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A FOUR-STAGE MULTIDERIVATIVE EXPLICIT RUNGE-KUTTA METHOD FOR THE SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The development of numerical methods for the solution of initial value problems in ordinary differential equation have turned out to be a very rapid research area in recent decades due to the difficulties encountered in finding solutions to some mathematical models composed into differential equations from real life situations. Researchers have in recent times, used higher derivatives in the derivation of numerical methods to produce totally new ways of solving these equations. In this article, a new Runge-Kutta type methods with reduced number of function evaluations in the increment function is constructed, analyzed and implemented. This proposed method border on the use of higher derivatives up to the second derivative in the k_i terms of Runge-Kutta method in order to achieve a higher order of accuracy. The qualitative features: local truncation error, consistency, convergence and stability of the new method were investigated and established. Numerical examples were also performed on some initial value problems and compared with some existing methods to confirm the accuracy of the new method.

1. Introduction

Differential equations are used in real-life as models to problems emanating from physical science, biological science and engineering. Despite the frequent occurrence of these mathematical models, it cannot be denied that most of them do not have analytical solutions hence the need to explore numerical techniques to get accurate approximate solutions in a discrete manner which can easily be tabulated for graphical interpretation of the solutions. New numerical techniques have been constructed by various scholars to solve these models or better the existing ones in terms of order of accuracy, rate of convergence, stability, efficiency and computational cost. Among these standard numerical techniques, there have been growing interest in multiderivative methods to deal with these models that are in form of initial value problems. The list of different multiderivative methods can be found in [1], [2], [3], [5], [8], [10], [12], [13], [17] to mention a few. In this paper, our motivation is drawn from the research works of Goeken [10], Akanbi

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[3] and Wusu [17] to derive a 4-stage multiderivative method that will compete well with some 4-stage methods. We also established the qualitative features of the method and computational experiment were carried out to ascertain the efficiency of the method.

2. Materials and Methods

Consider the general form of a first order initial value problem:

$$y'(x) = f(x, y(x)); \quad y(a) = y_0, \quad a \leq x \leq b \tag{2.1}$$

Numerous numbers of researchers proposed several multiderivative methods in the sense of Runge-Kutta method to solve (2.1). Goeken and Johnson [10] developed a third order method which requires only two evaluations of f, a fourth order method which requires three and fifth order method that requires four. These methods add higher order derivative terms to the Runge-Kutta k_i terms $(i > 1)$ that is they exploit the use of higher order derivative, specifically f_y,to achieve higher order of accuracy. The methods are expressed below:

Third Order Method:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{4}k_1 + \frac{3}{4}k_2 \\ k_1 &= hf(y_n) \\ k_2 &= hf\left(y_n + \frac{2}{3}hk_1 + \frac{2}{9}hf_yk_1\right) \end{aligned} \tag{2.2}$$

Fourth Order Method:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}k_1 + \frac{1}{6}k_2 + \frac{2}{3}k_3 \\ k_1 &= hf(y_n) \\ k_2 &= hf\left(y_n + k_1 + \frac{1}{2}hf_yk_1\right) \\ k_3 &= hf\left(y_n + \frac{3}{8}k_1 + \frac{1}{8}k_2\right) \end{aligned} \tag{2.3}$$

Fifth Order Method:

$$\begin{aligned} y_{n+1} &= y_n + \frac{5}{48}k_1 + \frac{27}{56}k_2 + \frac{125}{336}k_3 + \frac{1}{24}k_4 \\ k_1 &= hf(y_n) \\ k_2 &= hf\left(y_n + \frac{1}{3}k_1 + \frac{1}{18}hf_yk_1\right) \\ k_3 &= hf\left(y_n - \frac{152}{125}k_1 + \frac{252}{125}k_2 - \frac{44}{125}hf_yk_1\right) \\ k_4 &= hf\left(y_n + \frac{19}{2}k_1 - \frac{72}{7}k_2 + \frac{25}{14}k_3 + \frac{5}{2}hf_yk_1\right) \end{aligned} \tag{2.4}$$

In the same vein, Akanbi [3] developed a 2-stage order 4 multiderivative explicit Runge-Kutta methods by utilizing both the first and second derivatives in internal stage that is the k_r term for $r > 1$. The method is given as

$$y_{n+1} = y_n - \frac{1}{2}k_1 + \frac{3}{2}k_2$$

where

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}k_1 + \frac{1}{9}hf_y k_1 + \frac{1}{36}h^2(f_y^2 + ff_{yy})k_1\right) \end{aligned}$$

Also, Wusu A.S. et al.[17] applied the same notion of incorporating higher order derivatives of $f(y)$ up to the second derivative to a construct a new class of three stage Runge-Kutta method given as

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + k_2 + 4k_3)$$

where

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f\left(y_n + hk_1 + \frac{2}{5}h^2 ff_y + \frac{1}{10}h^3(ff_y^2 + f^2 f_{yy})\right) \\ k_3 &= f\left(y_n + \frac{3}{8}hk_1 + \frac{1}{8}hk_2 + \frac{1}{40}h^2 ff_y + \frac{1}{80}h^3(ff_y^2 + f^2 f_{yy})\right) \end{aligned}$$

In this study, attempt is been made to extend the work of Wusu A.S. et al.[17] to 4 stage method but with the order presented by Goeken[10] which would reduce the cost of functions evaluation in the internal stages at the same time maintain high level of accuracy.

3. Derivation of the Proposed Scheme

The proposed scheme tagged 4-Stage MERK method is of the form:

$$y_{n+1} - y_n = \Phi_{MERK}(y_n; h) \tag{3.1}$$

where

$$\Phi_{MERK}(y_n; h) = c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4$$

and

$$\begin{aligned} k_1 &= hf(y) \\ k_2 &= hf\left(y + ha_{21}k_1 + h^2a_{22}ff_y + \frac{h^3}{2}a_{23}(ff_y^2 + f^2f_{yy})\right) \\ k_3 &= hf\left(y + ha_{31}k_1 + ha_{32}k_2 + h^2a_{33}ff_y + \frac{h^3}{2}a_{34}(ff_y^2 + f^2f_{yy})\right) \\ k_4 &= hf\left(y + ha_{41}k_1 + ha_{42}k_2 + ha_{43}k_3 + h^2a_{44}ff_y + \frac{h^3}{2}a_{45}(ff_y^2 + f^2f_{yy})\right) \end{aligned}$$

Expanding K_2, K_3 and K_4, we have

$$\begin{aligned} k_2 &= hc_2f + h^2c_2a_{21}ff_y + \frac{h^3}{2}c_2a_{21}^2f^2f_{yy} + h^3c_2a_{22}ff_y^2 + \frac{h^4}{6}c_2a_{21}^3f^3f_{yyy} + \frac{h^4}{2}c_2(a_{23} \\ &\quad +2a_{21}a_{22})f^2f_yf_{yy} + \frac{h^4}{2}c_2a_{23}ff_y^3 + \frac{h^5}{24}c_2a_{21}^4f^4f_{yyyy} + \frac{h^5}{2}c_2a_{21}a_{23}f^3f_{yy}^2 \\ &\quad +\frac{h^5}{2}c_2a_{21}^2a_{22}f^3f_yf_{yyy} + \frac{h^5}{2}c_2\left(a_{22}^2 + a_{21}a_{23}\right)f^2f_y^2f_{yy}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} k_3 &= hc_3f + h^2c_3a_{31}ff_y + \frac{h^3}{2}c_3a_{31}^2f^2f_{yy} + h^3c_3(a_{21}a_{32} + a_{33})ff_y^2 + \frac{h^4}{6}c_3a_{31}^3f^3f_{yyy} \\ &\quad +\frac{h^4}{2}c_3(a_{21}^2a_{32} + a_{34} + 2a_{21}a_{31}a_{32} + 2a_{31}a_{33})f^2f_yf_{yy} + \frac{h^4}{2}c_3(2a_{22}a_{32} + a_{34})ff_y^3 \\ &\quad +\frac{h^5}{24}c_3a_{31}^4f^4f_{yyyy} + \frac{h^5}{2}c_3\left(a_{21}^2a_{31}a_{32} + a_{31}a_{34}\right)f^3f_{yy}^2 + \frac{h^5}{2}c_3(2a_{22}a_{31}a_{32} \\ &\quad +a_{31}a_{34} + a_{21}^2a_{32}^2 + 2a_{21}a_{32}a_{33} + a_{33}^2 + 2a_{21}a_{22}a_{32} + a_{23}a_{32})f^2f_y^2f_{yy} \\ &\quad +\frac{h^5}{2}c_3(a_{21}a_{32}a_{31}^2 + a_{33}a_{31}^2 + \frac{1}{3}a_{21}^3a_{32})f^3f_yf_{yyy} + \frac{h^5}{2}c_3a_{23}a_{32}ff_y^4, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} k_4 &= hc_4f + h^2c_4a_{41}ff_y + \frac{h^3}{2}c_4a_{41}^2f^2f_{yy} + h^3c_4(a_{21}a_{42} + a_{31}a_{43} + a_{44})ff_y^2 \\ &\quad +\frac{h^4}{6}c_4a_{41}^3f^3f_{yyy} + \frac{h^4}{2}c_4\left(a_{21}^2a_{42} + 2a_{21}a_{41}a_{42} + a_{31}^2a_{43} + 2a_{41}a_{44}\right. \\ &\quad \left.+2a_{31}a_{41}a_{43}\right)f^2f_yf_{yy} + \frac{h^4}{2}c_4\left(2a_{22}a_{42} + 2a_{21}a_{32}a_{43} + 2a_{33}a_{43} + a_{42}a_{45}\right)ff_y^3 \\ &\quad +\frac{h^5}{2}c_4(a_{31}^2a_{41}a_{43} + a_{41}a_{45} + a_{41}a_{42}a_{21}^2)f^3f_{yy}^2 + \frac{h^5}{24}c_4a_{41}^4f^4f_{yyyy} \\ &\quad +\frac{h^5}{2}c_4(2a_{21}a_{32}a_{41}a_{43} + 2a_{33}a_{41}a_{43} + a_{41}a_{45} + 2a_{22}a_{41}a_{42} + 2a_{21}a_{42}a_{44} \\ &\quad +a_{31}^2a_{43}^2 + 2a_{21}a_{31}a_{42}a_{43} + 2a_{31}a_{43}a_{44} + 2a_{33}a_{31}a_{43} + a_{21}^2a_{32}a_{43} + 2a_{22}a_{42}a_{21} \\ &\quad +a_{34}a_{43} + 2a_{21}a_{32}a_{31}a_{43} + a_{21}^2a_{42}^2 + a_{23}a_{42})f^2f_y^2f_{yy} + \frac{h^5}{2}c_4\left(\frac{1}{3}a_{42}a_{21}^3\right. \\ &\quad \left.+\frac{1}{3}a_{31}^3a_{43} + a_{21}a_{42}a_{41}^2 + a_{31}a_{43}a_{41}^2 + a_{44}a_{41}^2\right)f^3f_yf_{yyy} \\ &\quad +\frac{h^5}{6}c_4(3a_{23}a_{42} + 6a_{22}a_{32}a_{43} + 3a_{34}a_{43})ff_y^4 \end{aligned} \tag{3.4}$$

Substituting equations (3.2), (3.3) and (3.4) in (3.1) and comparing with the Taylor's series expansion of y_{n+1} about (x_n, y_n) up to order $O(h^5)$ to obtain some system of equations. With the help of some free parameters: $a_{21} = 1$ and $a_{32} = a_{43} = 0$, these system of equations were solved using MATHEMATICA to get the following parameters:$c_1 = \frac{6}{35}$,$c_2 = \frac{11}{35}$,$c_3 = \frac{3}{7}$,$c_4 = \frac{3}{35}$, $a_{22} = \frac{3}{4}$, $a_{23} = -\frac{1}{2}$,$a_{31} =$

$\frac{4}{5}$,$a_{33} = \frac{32}{25}$,$a_{34} = -\frac{48}{25}$, $a_{41} = \frac{6}{25}$,$a_{42} = -\frac{7}{5}$, $a_{44} = \frac{3}{5}$ and $a_{45} = \frac{32}{5}$. These parameters are then substituted in the proposed method (3.1) to get:

$$y_{n+1} - y_n = \frac{1}{70}\left(12k_1 + 22k_2 + 30k_3 + 6k_4\right) \tag{3.5}$$

where

$$\begin{aligned}
k_1 &= hf(y) \\
k_2 &= hf\left(y + hk_1 + \frac{3}{4}h^2 ff_y - \frac{1}{4}h^3(ff_y^2 + f^2 f_{yy})\right) \\
k_3 &= hf\left(y + \frac{4}{5}hk_1 + \frac{32}{25}h^2 ff_y - \frac{24}{25}h^3(ff_y^2 + f^2 f_{yy})\right) \\
k_4 &= hf\left(y + \frac{6}{25}hk_1 - \frac{7}{5}hk_2 + \frac{3}{5}h^2 ff_y + \frac{16}{5}h^3(ff_y^2 + f^2 f_{yy})\right)
\end{aligned}$$

4. Qualitative Features

We will consider some basic features which are very vital to the development of the constructed method. These features are local truncation error, consistency, stability and convergence.

4.1. Local Truncation Error.

Definition 4.1. (Lambert[11]). The local truncation error T_{n+1} at x_{n+1} of the general one step method is given as

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n), h)$$

where $y(x_n)$ is the theoretical solution.

The local truncation error of the constructed method in compliance with definition 4.1 can be expressed as

$$T_{n+1} = y(x_{n+1}) - y_{n+1}$$

Definition 4.2. (Lambert[11]). A numerical method is said to be of order p if p is the largest integer for which $T_{n+1} = O(h^{p+1})$ for every n and $p \geq 1$.

Consequently, the local truncation error of the proposed method is as follows:

$$\begin{aligned}
T_{n+1} = \frac{h^6}{1440}\big(&ff_y^5 - 2f^2 f_y^3 f_{yy} + 2f^3 f_y f_{yy}^2 f_{yyy} \\
&-5f^4 f_{yy} f_{yyy} - 4f^3 f_y^2 + 3f^4 f_y f_{yyyy} - f^5 f_{yyyyy}\big)
\end{aligned}$$

Theorem 4.3. *(Lambert[11]). Let $f(x,y)$ belongs to $C^4[a,b]$ and let it's partial derivatives be bounded and if if there exist L, M some positive constants such that*

$$|f(x,y)| < M, \qquad \left|\frac{\delta^{i+j}}{\delta x^i \delta y^j}\right| < \frac{L^{i+j}}{M^{i-j}}, \qquad i + j < M.$$

Hence, by virtue of Lotkin in Lambert [11], the strict upper bound with respect to y for 4 - stage MERK method is given as:

$$|LTE_{MERK}| \leq \frac{1}{240}h^5 ML^4 + O(h)^6$$

4.2. **Consistency.**

Definition 4.4. (Lambert[11]). A numerical method is said to be consistent with an initial value problem if

$$\phi(x, y, 0) \equiv f(x, y)$$

Thus a consistent method has at least order one.

Definition 4.5. (Lambert[11]). A method is said to be consistent if the difference equation of the integrating formula exactly approximates the differential equation it intends to solve as the step size approaches zero.

In order to establish the consistency property of the proposed method it is sufficient to show that

$$\lim_{h\to 0} \phi(x_n, y_n; h) = f(x_n, y_n)$$

where $\phi(x_n, y_n; h)$ is the increment function of the numerical method. Dividing $\phi(x_n, y_n; h)$ all through by h and taking limit as h tends to zero on both sides, we have

$$\lim_{h\to 0} \phi_{4MERK}(y_n; h) = (c_1 + c_2 + c_3 + c_4)f = f(x_n, y_n)$$

where $c_1 + c_2 + c_3 + c_4 = 1$. Hence, 4MERK method is consistent.

4.3. **Stability.** The stability of numerical methods for solving an IVP in ODE can be analyzed using the linear test problem $y' = \lambda y$ proposed by Dalquist [6], where the solution is $y = e^{\lambda y}$ and λ a complex variable. Upon applying $\phi_{MERK}(y_n; h)$ of 4-Stage MERK method to the linear test problem, the stability polynomial $R(z) = \frac{y_{n+1}}{y_n}$ where $z = \lambda h$ is obtained as follows:

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{124}z^5$$

and the stability region is expressed in the figure below.

4.4. **Convergence.** We will test for the convergence of the derived method using the following definitions and theorem.

Definition 4.6. (Dalquist[6]). A numerical method is said to be convergent if for all initial value problems satisfying the hypothesis of the Lipschitz condition given by

$$|f(x, y) - f(x, y^*)| \leq L|y - y^*|$$

where the Lipschitz constant L is denoted by $L = \max |f_y(x, y)|$.

Theorem 4.7. *(Dalquist[6]). The necessary and sufficient conditions for a numerical method to be convergent is for it to be consistent and stable.*

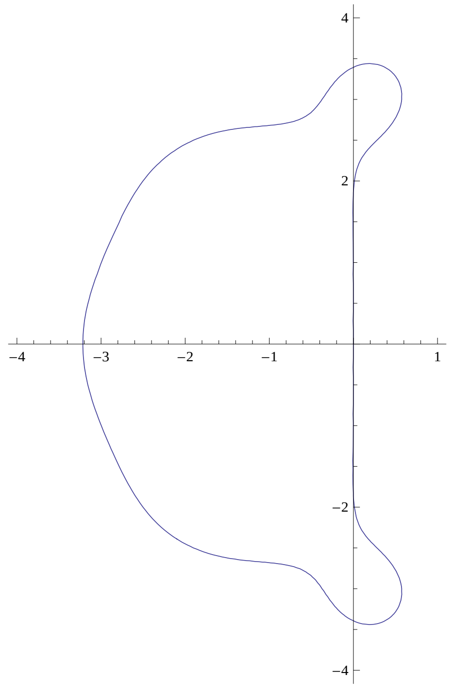

Figure 1. Stability Region of the Derived Scheme

Definition 4.8. (Lambert[11]). A numerical method is said to be convergent if it is consistent and has an order greater than one.
From theorem 4.7 and definition 4.8, we can conclude that the derived method is convergent.

5. Numerical Experiment

For the purpose of testing the performance and suitability of the method scheme, some standard initial value problems were solved with the aid of MATHEMATICA. Comparisons are made with some existing methods such as the methods derived in Wusu et al.[15] (4HERK) and Goeken D. and Johnson O.[10] (4GM) to ascertain the level of accuracy of the derived method.

5.1. Numerical Examples.

In order to prove the efficiency of the derived method, the following initial value problems will be solved.

5.1.1. *Problem 1:Source(Wusu et al. [15]).* Consider the IVP:

$$y' = -y, \quad y(0) = 1$$

whose analytical solution is given as

$$y(x) = e^{-x}$$

The global error values obtained with step sizes 0.1, 0.125, 0.01 and 0.0125 are presented in Table 1.

Table 1. Numerical Results for Problem 1

Stepsizes	Methods	Global Error for the Methods
	$4GM$	$5.996203428208E-6$
0.125	$4HERK$	$4.257625053977E-6$
	$4MERK$	$2.750255004251E-6$
	$4GM$	$3.996620342820E-5$
0.1	$4HERK$	$1.253811455002E-6$
	$4MERK$	$1.075025457425E-6$
	$4GM$	$7.910026861876E-8$
0.0125	$4HERK$	$9.890201216826E-8$
	$4MERK$	$5.486944210663E-9$
	$4GM$	$5.996020342820E-8$
0.01	$4HERK$	$8.349765306846E-8$
	$4MERK$	$3.750255004235E-9$

5.1.2. *Problem 2:Source(Wusu et al.* [15]*).* Consider the IVP:

$$y' = \frac{y}{4} - \frac{y^2}{80}, \qquad y(0) = 1$$

whose analytical solution is given as

$$y(x) = \frac{20}{1 + 19e^{\frac{-x}{4}}}$$

The global error values obtained with step sizes 0.1, 0.125, 0.01 and 0.0125 are presented in Table 2 .

6. Conclusion

In this paper, our research has been devoted to deriving a numerical method to approximate an initial value problem in ordinary differential equations. This method is capable of solving initial value problems arising in various fields of science and engineering. The derivation of the method is followed by the qualitative analysis: local truncation error, consistency, convergence and stability wherein it exhibited satisfactory performance. The stability region of the derived method displayed on Figure 1 revealed that the new method is stable like every other existing numerical methods in the relevant literature. The test problems under consideration were solved by the derived method and some standard methods of the same stages where it is easy to observe from Tables 1 and 2 that the new method produced smaller error when compared with some existing four stage numerical methods . Thus, the new developed method can reliably be used as

Table 2. Numerical Results for Problem 2

Stepsizes	Methods	Global Error for the Methods
	$4GM$	$9.108376540938E-5$
0.125	$4HERK$	$8.840629813946E-6$
	$4MERK$	$3.520985508217E-7$
	$4GM$	$7.620207391868E-5$
0.1	$4HERK$	$6.209863820146E-6$
	$4MERK$	$1.298400847625E-7$
	$4GM$	$7.910026861876E-8$
0.0125	$4HERK$	$9.079219748265E-7$
	$4MERK$	$8.610795732186E-9$
	$4GM$	$5.996020342820E-7$
0.01	$4HERK$	$6.896327039517E-7$
	$4MERK$	$5.057395986109E-9$

a substitute for some of the existing 4 stage explicit numerical methods to solve some initial value problems in ordinary differential equations.

References

1. O. Adeyeye, S. Qureshi and A. A. Shaikh, *Use of Partial Derivative to Derive a Convergent Numerical Scheme With Its Error Estimate*, Journal of Applied Mathematics and Computational Mechanics, 18 (2019), 73-83. https://doi.org/10.17512/jamcm.2019.4.07
2. I.B. Aiguobasimwin and R.I. Okuonghae, *A Class of Two-Derivative Two-Step Runge-Kutta Method for Non-Stiff ODEs*, Journal of Applied Mathematics, 33 (2019), 153-162. https://doi.org/10.1155/2019/2459809
3. M. A. Akanbi, S. A. Okunuga and A. B. Sofoluwe, *A family of 2-Stage Multiderivative Explicit Runge–Kutta Methods for Non-Stiff Ordinary Differential Equations*, National Mathematical Centre, Vol. 6 (2005), No. 1, 87-100.
4. J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*. Journal of Computational and Applied Mathematics, Vol. 125 (2003), No. 1-2, pp. 1-29. https://doi.org/10.1002/0470868279
5. R. P. Chan and A. Y. Tsai, *On explicit two derivative Runge-Kutta methods*, Applied Numerical Algorithm, Vol. 53 (2010), pp. 171–194. https://doi.org/10.1007/s11075-009-9349-1
6. G. Dahlquist, *Convergence and Stability in the Numerical Integration of ODEs*, Math. Scand. 4:33-53, 1956. https://doi.org/10.7146/math.scand.a-10454
7. S. O. Fatunla, *Numerical Methods for Initial Value Problems in Ordinary Differential Equations*, Academic Press, New York, 1988. https://doi.org/10.1016/c2013-0-10643-5
8. S. Geleker and T. Widmann, *On the Order Conditions of Runge-Kutta Method with Higher Derivatives*, Numerical Mathematics 50 2 (1986),183-203. https://doi.org/10.1007/bf01390429

9. D. Goeken and O. Johnson, *Fifth-Order Runge-Kutta with Higher Order Derivative Approximations*, Electronic Journal of Differential Equations, Vol. **2** (1999), 1-9. https://doi.org/10.1016/s0168-9274(99)00128-2
10. D. Goeken and O. Johnson, *Runge-Kutta with Higher Order Derivative Approximations*,Applied Numerical Mathematics, **34** (2000), 207-218. https://doi.org/10.1016/s0168-9274(99)00128-2
11. J.D. Lambert, *Numerical Methods for Ordinary Differential Systems*, John Wiley and Sons, 1991. https://doi.org/10.1002/0470868279
12. M.I. Mohammed, *Goeken-Johnson Sixth Order Runge-Kutta Method*, J. Edu. & Sci., Vol. **24** No. 1, 2011. https://doi.org/10.33899/edusj.2011.51502
13. O. T. Mukaddes and O. Turgut, *Derivation of Third derivative Runge-Kutta Method*, Numerical Algorithms, **74** (2017), 247-265. https://doi.org/10.1007/s11075-016-0147-2
14. A.S. Olaniyan, M. A. Akanbi and O. F. Bakre, 2-*Stage Implicit Runge-Kutta Methods Based on Heronian Mean for Solving Ordinary Differential equations*, Pure and Applied Mathematics Journal, Vol. **9** No. 5(2020), 84-90. https://doi.org/10.11648/j.pamj.20200905.11
15. A.S. Wusu., M. A. Akanbi and O. F. Bakre, *On the Derivation and Implementation of a Four-Stage Harmonic Explicit Runge-Kutta Method*, Applied Mathematics, **6** (2015), 694-699. https://doi.org/10.4236/am.2015.64064
16. A.S. Wusu, S. A. Okunuga and A. B. Sofoluwe, *A Third-Order Harmonic Explicit Runge-Kutta Method for Autonomous Initial Value Problems*, Global Journal of Pure and Applied Mathematics, **8** (2012), 441-451. https://doi.org/10.1007/s00607-010-0139-3
17. A.S. Wusu, M. A. Akanbi and S. A. Okunuga, *A Three Stage Multiderivative Explicit Runge-Kutta Method*, American Journal of Computational Mathematics (AJCM), Vol.**3** (2013), 121-126. https://doi.org/10.4236/ajcm.2013.32020

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