n-1}w),T^2(T^{n-1}w)))\\ &\leq F(M(T^{n-1}w,T^nw)),\end{aligned}$$

where

$$M(T^{n-1}w,T^nw)$$

$$= \max\left\{\begin{array}{l} d(T^{n-1}w,T(T^{n-1}w)),d(T^{n-1}w,T(T^{n-1}w)),d(T(T^{n-1}w),T^2(T^{n-1}w)),\\ \dfrac{d(T^{n-1}w,T(T^{n-1}w))d(T^{n-1}w,T^2(T^{n-1}w))}{d(T(T^{n-1}w),T(T^{n-1}w))+d(T^{n-1}w,T^2(T^{n-1}w))}\\ +\dfrac{d(T(T^{n-1}w),T(T^{n-1}w))d(T(T^{n-1}w),T^2(T^{n-1}w))}{d(T(T^{n-1}w),T(T^{n-1}w))+d(T^{n-1}w,T^2(T^{n-1}w))},\\ \dfrac{d(T^{n-1}w,T(T^{n-1}w))d(T(T^{n-1}w),T^2(T^{n-1}w))}{d(T^{n-1}w,T(T^{n-1}w))+d(T^{n-1}w,T^2(T^{n-1}w))+d(T(T^{n-1}w),T(T^{n-1}w))}\end{array}\right\}$$

$$= \max \begin{Bmatrix} d(T^{n-1}w, T(T^{n-1}w)), d(T^{n-1}w, T(T^{n-1}w)), d(T(T^{n-1}w), T^2(T^{n-1}w)), \\ d(T^{n-1}w, T(T^{n-1}w)), d(T^{n-1}w, T(T^{n-1}w)) \end{Bmatrix}$$

$$= \max \left\{ d(T^{n-1}w, T(T^{n-1}w)), d(T(T^{n-1}w), T^2(T^{n-1}w)) \right\} \tag{3.8}$$

If there exists $n \in N$ such that
$max \{d(T^{n-1}w, T(T^{n-1}w)), d(T(T^{n-1}w), T^2(T^{n-1}w))\} = d(T(T^{n-1}w), T^2(T^{n-1}w))$,
then (3.8) becomes

$$F(d(w,Tw)) = F(d(T(T^{n-1}w), T^2(T^{n-1}w))) \leq F(d(T(T^{n-1}w), T^2(T^{n-1}w))) - \tau \ \forall\, n \in N, \tag{3.9}$$

which is a contradiction. Therefore

$max \left\{ d(T^{n-1}w, T(T^{n-1}w)), d(T(T^{n-1}w), T^2(T^{n-1}w)) \right\} = d(T^{n-1}w), T(T^{n-1}w)) \ \forall\, n \in N.$

Hence, from (3.8), we get

$$F(d(w,Tw)) = F(d(T(T^{n-1}w), T^2(T^{n-1}w))) \leq F(d(T^{n-1}w, T(T^{n-1}w))) - \tau \ \forall\, n \in N, \tag{3.10}$$

or equivalently,

$$F(d(w,Tw)) \leq F(d(T^{n-1}w, T^n w)) - \tau \ \forall\, n \in N. \tag{3.11}$$

Consequently, by induction, we have

$$\begin{aligned} F(d(w,Tw)) &\leq F(d(T^{n-1}w, T^n w)) - \tau \\ &\leq F(d(T^{n-1}w, T^n w)) - 2\tau \\ &\leq \ldots \\ &\leq F(d(T^{n-1}w, T^n w)) - n\tau. \end{aligned} \tag{3.12}$$

Taking the limit as $n \to +\infty$ in (3.12), we have $F(d(w,Tw)) = -\infty$, which is a contradiction. Thus, $F(w,Tw) = 0$. Hence, $Fix(T^n) = Fix(T)$ for all $n \in N$. □

Corollary 3.3. Let (X,d) be a complete metric space. $T : X \in X$ is said to be a mapping satisfying the following conditions:

i. there exists $\tau > 0$ a function $F \in F$ such that,

$$\tau + F(d(Tx, T^2x)) \leq F(M(x,Tx)), \tag{3.13}$$

where

$$M(x,Tx) = \ max \begin{Bmatrix} d(x,Tx), d(x,Tx), d(Tx,T^2x), \\ \dfrac{d(x,Tx)d(x,T^2x) + d(Tx,Tx)d(Tx,T^2x)}{d(Tx,Tx) + d(x,T^2x)}, \\ \dfrac{d(x,Tx)d(Tx,T^2x)}{d(x,Tx) + d(x,T^2x) + d(Tx,Tx)} \end{Bmatrix}$$

holds for $x \in X$ with $F(d(Tx,T^2x)) > 0$. Then T has a property (P).

Proof. Let $\alpha(x, Tx) = 1$ for all $x \in X$ in Theorem 3.2, we get the result immediately. □

Corollary 3.4. Let (X, d) be a complete metric space. $T : X \to X$ is said to be a mapping satisfying the following conditions:

i. there exists $\tau > 0$ a function $F \in F$ such that,

$$\tau + F(d(Tx, T^2x)) \leq F(d(x, Tx)), \tag{3.14}$$

holds for $x \in X$ with $F(d(Tx, T^2x)) > 0$. Then T has property (P).

Proof. Let $\alpha(x, Tx) = 1$ for all $x \in X$ and $d(x, Tx) = M(x, Tx)$ in Theorem 3.2, we get the result immediately. □

Corollary 3.5. Let (X, d) be a complete metric space. $T : X \to$ is said to be a mapping satisfying the following conditions:

i. if there exist $\tau > 0$ and two functions $F \in F$ and a function $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that,

$$\tau + \alpha(x, Tx)F(d(Tx, T^2x)) \leq F(d(x, Tx)), \tag{3.15}$$

holds for $x \in X$ with $F(d(Tx, T^2x)) > 0$,

ii. there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

iii. T is α-admissible,

iv. if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then, $Tx_n \to Tx$ as $n \to +\infty$,

v. if $w \in Fix(T^n)$ and $w \notin Fix(T)$, then $\alpha(T^{n-1}w, T^n w) \geq 1$.
Then T has a property (P).

Proof. For all $x \in X$ and $d(x, Tx) = M(x, Tx)$, we get the result immediately from Theorem 3.2. □

Acknowledgement. The authors would like to extend their sincere appreciation to all that have contributed in one way or the other for the actualization of this work.

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