

FIXED POINT RESULTS FOR GENERALIZED F_α -CONTRACTION IN COMPLETE S -METRIC SPACES

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ABSTRACT. In this paper, we prove some fixed point theorems for generalized F_α -contraction in the setting of complete S -metric spaces and give some consequences as corollaries of the main results. Also, we furnish an illustrative example in support of the result. Our results extend, generalize and enrich several previous works from the existing literature (see, for example [6], [21], [33], [44] and some others).

1. INTRODUCTION

Fixed point theory originated with the method of successive approximations. In 1837, *Liouville* [26] and *Picard* [13] used the method of successive approximation to prove the existence of solutions for differential equations. In 1992, the Polish mathematicians *Banach* [5] established an important metric fixed point result regarding a contraction mapping, known as the Banach contraction mapping principle (or in short BCP). This principle is considered as one of the most remarkable results in analysis. It confirms the existence and uniqueness of fixed point of certain self maps on metric spaces. Due to its importance, several authors have achieved many interesting extensions and generalizations of this contraction principle. Many mathematicians studied the Banach contraction principle and used it in many mathematical rules and mathematical science, like approximation theory, game theory, quantum theory, economics, differential equations and integral equations. We also refer to the reader to see the following: [11], [20], [24], [25], [29], [30].

One way to generalize the Banach contraction principle is to enlarge the class of metric spaces. In 2000, *Dhage* [9] introduced the notion of D -metric space as a generalization of the metric space and proved some fixed point results in this setting. In 2012, *Sedghi et al.* [37] introduced the concept of an S -metric space which is a generalization of G -metric space [9] and D^* -metric space [36]. They proved some fixed point theorems for a self-map on an S -metric space. Also, they proved properties of S -metric space. Later on, *Sedghi and Dung* [38] proved

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generalized fixed point theorems in S -metric spaces which is a generalization of the results of [37].

In [39], *Sedghi et al.* proved some common fixed point theorems for the pair of weakly compatible self mappings satisfying some Φ -type contractive conditions in the setting of S -metric spaces which generalize and extend several results from the exiting literature.

In [40], *Sedghi et al.* presented some common fixed point results for four mappings satisfying generalized contractive condition in the framework of S -metric spaces. The results obtained in this paper extend and generalize several previously published results from the existing literature.

On the other hand, *Wardowski* [44] introduced a new type of contraction called F -contraction and proved a new fixed point result regarding F -contraction. In this way, *Wardowski* [44] generalized the Banach contraction principle in a different manner from the well-known results in the literature.

In 2012, *Samet et al.* [35] introduced the concept of α -admissible mappings. The α -admissible mappings notion has been used in many works, see for example [3, 4, 14, 22, 31, 34].

In [27], *Mlaiki* introduced the notion of α -admissible mapping in the setting of S -metric spaces. Recently, *Javed et al.* [17] introduced the concept of F_α -contraction which is a generalization of F -contraction and proved a fixed point theorem in the setting of S -metric spaces.

Motivated by the works of [17, 27, 37, 44] and some others, we generalize the notion of F_α -contraction and prove some fixed point theorems for generalized F_α -contraction in the framework of S -metric spaces. Our results extend and generalize several results from the existing literature.

2. PRELIMINARIES

We need the following definitions and lemmas in the sequel.

Definition 2.1. ([37]) Let $\Xi \neq \emptyset$ be a set and let $\tilde{S}: \Xi^3 \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, t \in \Xi$:

- (S1) $\tilde{S}(x, y, z) = 0$ if and only if $x = y = z$;
- (S2) $\tilde{S}(x, y, z) \leq \tilde{S}(x, x, t) + \tilde{S}(y, y, t) + \tilde{S}(z, z, t)$.

Then the function $\tilde{S}$ is called an S -metric on Ξ and the pair $(\Xi, \tilde{S})$ is called an S -metric space (in short SMS).

Immediate examples of such S -metric spaces are as follows.

Example 2.2. ([37])

- (1) Let $\Xi = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Ξ , then $\tilde{S}(u, v, z) = \|v + z - 2u\| + \|v - z\|$ is an S -metric on Ξ .
- (2) Let $\Xi = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Ξ , then $\tilde{S}(u, v, z) = \|u - z\| + \|v - z\|$ is an S -metric on Ξ .

Example 2.3. ([38]) Let $\Xi = \mathbb{R}$ be the real line. Then $\tilde{S}(u, v, z) = |u - z| + |v - z|$ for all $u, v, z \in \mathbb{R}$ is an S -metric on Ξ . This S -metric on Ξ is called the usual S -metric on Ξ .

Example 2.4. ([23]) Let $\Xi \neq \emptyset$ be a set and d be an ordinary metric on Ξ . Then $\tilde{S}(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in \Xi$ is an S -metric on Ξ .

Example 2.5. ([40]) Let $\Xi \neq \emptyset$ be a set and d_1, d_2 be two ordinary metrics on Ξ . Then $\tilde{S}(u, v, z) = d_1(u, z) + d_2(v, z)$ for all $u, v, z \in \Xi$ is an S -metric on Ξ .

Definition 2.6. Let $(\Xi, \tilde{S})$ be an S -metric space. For $r > 0$ and $e \in \Xi$ we define the open ball $B_{\tilde{S}}(e, r)$ and closed ball $B_{\tilde{S}}[e, r]$ with center e and radius r as follows, respectively:

(B_1)

$$B_{\tilde{S}}(e, r) = \{e' \in \Xi : \tilde{S}(e', e', e) < r\},$$

(B_2)

$$B_{\tilde{S}}[e, r] = \{e' \in \Xi : \tilde{S}(e', e', e) \leq r\}.$$

Example 2.7. ([38]) Let $\Xi = \mathbb{R}$. Denote $\tilde{S}(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} B_{\tilde{S}}(1, 2) &= \{e \in \mathbb{R} : \tilde{S}(e, e, 1) < 2\} = \{e \in \mathbb{R} : |e - 1| < 1\} \\ &= \{e \in \mathbb{R} : 0 < e < 2\} = (0, 2), \end{aligned}$$

and

$$\begin{aligned} B_{\tilde{S}}[2, 4] &= \{e \in \mathbb{R} : \tilde{S}(e, e, 2) \leq 4\} = \{e \in \mathbb{R} : |e - 2| \leq 2\} \\ &= \{e \in \mathbb{R} : 0 \leq e \leq 4\} = [0, 4]. \end{aligned}$$

Definition 2.8. ([37], [38]) Let $(\Xi, \tilde{S})$ be an S -metric space and $\mathcal{Y} \subset \Xi$.

(Λ_1) The subset $\mathcal{Y}$ is said to be an open subset of Ξ , if for every $e \in \mathcal{Y}$ there exists $r > 0$ such that $B_{\tilde{S}}(e, r) \subset \mathcal{Y}$.

(Λ_2) A sequence $\{e_n\}$ in Ξ converges to $e \in \Xi$ if $\tilde{S}(e_n, e_n, e) \rightarrow 0$ as $n \rightarrow \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\tilde{S}(e_n, e_n, e) < \varepsilon$. We denote this by $\lim_{n \rightarrow \infty} e_n = e$ or $e_n \rightarrow e$ as $n \rightarrow \infty$.

(Λ_3) A sequence $\{e_n\}$ in Ξ is called a Cauchy sequence if $\tilde{S}(e_n, e_n, e_m) \rightarrow 0$ as $n, m \rightarrow \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $\tilde{S}(e_n, e_n, e_m) < \varepsilon$.

(Λ_4) The S -metric space $(\Xi, \tilde{S})$ is called complete if every Cauchy sequence in Ξ is convergent in Ξ .

(Λ_5) Let τ be the set of all $\mathcal{Y} \subset \Xi$ with $e \in \mathcal{Y}$ and there exists $r > 0$ such that $B_{\tilde{S}}(e, r) \subset \mathcal{Y}$. Then τ is a topology on Ξ (induced by the S -metric space).

(Λ_6) A nonempty subset $\mathcal{Y}$ of Ξ is S -closed if closure of $\mathcal{Y}$ coincides with $\mathcal{Y}$.

Definition 2.9. ([37]) Let $(\Xi, \tilde{S})$ be an S -metric space. A mapping $\mathcal{R}: \Xi \rightarrow \Xi$ is said to be a contraction if there exists a constant $0 \leq \delta < 1$ such that

$$\tilde{S}(\mathcal{R}x, \mathcal{R}y, \mathcal{R}z) \leq \delta \tilde{S}(x, y, z) \quad (2.1)$$

for all $x, y, z \in \Xi$.

Note: If the S -metric space $(\Xi, \tilde{S})$ is complete, then the mapping defined as above has a unique fixed point (see, [37]).

Definition 2.10. ([37]) Let $(\Xi, \tilde{S})$ and $(\Xi', \tilde{S}')$ be two S -metric spaces. A function $g: \Xi \rightarrow \Xi'$ is said to be continuous at a point $e_0 \in \Xi$ if for every sequence $\{e_n\}$ in Ξ with $\tilde{S}(e_n, e_n, e_0) \rightarrow 0$, $\tilde{S}'(g(e_n), g(e_n), g(e_0)) \rightarrow 0$ as $n \rightarrow \infty$. We say that g is continuous on Ξ if g is continuous at every point $e_0 \in \Xi$.

Definition 2.11. Let $\Xi \neq \emptyset$ be a set and let $g: \Xi \rightarrow \Xi$ be a self mapping of Ξ . Then a point $\mu \in \Xi$ is called a fixed point of operator g if $g(\mu) = \mu$.

Definition 2.12. ([35]) Let $\Xi \neq \emptyset$ be a set. Let $\mathcal{T}: \Xi \rightarrow \Xi$ and $\alpha: \Xi \times \Xi \rightarrow [0, +\infty)$ be given mappings. We say that $\mathcal{T}$ is α -admissible if for all $x, y \in \Xi$, we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(\mathcal{T}x, \mathcal{T}y) \geq 1.$$

Definition 2.13. ([27]) Let $(\Xi, \tilde{S})$ be an S -metric space and $\mathcal{T}: \Xi \rightarrow \Xi$ be a given mapping. We say that $\mathcal{T}$ is α -admissible if $x, y, z \in \Xi$, $\alpha(x, y, z) \geq 1$ implies that $\alpha(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \geq 1$.

Example 2.14. Let $\Xi = [0, \infty)$ and d be a metric on Ξ . A mappings $\tilde{S}: \Xi^3 \rightarrow [0, \infty)$ defined by $\tilde{S}(x, y, z) = d(x, z) + d(y, z)$ is an S -metric on Ξ . Let $\alpha: \Xi^3 \rightarrow [0, \infty)$. Let a mapping $\mathcal{T}$ given by $\mathcal{T}x = \sqrt{x}$ and define α by

$$\alpha(x, y, z) = \begin{cases} e^{\max\{x, y\} - z}, & \text{if } \max\{x, y\} \geq z, \\ 0, & \text{if } \max\{x, y\} < z. \end{cases}$$

So, it is easy to see that the mapping $\mathcal{T}$ is α -admissible.

In [44], Wardowski introduced a new concept of F -contraction on a complete metric space as follows.

Definition 2.15. ([44]) Let $F: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a mapping satisfying:

(F1) F is strictly increasing, that is, $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$.

(F2) For every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$, we have $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$.

(F3) There exists a number $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

In what follows, $\mathcal{F}$ stands for the family of all functions F which satisfies the above three conditions.

Let $F_1(\alpha) = \ln(\alpha)$, $F_2(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_3(\alpha) = \alpha + \ln(\alpha)$ for $\alpha > 0$, then $F_1, F_2, F_3 \in \mathcal{F}$.

Definition 2.16. ([6]) Let $(\Xi, \tilde{S})$ be an S -metric space. A mapping $\mathcal{T}: \Xi \rightarrow \Xi$ is said to be a F -contraction if there exists a number $\tau > 0$ such that

$$\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)) \leq F(\tilde{S}(x, y, z)),$$

for all $x, y, z \in \Xi$.

Remark 2.17. Clearly Definition 2.15 and (F1) implies that $\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) < \tilde{S}(x, y, z)$ for all $x, y, z \in \Xi$ with $\mathcal{T}x \neq \mathcal{T}y \neq \mathcal{T}z$. Hence every F -contraction mapping is continuous.

Definition 2.18. ([17]) Let $(\Xi, \tilde{S})$ be an S -metric space and $\alpha: \Xi^3 \rightarrow [0, \infty)$ be a function. A mapping $\mathcal{T}: \Xi \rightarrow \Xi$ is called an F_α -contraction if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that

$$\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) > 0 \Rightarrow \tau + F(\alpha(x, y, z)\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)) \leq F(\tilde{S}(x, y, z)),$$

for all $x, y, z \in \Xi$.

Every S -metric $(\Xi, \tilde{S})$ on Ξ defines a metric $d_{\tilde{S}}$ on Ξ by

$$d_{\tilde{S}} = \tilde{S}(x, x, y) + \tilde{S}(y, y, x) \quad \forall x, y \in \Xi. \quad (2.2)$$

Lemma 2.19. ([37], Lemma 2.5) Let $(\Xi, \tilde{S})$ be an S -metric space. Then, we have $\tilde{S}(x, x, y) = \tilde{S}(y, y, x)$ for all $x, y \in \Xi$.

Lemma 2.20. ([37], Lemma 2.12) Let $(\Xi, \tilde{S})$ be an S -metric space. If $e_n \rightarrow e$ and $e'_n \rightarrow e'$ as $n \rightarrow \infty$ then $\tilde{S}(e_n, e_n, e'_n) \rightarrow \tilde{S}(e, e, e')$ as $n \rightarrow \infty$.

Remark 2.21. ([38]) Every S -metric space is topologically equivalent to a B -metric space.

Lemma 2.22. ([12], Lemma 8) Let $(\Xi, \tilde{S})$ be an S -metric space and $\mathcal{Y}$ is a nonempty subset of Ξ . Then $\mathcal{Y}$ is said to be S -closed if and only if for any sequence $\{e_n\}$ in $\mathcal{Y}$ such that $e_n \rightarrow e$ as $n \rightarrow \infty$, then $e \in \mathcal{Y}$.

Lemma 2.23. ([37]) Let $(\Xi, \tilde{S})$ be an S -metric space. If $r > 0$ and $e \in \Xi$, then the ball $B_{\tilde{S}}(e, r)$ is an open subset of Ξ .

Lemma 2.24. ([38]) The limit of the sequence $\{e_n\}$ in an S -metric space $(\Xi, \tilde{S})$ is unique.

Lemma 2.25. ([37]) Let $(\Xi, \tilde{S})$ be an S -metric space. Then the convergent sequence $\{e_n\}$ in Ξ is Cauchy.

In the following lemma we see the relationship between a metric and S -metric.

Lemma 2.26. ([15])

Let (Ξ, d) be a metric space. Then the following properties are satisfied:

- (1) $\tilde{S}_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in \Xi$ is an S -metric on Ξ .
- (2) $e_n \rightarrow e$ in (Ξ, d) if and only if $e_n \rightarrow e$ in $(\Xi, \tilde{S}_d)$.
- (3) $\{e_n\}$ is Cauchy in (Ξ, d) if and only if $\{e_n\}$ is Cauchy in $(\Xi, \tilde{S}_d)$.
- (4) (Ξ, d) is complete if and only if $(\Xi, \tilde{S}_d)$ is complete.

We call the function $\tilde{S}_d$ defined in Lemma 2.26 (1) as the S -metric generated by the metric d . It can be found an example of an S -metric which is not generated by any metric in [15, 28].

Corollary 2.27. ([38]) Let $\mathcal{T}: \Xi \rightarrow \Xi'$ be a map from an S -metric space Ξ to an S -metric space Ξ' . Then $\mathcal{T}$ is continuous at $e \in \Xi$ if and only if $\mathcal{T}e_n \rightarrow \mathcal{T}e$ whenever $e_n \rightarrow e$ as $n \rightarrow \infty$.

3. MAIN RESULTS

In this section, we shall prove some fixed point theorems for generalized F_α -contraction in the setting of S -metric spaces. First, we define the following:

Definition 3.1. Let $(\Xi, \tilde{S})$ be an S -metric space and $\alpha: \Xi^3 \rightarrow [0, \infty)$ be a function. A mapping $\mathcal{T}: \Xi \rightarrow \Xi$ is called generalized F_α -contraction if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that

$$\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) > 0 \Rightarrow \tau + F(\alpha(x, y, z)\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)) \leq F(\Omega(x, y, z)), \quad (3.1)$$

for all $x, y, z \in \Xi$, where

$$\begin{aligned} \Omega(x, y, z) = \max \Big\{ & \tilde{S}(x, y, z), \tilde{S}(x, x, \mathcal{T}x), \tilde{S}(z, z, \mathcal{T}z), \frac{1}{2}[\tilde{S}(x, x, \mathcal{T}y) \\ & + \tilde{S}(z, z, \mathcal{T}x)] \Big\}. \end{aligned}$$

Theorem 3.2. Let $(\Xi, \tilde{S})$ be a complete S -metric space and $\mathcal{T}: \Xi \rightarrow \Xi$ be a generalized F_α -contraction satisfying the following conditions:

- (1) $\mathcal{T}$ is α -admissible;
- (2) there exists $x_0 \in \Xi$ such that $\alpha(x_0, x_0, \mathcal{T}x_0) \geq 1$;
- (3) $\mathcal{T}$ is continuous.

Then $\mathcal{T}$ has a fixed point.

Proof. Let $x_0 \in \Xi$ be an arbitrary point. Consider the sequence $\{x_n\}$ defined by

$$x_1 = \mathcal{T}x_0, x_2 = \mathcal{T}x_1 = \mathcal{T}^2x_0, \dots, x_n = \mathcal{T}x_{n-1} = \mathcal{T}^nx_0.$$

By hypothesis (2), we know that $\alpha(x_0, x_0, \mathcal{T}x_0) \geq 1$ and as $\mathcal{T}$ is α -admissible, therefore $\alpha(x_1, x_1, x_2) \geq 1$. So, using the fact that $\mathcal{T}$ is α -admissible and by induction on n , we conclude that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n . Now,

$$\begin{aligned} \tilde{S}(x_n, x_n, x_{n+1}) &= \tilde{S}(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}, \mathcal{T}x_n) \\ &\leq \alpha(x_{n-1}, x_{n-1}, x_n)\tilde{S}(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}, \mathcal{T}x_n). \end{aligned}$$

This implies that

$$F(\tilde{S}(x_n, x_n, x_{n+1})) \leq F(\alpha(x_{n-1}, x_{n-1}, x_n)\tilde{S}(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}, \mathcal{T}x_n)).$$

So, we have

$$F(\tilde{S}(x_n, x_n, x_{n+1})) \leq F(\Omega(x_{n-1}, x_{n-1}, x_n)) - \tau, \quad (3.2)$$

where

$$\begin{aligned}
\Omega(x_{n-1}, x_{n-1}, x_n) &= \max \left\{ \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_{n-1}, x_{n-1}, \mathcal{T}x_{n-1}), \tilde{S}(x_n, x_n, \mathcal{T}x_n), \right. \\
&\quad \left. \frac{1}{2} [\tilde{S}(x_{n-1}, x_{n-1}, \mathcal{T}x_{n-1}) + \tilde{S}(x_n, x_n, \mathcal{T}x_{n-1})] \right\} \\
&= \max \left\{ \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_n, x_n, x_{n+1}), \right. \\
&\quad \left. \frac{1}{2} [\tilde{S}(x_{n-1}, x_{n-1}, x_n) + \tilde{S}(x_n, x_n, x_n)] \right\} \\
&= \max \left\{ \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_n, x_n, x_{n+1}), \right. \\
&\quad \left. \frac{1}{2} \tilde{S}(x_{n-1}, x_{n-1}, x_n) \right\} \\
&= \max \left\{ \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_n, x_n, x_{n+1}) \right\}.
\end{aligned}$$

The following cases arise:

Case I: If $\max \left\{ \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_n, x_n, x_{n+1}) \right\} = \tilde{S}(x_n, x_n, x_{n+1})$, then from equation (3.2), we obtain

$$F(\tilde{S}(x_n, x_n, x_{n+1})) \leq F(\tilde{S}(x_n, x_n, x_{n+1})) - \tau,$$

which is a contradiction since $\tau > 0$.

Case II: If $\max \left\{ \tilde{S}(x_{n-1}, x_{n-1}, x_n), \tilde{S}(x_n, x_n, x_{n+1}) \right\} = \tilde{S}(x_{n-1}, x_{n-1}, x_n)$, then from equation (3.2), we obtain

$$F(\tilde{S}(x_n, x_n, x_{n+1})) \leq F(\tilde{S}(x_{n-1}, x_{n-1}, x_n)) - \tau. \quad (3.3)$$

Continuing in the same manner, we obtain

$$F(\tilde{S}(x_{n-1}, x_{n-1}, x_n)) \leq F(\tilde{S}(x_{n-2}, x_{n-2}, x_{n-1})) - \tau. \quad (3.4)$$

Using equations (3.3) and (3.4), we get

$$\begin{aligned}
F(\tilde{S}(x_n, x_n, x_{n+1})) &\leq F(\tilde{S}(x_{n-1}, x_{n-1}, x_n)) - \tau \\
&\leq F(\tilde{S}(x_{n-2}, x_{n-2}, x_{n-1})) - 2\tau \\
&\leq \dots \\
&\leq F(\tilde{S}(x_0, x_0, x_1)) - n\tau.
\end{aligned} \quad (3.5)$$

Set

$$\Lambda_n = \tilde{S}(x_n, x_n, x_{n+1}).$$

Therefore, from equation (3.5), we obtain

$$F(\Lambda_n) \leq F(\Lambda_0) - n\tau. \quad (3.6)$$

Then, it follows that

$$\lim_{n \rightarrow \infty} F(\Lambda_n) \leq \lim_{n \rightarrow \infty} [F(\Lambda_0) - n\tau].$$

This implies that

$$\lim_{n \rightarrow \infty} F(\Lambda_n) < -\infty.$$

By $F \in \mathcal{F}$ and (F2), we have

$$\lim_{n \rightarrow \infty} \Lambda_n = \lim_{n \rightarrow \infty} \tilde{S}(x_n, x_n, x_{n+1}) = 0. \quad (3.7)$$

Now, by $F \in \mathcal{F}$ and (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} (\Lambda_n)^k F(\Lambda_n) = 0. \quad (3.8)$$

Now, using equation (3.6), we have

$$(\Lambda_n)^k [F(\Lambda_n) - F(\Lambda_0)] \leq -n(\Lambda_n)^k \tau \leq 0. \quad (3.9)$$

Taking the limit as $n \rightarrow \infty$ on both sides of equation (3.9) and using equation (3.8), we obtain

$$\lim_{n \rightarrow \infty} n(\Lambda_n)^k = 0 = \lim_{n \rightarrow \infty} n(\tilde{S}(x_n, x_n, x_{n+1}))^k. \quad (3.10)$$

Therefore, there exists a positive integer $N_1 \in \mathbb{N}$ such that $n(\Lambda_n)^k < 1$ for all $n > N_1$, or

$$\Lambda_n = \tilde{S}(x_n, x_n, x_{n+1}) < \frac{1}{n^{1/k}}. \quad (3.11)$$

Let $m, n \in \mathbb{N}$ with $m > n > N_1$. Then, we have

$$\begin{aligned} \tilde{S}(x_m, x_m, x_n) &\leq \Lambda_{m-1} + \Lambda_{m-2} + \cdots + \Lambda_n \\ &\leq \sum_{r=n}^{m-1} \Lambda_r \leq \sum_{r=n}^{\infty} \Lambda_r \\ &= \sum_{r=n}^{\infty} \frac{1}{r^{1/k}}. \end{aligned} \quad (3.12)$$

As $k \in (0, 1)$ and the series $\sum_{r=n}^{\infty} \frac{1}{r^{1/k}}$ is convergent, so

$$\lim_{m, n \rightarrow \infty} \tilde{S}(x_m, x_m, x_n) = 0. \quad (3.13)$$

Thus $\{x_n\}$ is a Cauchy sequence in an S -metric space $(\Xi, \tilde{S})$. Since $(\Xi, \tilde{S})$ is complete, so there exists an $u \in \Xi$ such that $\lim_{n \rightarrow \infty} x_n = u$. Since by hypothesis $\mathcal{T}$ is continuous, we have

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \mathcal{T}x_n = \mathcal{T}\left(\lim_{n \rightarrow \infty} x_n\right) = \mathcal{T}u.$$

This implies that

$$\mathcal{T}u = u.$$

This shows that $\mathcal{T}$ has a fixed point in Ξ . This completes the proof. $\square$

If we take

$$\begin{aligned} \max \left\{ \tilde{S}(x, y, z), \tilde{S}(x, x, \mathcal{T}x), \tilde{S}(z, z, \mathcal{T}z), \frac{1}{2} [\tilde{S}(x, x, \mathcal{T}y) + \tilde{S}(z, z, \mathcal{T}x)] \right\} \\ = \tilde{S}(x, y, z), \end{aligned}$$

in Theorem 3.2, then we have the following result.

Corollary 3.3. ([17], Theorem 2.1) Let $(\Xi, \tilde{S})$ be a complete S -metric space and $\mathcal{T}: \Xi \rightarrow \Xi$ be a F_α -contraction as defined in Definition 2.18 satisfying the following conditions:

- (1) $\mathcal{T}$ is α -admissible;
- (2) there exists $x_0 \in \Xi$ such that $\alpha(x_0, x_0, \mathcal{T}x_0) \geq 1$;
- (3) $\mathcal{T}$ is continuous.

Then $\mathcal{T}$ has a fixed point.

Remark 3.4. Theorem 3.2 generalizes Theorem 2.1 of Javed et al. [17].

If we take

$$\begin{aligned} \max \left\{ \tilde{S}(x, y, z), \tilde{S}(x, x, \mathcal{T}x), \tilde{S}(z, z, \mathcal{T}z), \frac{1}{2} [\tilde{S}(x, x, \mathcal{T}y) + \tilde{S}(z, z, \mathcal{T}x)] \right\} \\ = \tilde{S}(x, y, z), \end{aligned}$$

and $\alpha(x, y, z) = 1$ in Theorem 3.2, then we have the following result.

Corollary 3.5. Let $(\Xi, \tilde{S})$ be a complete S -metric space and $\mathcal{T}: \Xi \rightarrow \Xi$ be an F -contraction defined as:

$$\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)) \leq F(\tilde{S}(x, y, z)),$$

for all $x, y, z \in \Xi$. Then $\mathcal{T}$ has a fixed point.

Remark 3.6. Corollary 3.5 generalizes Theorem 2.1 of Wardowski [44] from metric space to the setting of S -metric space.

Example 3.7. Let $\Xi = \mathbb{R}$ and $\tilde{S}(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ be an S -metric on Ξ . Suppose that

$$\alpha(x, y, z) = \begin{cases} e^{\max\{x, y\} - z}, & \text{if } \max\{x, y\} \geq z, \\ 0, & \text{if } \max\{x, y\} < z, \end{cases}$$

and

$$F(x) = \frac{1}{2} \sinh x, \quad \mathcal{T}(x) = 0.1.$$

We have to show that the equation (3.1) is satisfied.

Assume that $x \geq y \geq z$. Let $x = 0.3$, $y = 0.2$ and $z = 0.1$. Then, we have

$$\alpha(0.3, 0.2, 0.1) = e^{\max\{0.3, 0.2\} - 0.1} = e^{0.2} = 1.22.$$

Now, we have $\mathcal{T}(0.3) = 0.1$, $\mathcal{T}(0.2) = 0.1$ and $\mathcal{T}(0.1) = 0.1$. So,

$$\begin{aligned} \tilde{S}(x, y, z) &= \tilde{S}(0.3, 0.2, 0.1) = |0.3 - 0.1| + |0.2 - 0.1| = 0.3, \\ \tilde{S}(x, x, \mathcal{T}x) &= \tilde{S}(0.3, 0.3, 0.1) = |0.3 - 0.1| + |0.3 - 0.1| = 0.4, \\ \tilde{S}(z, z, \mathcal{T}z) &= \tilde{S}(0.1, 0.1, 0.1) = |0.1 - 0.1| + |0.1 - 0.1| = 0, \\ \tilde{S}(x, x, \mathcal{T}y) &= \tilde{S}(0.3, 0.3, 0.1) = |0.3 - 0.1| + |0.3 - 0.1| = 0.4, \\ \tilde{S}(z, z, \mathcal{T}x) &= \tilde{S}(0.1, 0.1, 0.1) = |0.1 - 0.1| + |0.1 - 0.1| = 0. \end{aligned}$$

Now by equation (3.1), we have

$$\begin{aligned}\Omega(0.3, 0.2, 0.1) &= \max \left\{ \tilde{S}(0.3, 0.2, 0.1), \tilde{S}(0.3, 0.3, 0.1), \tilde{S}(0.1, 0.1, 0.1), \right. \\ &\quad \left. \frac{1}{2} [\tilde{S}(0.3, 0.3, 0.1) + \tilde{S}(0.1, 0.1, 0.1)] \right\} \\ &= \max\{0.3, 0.4, 0, 0.2\} = 0.4.\end{aligned}$$

Again, we have

$$\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = \tilde{S}(0.1, 0.1, 0.1) = 0,$$

$$F(\alpha(x, y, z)\tilde{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)) = F((1.22).0) = F(0) = \frac{1}{2}\sinh(0) = 0,$$

and

$$\begin{aligned}F(\Omega(x, y, z)) &= F(\Omega(0.3, 0.2, 0.1)) = F(0.4) = \frac{1}{2}\sinh(0.4) = \frac{1}{2}(0.41075233) \\ &= 0.20537616.\end{aligned}$$

Putting these values in equation (3.1), we obtain for taking $\tau = 0.001$

$$0 + 0.001 \leq 0.20537616 \Rightarrow 0.001 \leq 0.20537616.$$

This shows that $\mathcal{T}$ is a generalized F_α -contraction. Now, we will show that $\mathcal{T}$ is α -admissible. Note that

$$\alpha(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = \alpha(0.1, 0.1, 0.1) = e^{\max\{0.1, 0.1\} - 0.1} = e^0 = 1.$$

Consequently,

$$\alpha(x, y, z) \geq 1 \Rightarrow \alpha(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \geq 1.$$

Thus, $\mathcal{T}$ is α -admissible. Now, let $x_0 = 1 \in \Xi = \mathbb{R}$, we have

$$\alpha(x_0, x_0, \mathcal{T}x_0) = \alpha(1, 1, 0.1) = e^{\max\{1, 1\} - 0.1} = e^{0.9} = 2.4596 \geq 1.$$

Also, $\mathcal{T}$ is continuous, since $\mathcal{T}(x) = 0.1$. Hence, $\mathcal{T}$ has a fixed point.

For our next result, we define the following.

Definition 3.8. Let $(\Xi, \tilde{S})$ be an S -metric space and $\alpha: \Xi^3 \rightarrow [0, \infty)$ be a function. A mapping $\mathcal{T}: \Xi \rightarrow \Xi$ is called *Hardy-Rogers-type F_α -contraction* if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that

$$\begin{aligned}\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) > 0 &\Rightarrow \tau + F(\alpha(x, x, y)\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)) \\ &\leq F(a\tilde{S}(x, x, y) + b\tilde{S}(x, x, \mathcal{T}x) + c\tilde{S}(y, y, \mathcal{T}y) \\ &\quad + d\tilde{S}(x, x, \mathcal{T}y) + e\tilde{S}(y, y, \mathcal{T}x)),\end{aligned}\tag{3.14}$$

for all $x, y \in \Xi$, where a, b, c, d, e are nonnegative constants such that $a + b + c + 3d = 1$ and $c \neq 1$.

If we take $\alpha(x, x, y) = 1$, then Definition 3.8 reduces to the following.

Definition 3.9. Let $(\Xi, \tilde{S})$ be an S -metric space and $\alpha: \Xi^3 \rightarrow [0, \infty)$ be a function. A mapping $\mathcal{T}: \Xi \rightarrow \Xi$ is called *Hardy-Rogers-type F -contraction* if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that

$$\begin{aligned} \tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) > 0 &\Rightarrow \tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)) \\ &\leq F(a\tilde{S}(x, x, y) + b\tilde{S}(x, x, \mathcal{T}x) + c\tilde{S}(y, y, \mathcal{T}y) \\ &\quad + d\tilde{S}(x, x, \mathcal{T}y) + e\tilde{S}(y, y, \mathcal{T}x)), \end{aligned} \quad (3.15)$$

for all $x, y \in \Xi$, where a, b, c, d, e are nonnegative constants such that $a + b + c + 3d = 1$ and $c \neq 1$.

Theorem 3.10. Let $(\Xi, \tilde{S})$ be a complete S -metric space and $\mathcal{T}: \Xi \rightarrow \Xi$ be a F_α -contraction of Hardy-Rogers-type satisfying the following conditions:

- (1) $\mathcal{T}$ is α -admissible;
- (2) there exists $x_0 \in \Xi$ such that $\alpha(x_0, x_0, \mathcal{T}x_0) \geq 1$;
- (3) $\mathcal{T}$ is continuous.

Then $\mathcal{T}$ has a fixed point. Moreover, if $a + d + e \leq 1$, then the fixed point of $\mathcal{T}$ is unique.

Proof. Let x_0 be arbitrary. Define a sequence $\{x_n\}$ in Ξ by $x_n = \mathcal{T}x_{n-1}$ for all $n \in \mathbb{N}$. Now, let $\eta_n = \tilde{S}(x_n, x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n \neq x_{n+1}$, that is, $\mathcal{T}x_{n-1} \neq \mathcal{T}x_n$ for all $n \in \mathbb{N}$. By hypothesis (2), we know that $\alpha(x_0, x_0, \mathcal{T}x_0) \geq 1$ and as $\mathcal{T}$ is α -admissible, therefore $\alpha(x_1, x_1, x_2) \geq 1$. So, using the fact that $\mathcal{T}$ is α -admissible and by induction on n , we conclude that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n . Now, using the contractive condition (3.14) with $x = x_{n-1}$, $y = x_n$ and the conditions (S1), (S2) and Lemma 2.19, we obtain

$$\begin{aligned} \tau + F(\eta_n) &= \tau + F(\tilde{S}(x_n, x_n, x_{n+1})) \\ &= \tau + F(\tilde{S}(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}, \mathcal{T}x_n)) \\ &\leq F(\alpha(x_{n-1}, x_{n-1}, x_n)\tilde{S}(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}, \mathcal{T}x_n)) \\ &\leq F(a\tilde{S}(x_{n-1}, x_{n-1}, x_n) + b\tilde{S}(x_{n-1}, x_{n-1}, \mathcal{T}x_{n-1}) \\ &\quad + c\tilde{S}(x_n, x_n, \mathcal{T}x_n) + d\tilde{S}(x_{n-1}, x_{n-1}, \mathcal{T}x_n) \\ &\quad + e\tilde{S}(x_n, x_n, \mathcal{T}x_{n-1})) \\ &= F(a\tilde{S}(x_{n-1}, x_{n-1}, x_n) + b\tilde{S}(x_{n-1}, x_{n-1}, x_n) \\ &\quad + c\tilde{S}(x_n, x_n, x_{n+1}) + d\tilde{S}(x_{n-1}, x_{n-1}, x_{n+1}) \\ &\quad + e\tilde{S}(x_n, x_n, x_n)) \\ &= F(a\eta_{n-1} + b\eta_{n-1} + c\eta_n + d\tilde{S}(x_{n-1}, x_{n-1}, x_{n+1})) \\ &\leq F(a\eta_{n-1} + b\eta_{n-1} + c\eta_n + d[2\eta_{n-1} + \eta_n]) \\ &= F((a + b + 2d)\eta_{n-1} + (c + d)\eta_n). \end{aligned} \quad (3.16)$$

Since F is strictly increasing (by (F1)), we deduce that

$$\eta_n < (a + b + 2d)\eta_{n-1} + (c + d)\eta_n,$$

and hence

$$(1 - c - d)\eta_n < (a + b + 2d)\eta_{n-1}, \text{ for all } n \in \mathbb{N}. \quad (3.17)$$

From assumption of the theorem $a + b + c + 3d = 1$ and $c \neq 1$, we deduce that $1 - c - d > 0$ and so

$$\eta_n < \left(\frac{a + b + 2d}{1 - c - d} \right) \eta_{n-1} = \eta_{n-1}, \text{ for all } n \in \mathbb{N}. \quad (3.18)$$

Consequently, we have

$$\tau + F(\eta_n) < F(\eta_{n-1}), \text{ for all } n \in \mathbb{N}. \quad (3.19)$$

This implies that

$$F(\eta_n) \leq F(\eta_{n-1}) - \tau \leq F(\eta_{n-2}) - 2\tau \leq \cdots \leq F(\eta_0) - n\tau. \quad (3.20)$$

In the same fashion to proof Theorem 3.2, we can conclude that $\mathcal{T}$ has a fixed point. Now, we prove the uniqueness of fixed point. Assume that $v \in \Xi$ is another fixed point of $\mathcal{T}$ such that $u \neq v$. This means that $\tilde{S}(u, u, v) > 0$. Taking $x = u$ and $y = v$ in the contractive condition (3.14) and using Lemma 2.19, we have

$$\begin{aligned} \tau + F(\tilde{S}(u, u, v)) &= \tau + F(\tilde{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v)) \\ &\leq F(\alpha(u, u, v)\tilde{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v)) \\ &\leq F(a\tilde{S}(u, u, v) + b\tilde{S}(u, u, \mathcal{T}u) + c\tilde{S}(v, v, \mathcal{T}v) \\ &\quad + d\tilde{S}(u, u, \mathcal{T}v) + e\tilde{S}(v, v, \mathcal{T}u)) \\ &= F(a\tilde{S}(u, u, v) + b\tilde{S}(u, u, u) + c\tilde{S}(v, v, v) \\ &\quad + d\tilde{S}(u, u, v) + e\tilde{S}(v, v, u)) \\ &= F((a + d + e)\tilde{S}(u, u, v)), \end{aligned}$$

which is a contradiction, if $a + d + e \leq 1$ and hence $u = v$. This shows that the fixed point of $\mathcal{T}$ is unique. This completes the proof. $\square$

Example 3.11. Let $\Xi = \mathbb{R}$ and $\tilde{S}(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ be an S -metric on Ξ . Suppose that

$$\alpha(x, x, y) = \begin{cases} e^{x-y}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases}$$

and

$$F(x) = \frac{1}{2} \sinh x, \quad \mathcal{T}(x) = 0.1.$$

We have to show that the equation (3.14) is satisfied.

Assume that $x \geq y$. Let $x = 0.3$ and $y = 0.2$. Then, we have

$$\alpha(x, x, y) = \alpha(0.3, 0.3, 0.2) = e^{0.3-0.2} = e^{0.1} = 1.10517.$$

Now, we have $\mathcal{T}(0.3) = 0.1$, $\mathcal{T}(0.2) = 0.1$ and $\mathcal{T}(0.1) = 0.1$. So,

$$\begin{aligned} \tilde{S}(x, x, y) &= \tilde{S}(0.3, 0.3, 0.2) = |0.3 - 0.2| + |0.3 - 0.2| = 0.2, \\ \tilde{S}(x, x, \mathcal{T}x) &= \tilde{S}(0.3, 0.3, 0.1) = |0.3 - 0.1| + |0.3 - 0.1| = 0.4, \\ \tilde{S}(y, y, \mathcal{T}y) &= \tilde{S}(0.2, 0.2, 0.1) = |0.2 - 0.1| + |0.2 - 0.1| = 0.2, \\ \tilde{S}(x, x, \mathcal{T}y) &= \tilde{S}(0.3, 0.3, 0.1) = |0.3 - 0.1| + |0.3 - 0.1| = 0.4, \\ \tilde{S}(y, y, \mathcal{T}x) &= \tilde{S}(0.2, 0.2, 0.1) = |0.2 - 0.1| + |0.2 - 0.1| = 0.2. \end{aligned}$$

Now by equation (3.14), we have

$$F(0.2a + 0.4b + 0.2c + 0.4d + 0.2e),$$

since $a + b + c + 3d = 1$ and $c \neq 1$, so if we take $a = c = d = \frac{1}{5}$ and $b = e = 0$ with $a + b + c + 3d = 1$ and $c = \frac{1}{5} \neq 1$, then we have

$$\begin{aligned} F\left(\frac{0.2}{5} + \frac{0.2}{5} + \frac{0.4}{5}\right) &= F\left(\frac{2}{50} + \frac{2}{50} + \frac{4}{50}\right) \\ &= F\left(\frac{8}{50}\right) = F(0.16). \end{aligned}$$

Again, we have

$$\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) = \tilde{S}(0.1, 0.1, 0.1) = 0,$$

$$F\left(\alpha(x, x, y)\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)\right) = F((1.10517).0) = F(0) = \frac{1}{2}\sinh(0) = 0,$$

and

$$\begin{aligned} F\left(a\tilde{S}(x, x, y) + b\tilde{S}(x, x, \mathcal{T}x) + c\tilde{S}(y, y, \mathcal{T}y) + d\tilde{S}(x, x, \mathcal{T}y) + e\tilde{S}(y, y, \mathcal{T}x)\right) \\ = F(0.16) = \frac{1}{2}\sinh(0.16) = \frac{1}{2}(0.16068) = 0.08034. \end{aligned}$$

Putting these values in equation (3.14), we obtain for taking $\tau = 0.001$

$$0 + 0.001 \leq 0.08034 \Rightarrow 0.001 \leq 0.08034.$$

This shows that $\mathcal{T}$ is *Hardy-Rogers-type* F_α -contraction. Now, we will show that $\mathcal{T}$ is α -admissible. Note that

$$\alpha(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) = \alpha(0.1, 0.1, 0.1) = e^{0.1-0.1} = e^0 = 1.$$

Consequently,

$$\alpha(x, x, y) \geq 1 \Rightarrow \alpha(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \geq 1.$$

Thus, $\mathcal{T}$ is α -admissible. Now, let $x_0 = 1 \in \Xi = \mathbb{R}$, we have

$$\alpha(x_0, x_0, \mathcal{T}x_0) = \alpha(1, 1, 0.1) = e^{1-0.1} = e^{0.9} = 2.4596 \geq 1.$$

Also, $\mathcal{T}$ is continuous, since $\mathcal{T}(x) = 0.1$. Hence, $\mathcal{T}$ has a fixed point. Likewise, we can verify the result for taking other values of a, b, c, d and e satisfying the given assumptions.

If we take $\alpha(x, x, y) = 1$ in Theorem 3.10, then we have the following result.

Corollary 3.12. ([6], Theorem 3.4) *Let $(\Xi, \tilde{S})$ be a complete S -metric space and $\mathcal{T}: \Xi \rightarrow \Xi$ be a F -contraction of Hardy-Rogers-type as defined in Definition 3.9, then $\mathcal{T}$ has a fixed point. Moreover, if $a + d + e \leq 1$, then the fixed point of $\mathcal{T}$ is unique.*

If we take $\alpha(x, x, y) = 1$, $a = 1$ and $b = c = d = e = 0$ in Theorem 3.10, then we have the following result.

Corollary 3.13. *Let $(\Xi, \tilde{S})$ be a complete S -metric space. Let $\mathcal{T}: \Xi \rightarrow \Xi$ be a F -contraction and there exists $\tau > 0$ such that*

$$\tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)) \leq F(\tilde{S}(x, x, y)),$$

for all $x, y \in \Xi$, $\mathcal{T}x \neq \mathcal{T}y$. Then $\mathcal{T}$ has a fixed point.

Remark 3.14. Corollary 3.13 generalizes Theorem 2.1 of Wardowski [44] from metric space to the setting of S -metric space.

If we take $\alpha(x, x, y) = 1$, $b + c = 1$, $b \neq 0$ and $a = d = e = 0$ in Theorem 3.10, then we obtain following version of Kannan's result [21] in a complete S -metric space.

Corollary 3.15. *Let $(\Xi, \tilde{S})$ be a complete S -metric space. Let $\mathcal{T}: \Xi \rightarrow \Xi$ be a F -contraction and there exists $\tau > 0$ such that*

$$\tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)) \leq F(b\tilde{S}(x, x, \mathcal{T}x) + c\tilde{S}(y, y, \mathcal{T}y)),$$

for all $x, y \in \Xi$, $\mathcal{T}x \neq \mathcal{T}y$, where $b + c = 1$, $c \neq 1$. Then $\mathcal{T}$ has a unique fixed point.

The Chatterjoe [7] version of fixed point theorem in a complete S -metric space is obtained from Theorem 3.10 by putting $\alpha(x, x, y) = 1$, $a = b = c = 0$, and $d = e = \frac{1}{2}$.

Corollary 3.16. *Let $(\Xi, \tilde{S})$ be a complete S -metric space. Let $\mathcal{T}: \Xi \rightarrow \Xi$ be a F -contraction and there exists $\tau > 0$ such that*

$$\tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)) \leq F\left(\frac{1}{2}[\tilde{S}(x, x, \mathcal{T}y) + \tilde{S}(y, y, \mathcal{T}x)]\right),$$

for all $x, y \in \Xi$, $\mathcal{T}x \neq \mathcal{T}y$. Then $\mathcal{T}$ has a unique fixed point in Ξ .

Finally, if we set $\alpha(x, x, y) = 1$ and $d = e = 0$ in Theorem 3.10, then we obtain a Reich [33] version of fixed point theorem in complete S -metric space.

Corollary 3.17. *Let $(\Xi, \tilde{S})$ be a complete S -metric space. Let $\mathcal{T}: \Xi \rightarrow \Xi$ be a F -contraction and there exists $\tau > 0$ such that*

$$\tau + F(\tilde{S}(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)) \leq F\left(a\tilde{S}(x, x, y) + b\tilde{S}(x, x, \mathcal{T}x) + c\tilde{S}(y, y, \mathcal{T}y)\right),$$

for all $x, y \in \Xi$, $\mathcal{T}x \neq \mathcal{T}y$, where $a + b + c = 1$, $c \neq 1$. Then $\mathcal{T}$ has a unique fixed point in Ξ .

4. CONCLUSION

In this article, we establish some fixed point results for generalized F_α -contraction in the setting of S -metric spaces. Furthermore, we give some corollaries of the main results as a consequence. Also, we give an illustrative example in support of the result. Our results extend, generalize and improve several results from the existing literature (see, for example [6], [21], [33], [44] and some others).

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