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NIJENHUIS OPERATOR AND LIE LIKE BRACKET ON GENERALISED TANGENT BUNDLE

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Abstract. The evolution of Nijenhuis operators has started during nineties. The Nijenhuis operator is formulated using the deformation on Poisson Nijenhuis manifolds and Lie algebras. In this paper, the Nijenhuis operator is suggested on the generalised tangent bundle $TM \oplus T^*M$ followed by the deformation on Lie algebra. A new geometric structure is formulated in association with Nijenhuis relation. It is proved that if the Nijenhuis operator, $\mathfrak{N} : \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$ is restricted to the Dirac structure of the generalised tangent bundle $TM \oplus T^*M$, then the deformation is a trivial. However, on the whole space $\Gamma(TM \oplus T^*M)$, it is only a deformation weaker than the trivial deformation. A bracket like a Lie bracket is defined on $\Gamma(TM \oplus T^*M) \oplus \Gamma(T^*M)$. The bracket is skew symmetric and does not satisfy Jacobi identity property. A structure like Dirac structure is also defined on that space.

1. Introduction

The work on Nijenhuis structures has started from mid nineties. Nijenhuis himself studied on Nijenhuis relation around 1954. In 1984, Magri and Moroshi worked on a Poisson-Nijenhuis manifold via integrability of Hamiltonian system. This was further explained in the work of Kosmann and Magri in the year 1990. The famous paper of T. J. Courant on "Dirac Manifolds" [5] in 1990 introduced Dirac structures as a generalization of symplectic and Poisson structures. Irene Dorfman, a student of Gelfand worked independently and introduced Dirac structures in another set up [6] at around the same time. She constructed Nijenhuis operator on a finite dimensional manifold through deformation of the Lie algebra on that manifold. Further she associated a pair of Dirac structures with the Nijenhuis relation. Later the construction of Nijenhuis operator other than on Lie algebra set up was studied by many authors such as Fuchssteiner [8], Cariñena, Grabowski, Marmo [2, 3], Weinstein, Xu [20] etc. Dirac Nijenhuis structures were introduced [4] in 2004. Guang and Kang developed the concept of Dirac Nijenhus manifolds in [11] in 2004 and they subsequently introduced Dirac Nijenhuis

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structures on Lie bialgebroids. In 2011 Kosmann [18] studied on Nijenhuis structures on Courant algebroid in the light of works of Bedjoui-Tebbal [1], Grabowski, Khudaverdyan [10], Loday, Pirashvili [21], Grabowski [?], Keller and Weldmann [14]. The relation of Nijenhuis operator on Dirac pairs and Hamiltonian systems has been studied recently [6].

In this article, a Nijenhus operator on the generalised tangent bundle $TM \oplus T^*M$ following Dorfman's Work is constructed. This approach is different. We have associated a new geometric structure with Nijenhuis relation. The work on the deformation of the space $\Lambda(TM) \oplus \Lambda(T^*M)$ and on different geometric structures using Nijenhuis operator is studied.

The remaining of the paper is progressed as: in section 2, the related work on Nijenhuis operator on TM is discussed. The proposed Nijenhuis operator on $TM \oplus T^*M$ is inroduced in section 3. Further, the proof of theorems and some relevant definitions are discussed in this section. Nijenhuis relation both on TM and $TM \oplus T^*M$ are discussed in section 4. A new bracket is also introduced in this section. It is wiser to state that the new bracket is skew symmetric and does not satisfy Jacobi identity.

2. Nijenhuis operator on TM

Let M be a finite dimensional differentiable manifold and TM be its tangent bundle. $\Gamma(TM)$ be the space of sections of TM on M, which is a Lie algebra with the usual Lie bracket $[.,.]$ on it.

Definition 2.1. Nijenhuis operator:[4, 18]
An endomorphism $N : \Gamma(TM) \to \Gamma(TM)$ is said to be a Nijenhuis operator if and only if the torsion tensor $\mathfrak{T}(N)$ vanishes, where $\mathfrak{T}(N) = [N(X), N(Y)] - N([X, N(Y)]) - N([N(X), Y]) + N^2([X, Y]), \forall X, Y \in \Gamma(TM)$.

Dorfman [7] has studied Nijenhuis operator via deformation of Lie bracket.

Theorem 2.2. [4] *Consider $\Gamma(TM)$, the space of all vector fields on M and a Nijenhuis operator $N : \Gamma(TM) \to \Gamma(TM)$. Define a bilinear bracket $[X, Y]_N = -N([X, Y]) + [X, N(Y)] + [N(X), Y]$ on $\Gamma(TM)$ and a mapping $\acute{N} = \rho o N$ as the anchor map. Then $(\Gamma(TM), [X, Y]_N, \acute{N})$ represents a Lie algebroid.*

2.1. **Construction of Nijenhuis operator in the sense of Dorfman[7].** Let $\mathfrak{A}$ be any Lie algebra with bracket $[.,.]$. A bilinear operation $\omega : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ generates a deformation of Lie algebra $\mathfrak{A}$ if all the μ parametrised family of brackets $[X, Y]_\mu = [X, Y] + \mu\omega(X, Y)$ equip $\mathfrak{A}$ with a Lie algebra structure. For this ω is skew symmetric and satisfies Jacobi identity as given below.

(1) $\omega(X, Y) = -\omega(Y, X)$.
(2) ω must satisfy Jacobi identity. $\omega(\omega(X, Y), Z) + c.p. = 0$.

From the definition of the co boundary operator in the standard Lie algebra complex, the Jacobi identity reduces to $d\omega = 0$, i.e. $d\omega(X, Y, Z) = [X, \omega(Y, Z)] - \omega([X, Y], Z) + c.p. = 0$. For a fixed $N : \Gamma(TM) \to \Gamma(TM)$, the deformation of Lie algebra is said to be trivial if for $T_\mu = id + \mu N$ and the following condition holds:

$$T_\mu[X,Y]_\mu = [T_\mu X, T_\mu Y]. \tag{2.1}$$

Now expanding L.H.S. and R.H.S. both and comparing them we get

$$\omega(X,Y) = [X, N(Y)] + [N(X), Y] - N[X,Y] \tag{2.2}$$

$$N\omega(X,Y) = [N(X), N(Y)] \tag{2.3}$$

From the above two equations (2.2), (2.3) we have

$$[NX, NY] - N[NX, Y] - N[X, NY] + (N)^2[X,Y] = 0. \tag{2.4}$$

A linear operator N satisfying (2.4) is called a **Nijenhuis Operator**.

Theorem 2.3 ([7]). *Let $N : \Gamma(TM) \to \Gamma(TM)$ be a Nijenhuis Operator. Then a trivial deformation of $\Gamma(TM)$ can be obtained by putting $\omega(X,Y) = [Na, b] + [a, Nb] - N[a,b]$.*

3. Nijenhuis operator on the generalised tangent bundle $TM \oplus T^*M$(Via deformation)

Let us consider the generalised tangent bundle $TM \oplus T^*M$of M, whose section is a double of Lie bialgebroid. i.e. $\Gamma(TM \oplus T^*M) = \Gamma(TM) \oplus \Gamma(T^*M) = \{(X,\xi) | X \in \Gamma(TM), \xi \in \Gamma(T^*M)$. $\Gamma(TM \oplus T^*M)$ is naturally endowed with the symmetric and skew symmetric pairing:

$$\langle (X,\alpha), (Y,\beta) \rangle_\pm = \frac{1}{2} i_Y \alpha \pm i_X \beta.$$

. $\Gamma(TM) \oplus \Gamma(T^*M)$ is an algebra with a bracket operation on it, which is known as Courant bracket.

Definition 3.1. Courant Bracket:[5]
For differentiable manifold M, the Courant bracket$[X,Y]_c$ on $\Gamma(TM \oplus T^*M)$ is a bilinear operation defined by $[X+\xi, Y+\eta]_c = ([X,Y], L_Y\xi - L_X\eta - d\langle X + \xi, Y+\eta\rangle_+)$, where $X+\xi, Y+\eta \in \Gamma(TM \oplus T^*M)$, L_X is the Lie derivative and $\langle X+\xi, Y+\eta\rangle_+$ is the symmetric pairing on $\Gamma(TM \oplus T^*M)$.

Let us assume a λ-parametrised family of brackets on $\Gamma(TM \oplus T^*M)$, i.e. $[Y_1, Y_2]_\lambda = [Y_1, Y_2]_c + \lambda\varphi(Y_1, Y_2)$, where φ is a bilinear operation on $\Gamma(TM \oplus T^*M)$, $Y_1, Y_2 \in \Gamma(TM \oplus T^*M)$ and $[Y_1, Y_2]_c$ is the Courant bracket on $\Gamma(TM \oplus T^*M)$. Here $[.,.]_\lambda$ describes deformation of the algebra $\Gamma(TM \oplus T^*M)$ if $\varphi(Y_1, Y_2)$ has a Courant bracket structure, where Courant bracket satisfies skew symmetric property and does not satisfy Jacobi identity.

Let $\mathfrak{N} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ be a linear map and define $\mathfrak{T}_\lambda = id + \lambda\mathfrak{N}$ on $\Gamma(TM \oplus T^*M)$. For $\mathfrak{T}_\lambda$, this deformation is said to be trivial if

$$\mathfrak{T}_\lambda[X,Y]_\lambda = [\mathfrak{T}_\lambda X, \mathfrak{T}_\lambda Y]. \tag{3.1}$$

Comparing both sides of the above equation we have

$$\varphi(X,Y) = [X, \mathfrak{N}Y]_c + [\mathfrak{N}X, Y]_c - \mathfrak{N}[X,Y]_c \tag{3.2}$$

$$\mathfrak{N}\varphi(X,Y) = [\mathfrak{N}X, \mathfrak{N}Y]_c \tag{3.3}$$

From the equation (3.3) we can conclude that

$$\mathfrak{N}\varphi(X,Y) - [\mathfrak{N}X, \mathfrak{N}Y]_c = 0$$
$$\mathfrak{N}([X, \mathfrak{N}Y]_c + [\mathfrak{N}X, Y]_c - \mathfrak{N}[X,Y]_c) - [\mathfrak{N}X, \mathfrak{N}Y]_c = 0 \tag{3.4}$$

With the equation (3.4), $\mathfrak{N}$ is called a Nijenhuis Operator on $\Gamma(TM \oplus T^*M)$.

A linear operator $\mathfrak{N}$ in $\Gamma(TM \oplus T^*M)$ is called a Nijenhuis operator if the equation (3.4) holds.

Definition 3.2. A linear operator $\mathfrak{N}$ in $\Gamma(TM \oplus T^*M)$ is called a Nijenhuis operator on $\Gamma(TM \oplus T^*M)$ if

$$\mathfrak{N}([X, \mathfrak{N}Y]_c + [\mathfrak{N}X, Y]_c - \mathfrak{N}[X,Y]_c) - [\mathfrak{N}X, \mathfrak{N}Y]_c = 0,$$

where $[.,.]_c$ is the Courant bracket and $X, Y \in \Gamma(TM \oplus T^*M)$.

Remark 3.3 (Structure of Nijenhuis operator on $\Gamma(TM \oplus T^*M)$). The Nijenhuis Operator $\mathfrak{N}$ is a linear mapping from $\Gamma(TM \oplus T^*M)$ to itself.

- It is skew symmetric if $\mathfrak{N} = -(\mathfrak{N})^t$.
- $\mathfrak{N}$ can be written as
$$\begin{pmatrix} N & \pi \\ \omega & N^* \end{pmatrix},$$
where $N : \Gamma(TM) \to \Gamma(TM), N^* : \Gamma(T^*M) \to \Gamma(T^*M), \pi : \Gamma(T^*M) \to \Gamma(TM)$ and $\omega : \Gamma(TM) \to \Gamma(T^*M)$.

Here $N^2 = -Id_M$. Kosmann-Schwarzbach in her article [18] has shown that $\Gamma(TM \oplus T^*M)$ has a weak deformation with respect to a Nijenhuis Operator $\mathfrak{N}$ only if $\mathfrak{N}$ is equivalent to a almost complex structure, i.e. $\mathfrak{N}^2 = -Id_M$. $\mathfrak{N}$ satisfies almost complex structure iff π, ω vanishes on $\Gamma(TM \oplus T^*M)$.

Definition 3.4. Dirac Structure:[5]
A sub bundle $L \subset TM \oplus T^*M$ is said to be a Dirac structure on M if $L = L^{\perp}$ and $[\chi_1, \chi_2]_c = \chi_3$, where $\chi_1, \chi_2, \chi_3 \in \Gamma(L)$.

Dirac structure is a maximally isotropic subbundle of $TM \oplus T^*M$, as the symmetric pairing on $TM \oplus T^*M$ vanishes on L. On $TM \oplus T^*M$, Courant bracket does not satisfy Jacobi identity , it satisfies an anomaly known as Jacobi Anomaly,which gives rise to a non vanishing three tensor $T(Z_1, Z_2, Z_3)$ on $TM \oplus T^*M$. $T(Z_1, Z_2, Z_3) = \langle [Z_1, Z_2], Z_3 \rangle_+ + c.p.$.

Some authors like Clemente-Gallardo, Nunes da Costa [4], Long-Guang, Bao-kang [?] have studied independently on deformation of Dirac structure on 2004. Effort has been done to give a new approach following Dorfman's construction.

Theorem 3.5. *Let $\mathfrak{N} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ be a Nijenhuis operator, then the deformation of $\Gamma(TM \oplus T^*M)$ is not a weak deformation if and only if both $\mathfrak{N}$ and $\varphi(X,Y)$ are restricted to Dirac structures, otherwise the deformation is weak deformation on the whole space.*

Proof. We have $\varphi(X,Y) = [X, \mathfrak{N}Y]_c + [\mathfrak{N}X, Y]_c - \mathfrak{N}[X,Y]_c$. φ is skew symmetric as Courant bracket is skew symmetric. For the coboundary operator δ we have, $\delta\varphi(X,Y,Z) = [X, \varphi(Y,Z)] + \varphi([X,Y],Z) + c.p. \neq 0$ as Courant bracket does not satisfy Jacobi identity property. Therefore on $\Gamma(TM \oplus T^*M)$ deformation is not trivial. Suppose both $\mathfrak{N}, \varphi(X,Y)$ are restricted to the sections of Dirac structure L on $\Gamma(TM \oplus T^*M)$. That means $\mathfrak{N} : \Gamma(L) \to \Gamma(L)$ and $\varphi : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$.

As we know that Dirac Structure L is a maximally isotropic subbundle of $(TM \oplus T^*M)$, the natural symmetric pairing on it vanishes. Courant bracket restricted to the Dirac structures satisfies Jacobi Identity. Therefore on L, $\delta\varphi = 0$ as $\delta\varphi$ is the representation of Jacobi identity on the given space. □

4. Nijenhuis Relation

4.1. **Nijenhuis Relation on $\Gamma(TM)$.** Let $A \subset \Gamma(TM) \oplus \Gamma(TM)$ and $A^* \subset \Gamma(T^*M) \oplus \Gamma(T^*M)$. Let us take $a_1 \oplus a_2 \in \Gamma(TM) \oplus \Gamma(TM)$ and $\zeta_1 \oplus \zeta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$. Let $\zeta_1 \oplus \zeta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$ be such that $(\zeta_1, a_2) = (\zeta_2, a_1)$ for arbitrary $a_1 \oplus a_2 \in A$. we have

Definition 4.1. Nijenhuis relation:[7]
A relation $A \subset \Gamma(TM) \oplus \Gamma(TM)$ is said to be a Nijenhuis relation for arbitrary $a_1, a_2, b_1, b_2 \in \Gamma(TM)$ and $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(T^*M)$ satisfying $a_1 \oplus a_2, b_1 \oplus b_2 \in A$ and $\zeta_1 \oplus \zeta_2, \zeta_2 \oplus \zeta_3 \in A^*$ if $(\zeta_1, [a_2, b_2]) - (\zeta_2, [a_2, b_1] + [a_1, b_2]) + (\zeta_3, [a_1, b_1]) = 0$ holds.

Theorem 4.2. [7] *The graph of a Nijenhuis operator $N : \Gamma(TM) \to \Gamma(TM)$ is a Nijenhuis relation. Conversely if a graph of some operator $N : \Gamma(TM) \to \Gamma(TM)$ is a Nijenhuis relation, then N is a Nijenhuis operator.*

Definition 4.3. Pair of Dirac structures:[7]
Two Dirac structures $L, N \subset \Gamma(TM) \oplus \Gamma(T^*M)$ are said to be a pair of Dirac structures, if the set

$$A_{L,N} = \{a_1 \oplus a_2 : \exists \zeta \in \Gamma(T^*M), a_1 \oplus \zeta \in N, a_2 \oplus \zeta \in L\}$$

is a Nijenhuis relation.

4.2. **Nijenhuis relation on $TM \oplus T^*M$.** Let $\mathfrak{A} \subset \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ and $A^* \subset \Gamma(T^*M) \oplus \Gamma(T^*M)$. Let $\alpha_1 \oplus \alpha_2 \in \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ and $\eta_1 \oplus \eta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$. The pairing is defined as

$$\begin{aligned} &\langle \alpha_2, \eta_1 \rangle \\ &\Rightarrow \langle X_2 + \xi_2, \eta_1 \rangle \\ &\Rightarrow (i_{X_2}\eta_1 + \eta_1 \wedge \xi_2) \end{aligned}$$

Definition 4.4. Nijenhuis relation on $TM \oplus T^*M$:
A relation $\mathfrak{A} \subset \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ is said to be a Nijenhuis relation for arbitrary $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma(TM \oplus T^*M)$ and $\eta_1, \eta_2, \eta_3 \in \Gamma(T^*M)$ satisfying $\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2 \in A$ and $\eta_1 \oplus \eta_2, \eta_2 \oplus \eta_3 \in A^*$ if $\eta_1, [\alpha_2, \beta_2]) - (\eta_2, [\alpha_2, \beta_1] + [\alpha_1, \beta_2]) + (\eta_3, [\alpha_1, \beta_1]) = 0$.

Definition 4.5. Symmetric and skew symmetric pairings:
Let $((X_1, \xi_1), \eta_1), ((X_2, \xi_2), \eta_2) \in \Gamma(TM \oplus T^*M) \oplus \Gamma(T^*M)$. A pair of symmetric

and skew symmetric pairing is defined as:

$$\begin{aligned}
&\langle((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)\rangle_{\pm} \\
&=\frac{1}{2}(\langle(X_1+\xi_1),\eta_2\rangle \pm \langle((X_2+\xi_2),\eta_1\rangle)+\frac{1}{2}(\langle X_1,\xi_2\rangle \pm \langle X_2,\xi_1\rangle) \\
&=\frac{1}{2}((i_{X_1}\eta_2+\xi_1\wedge\eta_2)\pm(i_{X_2}\eta_1+\xi_2\wedge\eta_1))+\frac{1}{2}(i_{X_1}\xi_2 \pm i_{X_2}\xi_1)
\end{aligned} \tag{4.1}$$

Depending on the skew symmetric paring on $\Gamma(TM \oplus T^*M) \oplus \Gamma(T^*M)$ and the skew symmetric pairing on $\Gamma(TM \oplus T^*M)$ a new bracket is defined on the respective space.

Definition 4.6. Lie like bracket:
For $((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2) \in \Gamma(TM\oplus T^*M)\oplus\Gamma(T^*M)$, the Lie like bracket is defined as:

$$\begin{aligned}
&[((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)] \\
&=(([X_1,X_2]+(L_{X_2}\xi_1-L_{X_1}\xi_2+d\langle(X_1+\xi_1),(X_2+\xi_2)\rangle_-)), \\
&L_{X_2}\eta_1-L_{X_1}\eta_2+d\langle((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)\rangle_-),
\end{aligned} \tag{4.2}$$

where $\langle(X_1,\xi_1),(X_2,\xi_2)\rangle_-$ is the natural skew symmetric pairing on $\Gamma(TM \oplus T^*M)$.

Definition 4.7. Dirac like structure:
A subbundle $N \subset \Gamma(TM\oplus T^*M)\oplus\Gamma(T^*M)$ is called a Dirac like structure on M if and only if

- $N = N^{\perp}$.
- $\Gamma(N)$ is closed under Lie like bracket (4.2).

Theorem 4.8. *The Lie like bracket* $[((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)]$ *is skew symmetric and does not satisfy Jacobi identity property on the space* $\Gamma(TM\oplus T^*M)\oplus\Gamma(T^*M)$.

- $[((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)] = -[((X_2+\xi_2),\eta_2),((X_1+\xi_1),\eta_1)]$.
- $[[((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)],((X_3+\xi_3),\eta_3)] + c.p. \neq 0,$

$\forall ((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2),((X_3+\xi_3),\eta_3) \in \Gamma(TM\oplus T^*M)\oplus\Gamma(T^*M)$.

Proof. To show the Lie like bracket is skew symmetric, consider

$$\begin{aligned}
&[((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)] \\
&=(([X_1,X_2],L_{X_2}\xi_1-L_{X_1}\xi_2+d\langle(X_1+\xi_1),(X_2+\xi_2)\rangle_-), \\
&L_{X_2}\eta_1-L_{X_1}\eta_2+d\langle((X_1+\xi_1),\eta_1),((X_2+\xi_2),\eta_2)\rangle_-) \\
&=((-[X_2,X_1],-L_{X_2}\xi_1+L_{X_1}\xi_2-d\langle(X_2+\xi_2),(X_1+\xi_1)\rangle_-), \\
&-L_{X_2}\eta_1+L_{X_1}\eta_2-d\langle((X_2+\xi_2),\eta_2),((X_1+\xi_1),\eta_1)\rangle_-) \\
&=-[((X_2+\xi_2),\eta_2),((X_1+\xi_1),\eta_1)],
\end{aligned}$$

which proves that the bracket is skew symmetric.
For Jacobi identity consider

$$[[((X_1,\xi_1),\eta_1),((X_2,\xi_2),\eta_2)]_{cl},((X_3,\xi_3),\eta_3)]_{cl}+c.p.$$
$$=[((([X_1,X_2],L_{X_2}\xi_1-L_{X_1}\xi_2+d\langle(X_1,\xi_1),(X_2,\xi_2)\rangle_-),$$
$$L_{X_2}\eta_1-L_{X_1}\eta_2+d\langle((X_1,\xi_1),\eta_1),((X_2,\xi_2),\eta_2)\rangle_-),((X_3,\xi_3),\eta_3)]_{cl}+c.p.$$
$$=((0,\langle[((X_1,\xi_1),(X_2,\xi_2)]_c,((X_3,\xi_3)\rangle_+),E_N),$$

where $E_N=\frac{1}{2}(i_{X_1}\eta_2+\xi_1\wedge\eta_2)\pm\frac{1}{2}(i_{X_2}\eta_1+\xi_2\wedge\eta_1))+c.p.$ □

5. Conclusion and future scope

In this paper, Nijenhuis operator on a generalised tangent bundle is introduced. It is verified that the deformation due to Nijenhuis operator is weak through out the generalised tangent bundle $\Gamma(TM\oplus T^*M)$ as compared to the trivial deformation. In addition to this, the deformation is strong when the Nijenhuis operator is restricted to Dirac structure of that space. It is due to its maximal isotropic property. A respective Nijenhuis relation is also discussed on the generalised tangent bundle $\Gamma(TM\oplus T^*M)$. A new Lie like bracket is introduced on the space $\Gamma(TM\oplus T^*M)\oplus\Gamma(T^*M)$. The proposed bracket is skew symmetric and does not satisfy Jacobi identity. A Dirac like structure is introduced on that space.

Further, a pair of such Dirac like structures may be associated to the defined Nijenhuis relation with some interesting geometric properties.

Declaration

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