Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 20 (2024) 1-15 https://doi.org/10.56947/amcs.v20.223

BI-UNIQUE RANGE SETS FOR POWERS OF MEROMORPHIC FUNCTIONS

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Abstract. In the paper, we study the uniqueness of powers of meromorphic functions for sharing two finite sets with finite weights and find Bi-unique range sets for meromorphic functions of power at least d. Examples are given in support of our results. We are inspired by the results due to A. Banerjee [Nihonkai Math. J., 24 (2013), 121-134] and that of S. Mallick and D. Sarkar [Mat. Stud., 50 (2018), 143-157.]

1. Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [11, 21, 22]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $(r \to \infty, r \notin E)$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$, by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ as $(r \to \infty, r \notin E)$. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to f, provided that $T(r, a) = S(r, f)$. We say that two nonconstant meromorphic functions f and g share a value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with same multiplicities. Similarly, we say that f and g share the value a IM (ignoring multipliticities) if $f - a$ and $g - a$ have the same zeros ignoring multiplicities.

In the beginning of this century, I. Lahiri has introduced the concept of weighted sharing which is a scaling between CM and IM. The definition is as follows.

Definition 1.1. [14, 15] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with

Date: Received: Oct 8, 2023; Accepted: Nov 5, 2023.
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2010 *Mathematics Subject Classification.* Primary 30D35.
Key words and phrases. Uniqueness, Meromorphic function, Set sharing, Bi-Unique range sets, Weighted sharing.

multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [14] For $S \subset \mathbb{C} \cup \{\infty\}$ and a nonconstant meromorphic function f, we define $E_f(S, k) = \cup_{a \in S} E_k(a; f)$, where k is a non-negative integer. Two nonconstant meromorphic functions f and g share the set S with weight k, if $E_f(S, k) = E_g(S, k)$. For $k = \infty$, i.e., for CM sharing we write $E_f(S) = E_g(S)$.

Most of the research works related to set sharing problems was broadly initiated due to the following question raised by F. Gross [10] in 1976.

Question 1.3. *Can one find two finite sets $S_i (i = 1, 2)$ of $\mathbb{C} \cup \{\infty\}$ such that any two nonconstant entire functions f and g satisfying $E_f(S_i) = E_g(S_i)$, $i = 1, 2$ must be identical?*

In 1994, H.X. Yi [24] gave an affirmative answer to the above question by proving the following theorem.

Theorem 1.4. *Let $S_1 = \{\omega \mid \omega^n - 1 = 0\}$ and $S_2 = \{a\}$, where $n(\geq 5)$ is an integer, $a \neq 0$ and $a^{2n} \neq 1$. If f and g are entire functions such that $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$, then $f \equiv g$.*

In [10] F. Gross also expressed his quest about how large the sets can be if the answer of Question 1.3 is affirmative. In 1998, H.X. Yi [26] proved the following theorem regarding to the above comment.

Theorem 1.5. *Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^2(\omega + a) - b = 0\}$, where a and b are nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If f and g are entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In 2003, the following question, analogous to Gross' question, was asked by W.C. Lin and H.X. Yi in [18].

Question 1.6. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, must be identical ?*

Beside the URSE, the concepts of URSE-IM, URSM, URSM-IM have been introduced and a lot of research has been done in this direction (see [2]-[9],[12],[19], [23]-[28]). From those results it can observed that the cardinality of any unique range set for meromorphic (resp. entire) function must be greater than 5 (resp. 4) (see [21], pages 527, 517) but the smallest URSM (resp. URSE) and URSM-IM (resp. URSE-IM) obtained were of cardinality 11 (resp. 7) (see [8, 16, 25]) and 17 (resp. 10) (see [6, 9, 27]) respectively. In 2012, A. Banerjee and I. Lahiri [4] introduced the concept of URSMk (resp. URSEk) called URSM (resp. URSE) with weight k. This attempt relaxed the nature of sharing but cardinality could not be diminished. In 2013, A. Banerjee [2] introduced the concept of Bi-unique range sets for meromophic functions with weights l, m as follows:

Definition 1.7. Let $M(\mathbb{C})$ (resp. $E(\mathbb{C})$) be the set of all meromorphic (resp. entire) functions. A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is said to be the bi-unique range sets for meromorphic (resp. entire) functions with weights l, m respectively, in notation $BURSM(l, m)$ (resp. $BURSE(l, m)$), if for any two nonconstant meromorphic (resp. entire) functions f, g; $E_f(S_1, l) = E_g(S_1, l)$ and $E_f(S_2, m) = E_g(S_2, m)$ imply $f = g$.

Remark 1.8. Note that if both the weight be ∞, then that pair of sets will be called BURSM (resp. BURSE).

In 2012, B. Yi and Y.H. Li [23] have found BURSM, prior to its naming, with collective cardinality 7. In [2] the author proved the following result.

Theorem 1.9. *Let* $S_1 = \{0, 1\}$, $S_2 = \left\{z : \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c = 0\right\}$, *where* $n(\geq 5)$ *is an integer and* $c(\neq 0, 1, \frac{1}{2})$ *is a complex number such that* $c^2 - c + 1 \neq 0$. *Then* S_i*'s* $(i = 1, 2)$ *are* $BURSM(1, 3)$ *and* $BURSM(3, 2)$.

Remark 1.10. In this result the collective cardinality is 7 like that of [23] but the nature of sharing has been relaxed.

In 2016, A. Banerjee and S. Mallick [5] found Bi-unique range sets with weights having collective cardinality 5 for k-th derivatives of meromorphic functions. In 2018, S. Mallick and D. Sarkar [19] have introduced the concept of unique range sets for power of meromorphic functions with weights k (in notation $URSP^dMk$), which is as follows:

Definition 1.11. Let $M(\mathbb{C})$ denotes the set of all meromorphic functions in $\mathbb{C}$. We define $M^d(\mathbb{C})$ for $d \in \mathbb{N}$, as the set $M^d(\mathbb{C}) = \{f^{d+r} | f \in M(\mathbb{C}), r \in \mathbb{N} \cup \{0\}\}$.

Similar notions can be defined for entire functions and is denoted by $E^d(\mathbb{C})$.

Definition 1.12. A non empty set $S \subset \mathbb{C}$ is said to be unique range set for meromorphic (resp. entire) functions of power atleast d with weight k, in notation $URSP^dMk$ (resp. $URSP^dEk$), if for any two nonconstant meromorphic (resp. entire) functions $f, g \in M^d(\mathbb{C})$ (resp. $E^d(\mathbb{C})$); $E_f(S, k) = E_g(S, k)$.

In the same paper the authors found following $URSP^dMk$ and $URSP^dEk$ with cardinality 5 or 4 :

(1) $URSP^5M0$, $URSP^4M1$, $URSP^3M2$, $URSP^4E0$, $URSP^2E1$, with cardinality 5.
(2) $URSP^8M0$, $URSP^5M1$, $URSP^4E0$, $URSP^3E1$, with cardinality 4.

In this connection question arises about existence of Bi-unique range sets with weight having collective cardinality less than 5 for some special group of meromorphic (resp. entire) functions. In this direction we define Bi-unique range sets for power of meromorphic (resp. entire) functions with weights as follows:

Definition 1.13. A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is said to be the Bi-unique range sets for meromorphic (resp. entire) functions of power atleast d with weights l, m respectively, in notation $BURSP^dM(l,m)$ (resp. $BURSP^dE(l,m)$), if for any two nonconstant meromorphic (resp. entire) functions $f, g \in M^d(\mathbb{C})$ (resp. $E^d(\mathbb{C})$); $E_f(S_1,l) = E_g(S_1,l)$ and $E_f(S_2,m) = E_g(S_2,m)$ imply $f = g$.

Now our objective is to find a possible answer of the following question:

Question 1.14. *Can one find $BURSP^dM(l,m)$ and $BURSP^dE(l,m)$ with collective cardinality less than 5 ? If yes, what will be the nature of sharing ?*

Here we present an affirmative answer for this question. Our findings are as follows:

Theorem 1.15. *Let $P(z) = z^n + az^{n-1} + b$ be a polynomial having only simple zeros. Also, let $S_1 = \{0\}$ and $S_2 = \{\omega | P(\omega) = 0\}$. Then for $d \geq 2$, S_i's $(i = 1, 2)$ are*

(1) *$BURSP^dM(0,2)$ for $n > \max\left\{\frac{3}{2} + \frac{6}{d}, 2 + \frac{1}{d}\right\}$,*
(2) *$BURSP^dM(0,1)$ for $n > \max\left\{\frac{3}{2} + \frac{36}{5d}, 2 + \frac{1}{d}\right\}$.*

Remark 1.16. From Theorem 1.15 we see that S_1 and S_2 are :

(1) $BURSP^2M(0,1)$ for $n \geq 6$.
(2) $BURSP^3M(0,1)$ for $n \geq 4$.
(3) $BURSP^5M(0,1)$ for $n \geq 3$.
(4) $BURSP^2M(0,2)$ for $n \geq 5$.
(5) $BURSP^3M(0,2)$ for $n \geq 4$.
(6) $BURSP^5M(0,2)$ for $n \geq 3$.

Thus we get Bi-unique range sets with weights 0 and 1 having collective cardinality 4 for at least 5^{th} power of meromorphic functions.

The following example shows that the assumption $d \geq 2$ in Theorem 1.15 is necessary.

Example 1.17. Let $S_1 = \{0\}$ and $S_2 = \{z : P(z) = 0\}$, where $P(z)$ is defined as in Theorem 1.15. Also let $h(z) = \frac{e^z - 1}{e^{nz} - 1}$. Then, the functions $f(z) = a\{h(z) - 1\}$ and $g(z) = e^{-z}f(z)$ share S_1, S_2 CM but $f \not\equiv g$.

Theorem 1.18. *Let S_1, S_2 be defined as in Theorem 1.15. Then S_i's $(i = 1, 2)$ are*

(1) *$BURSP^dE(0,1)$ and $BURSE(0,1)$ for $n \geq 4$,*
(2) *$BURSP^dE(0,2)$ and $BURSE(0,2)$ for $n \geq 3$.*

The following example shows that the assumption $n \geq 3$ is mandatory for $BURSP^1E(0,2)$ (or $BURSE(0,2)$), $BURSP^5M(0,2)$ and $BURSP^5M(0,1)$.

Example 1.19. Let α, β be the roots of the equation $z^2 + az + b = 0$. Also let $S_1 = \{0\}$ and $S_2 = \{\alpha, \beta\}$. Then the functions f^d and g^d, where $f(z) = \frac{z - z_0}{z_1 - z_2}\left\{\frac{z - z_2}{z_1 - z_0} - \frac{z - z_1}{z_2 - z_0}\right\}e^z$ and $g(z) = b^{\frac{1}{d}}\frac{z - z_0}{z_1 - z_2}\left\{\frac{z - z_2}{z_1 - z_0} - \frac{z - z_1}{z_2 - z_0}\right\}e^{-z}$ for $z_1 = \frac{1}{d}\log\alpha$ and $z_2 = \frac{1}{d}\log\beta$, share S_i, $i = 1, 2$ CM but $f^d \not\equiv g^d$, for $d = 1, 5$ (as the case).

Now we present some useful definitions.

Definition 1.20. [13] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function. Analogously, we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 1.21. [1] Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a $1-$point of f with multiplicity p and a $1-$point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those $1-$points of f and g, where $p > q$, by $N_E^{(k}(r, 1; f)$ ($k \geq 2$ is an integer) the counting function of those $1-$points of f and g, where $p = q \geq k$, where each $1-$point is counted only once. In the same manner we can define $\overline{N}_L(r, 1; g)$ and $N_E^{(k}(r, 1; g)$.

Definition 1.22. [14, 15] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those $a-points$ of f and g whose multiplicities are not equal. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

We now give some lemmas which will be needed to prove the results of this paper. Let f_1 and g_1 be two nonconstant meromorphic (resp. entire) functions defined in the complex plane $\mathbb{C}$. Let us define f, g, F, G as

$$f = f_1^{d+r},\ g = g_1^{d+s},\ F = \frac{f^{n-1}(f+a)}{-b},\ G = \frac{g^{n-1}(g+a)}{-b}, \tag{2.1}$$

where d is a positive integer and r, s are nonnegative integers. Also we define H and Φ as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{2.2}$$

and

$$\Phi = \left(\frac{F'}{F-1} - \frac{G'}{G-1}\right). \tag{2.3}$$

Lemma 2.1. [20] *Let f be a nonconstant meromorphic function and $P(f)$ be a polynomial in f of degree n. Then*

$$T\left(r, P(f)\right) = nT(r, f) + S(r, f).$$

Lemma 2.2. [28] *Let F and G be two nonconstant meromorphic functions sharing $(1, 0)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r, 1; F) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.3. [3] *Let F and G be two nonconstant meromorphic functions sharing $(1,m)$ where $0 \leq m < \infty$. Then*

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\leq \frac{1}{2}[N(r,1;F)+N(r,1;G)]+N_E^{1)}(r,1;F)\\ &- \left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G).\end{aligned}$$

Lemma 2.4. *Let F and G be given by (2.1) and $H(\not\equiv 0)$ be given by (2.2). If f,g share $(0,l)$ and F,G share $(1,m)$ where $0 \leq l < \infty$ and $0 \leq m < \infty$, then*

$$\begin{aligned}N(r,H) &\leq \overline{N}\left(r,-\frac{a(n-1)}{n};f\right)+\overline{N}\left(r,-\frac{a(n-1)}{n};g\right)+\overline{N}_*(r,1;F,G)\\ &+ \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,0;f,g)\\ &+ \overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to those zeros of f' which are not the zeros of $f\left(f+\frac{a(n-1)}{n}\right)(F-1)$. $\overline{N}_0(r,0;g')$ is defined similarly.

Proof. Since F,G share $(1,m)$, and f,g share $(0,l)$, H has only simple poles. Poles of H come from the zeros of $F'G'(F-1)(G-1)$ and poles of f and g. Zeros of $F'G'$ comes from zeros of $ff'gg'\{nf+a(n-1)\}\{ng+a(n-1)\}$. Also noting that the zeros of f and g with same multiplicities and $1-$points of F and G with same multiplicities are not pole of H, the result is obvious. □

Lemma 2.5. *Let F and G be given by (2.1) and $H(\not\equiv 0)$ be given by (2.2). If f, g share $(0,l)$ and F, G share $(1,m)$ where $0 \leq l < \infty$ and $0 \leq m < \infty$, then*

$$\begin{aligned}\left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq 2[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]+\overline{N}_*(r,0;f,g)\\ &+ \overline{N}(r,0;f)+\overline{N}(r,0;g)\\ &- \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. Let ω_j, $j=1,2,\cdots,n$ be the zeros of the polynomial $P(z)$ defined as in Theorem 1.15. Then by Nevanlinna's second fundamental theorem, we get

$$\begin{aligned}(n+1)T(r,f) &\leq \sum_{j=1}^{n}\overline{N}(r,\omega_j;f)+\overline{N}\left(r,-\frac{a(n-1)}{n};f\right)\\ &+ \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;F)-\overline{N}_0(r,0;f')\\ &+ \overline{N}\left(r,-\frac{a(n-1)}{n};f\right)+S(r,f). \qquad (2.4)\end{aligned}$$

Similarly,

$$\begin{aligned}(n+1)T(r,g) &\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;G)+\overline{N}\left(r,-\frac{a(n-1)}{n};g\right)\\ &- \overline{N}_0(r,0;g')+S(r,g). \qquad (2.5)\end{aligned}$$

Combining (2.4) and (2.5) we get

$$\begin{aligned}(n+1)[T(r,f)+T(r,g)] &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;F)-\overline{N}_0(r,0;f')\\ &+ \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;G)-\overline{N}_0(r,0;g')\\ &+ \overline{N}\left(r,-\frac{a(n-1)}{n};f\right)+\overline{N}\left(r,-\frac{a(n-1)}{n};g\right)\\ &+ S(r,f)+S(r,g). \qquad (2.6)\end{aligned}$$

Now using Lemma 2.3, Lemma 2.2 and Lemma 2.4 successively we get,

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\leq \frac{1}{2}[N(r,1;F)+N(r,1;G)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &+ \overline{N}\left(r,-\frac{a(n-1)}{n};f\right)+\overline{N}\left(r,-\frac{a(n-1)}{n};g\right)\\ &- \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\\ &+ \overline{N}_*(r,0;f,g)+S(r,f)+S(r,g). \qquad (2.7)\end{aligned}$$

Combining (2.6) and (2.7) we get

$$\begin{aligned}(n+1)[T(r,f)+T(r,g)] &\leq \frac{1}{2}[N(r,1;F)+N(r,1;G)]+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\ &+ 2[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]+\overline{N}_*(r,0;f,g)\\ &+ 2\left[\overline{N}\left(r,-\frac{a(n-1)}{n};f\right)+\overline{N}\left(r,-\frac{a(n-1)}{n};g\right)\right]\\ &- \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\leq \left(\frac{n}{2}+2\right)[T(r,f)+T(r,g)]+\overline{N}_*(r,0;f,g)\\ &+ \overline{N}(r,0;f)+\overline{N}(r,0;g)+2[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]\\ &- \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Therefore

$$\begin{aligned}\left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,0;f,g)\\ &+ 2[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]\\ &- \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

□

Lemma 2.6. *Let F and G be given by (2.1) and $H(\not\equiv 0)$ be given by (2.2). If f, g share $(0,l)$ and F, G share $(1,m)$ where $0\leq l<\infty$ and $0\leq m<\infty$, then*

$$\begin{aligned}\{(n-1)l+(n-2)\}\overline{N}(r,0;f|\geq l+1) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &+ +\overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. Let us consider the following two cases.
Case 1: $\Phi\equiv 0$. Then

$$\frac{F'}{F-1}\equiv\frac{G'}{G-1}.$$

Integrating we get $F-1=A(G-1)$ where A is an arbitrary constant. Therefore $F'=AG'$ and $F''=AG''$. Then $H\equiv 0$, which is a contradiction.
Case 2: $\Phi\not\equiv 0$. Then

$$\Phi=\frac{f^{n-2}f'\{nf+a(n-1)\}}{-b(F-1)}-\frac{g^{n-2}g'\{ng+a(n-1)\}}{-b(G-1)}.$$

Let z_0 be a zero of f with multiplicity p. Since f and g share $(0,l)$, z_0 is a zero of Φ with multiplicity at least $(n-2)p+(p-1)=(n-1)p-1$ if $p\leq l$ and a zero with multiplicity at least $(n-2)(l+1)+l=(n-1)l+(n-2)$ if $p>l$. Therefore

$$\begin{aligned}\{(n-1)l+(n-2)\}\overline{N}(r,0;f|\geq l+1) &\leq N(r,0;\Phi)\\ &\leq T(r,\Phi)\\ &\leq N(r,\infty;\Phi)+S(r,f)+S(r,g).\end{aligned}$$

Since poles of Φ come from 1-points of F and G with distinct multiplicities, we get from above

$$\begin{aligned}\{(n-1)l+(n-2)\}\overline{N}(r,0;f|\geq l+1) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &+ \overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

This proves the lemma. □

Lemma 2.7. *Let S_1 and S_2 be defined as in Theorem 1.15 and F, G, H are given by (2.1) and (2.2). If $E_f(S_1,l)=E_g(S_1,l)$, $E_f(S_2,m)=E_g(S_2,m)$ and $H\not\equiv 0$, where $0\leq l<\infty$ and $1\leq m<\infty$, then*

$$\begin{aligned}\overline{N}_{*}(r,1;F,G) &\leq \frac{(n-1)d+1}{2\{m(n-1)d-(m+1)\}}[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]\\ &+ S(r,f)+S(r,g).\end{aligned}$$

Proof. Since $f=f_1^{d+r}$ and $g=g_1^{d+s}$ with $d\geq 1$, here $\overline{N}(r,0;f)=\overline{N}(r,0;f|\geq d)$ and $\overline{N}(r,0;g)=\overline{N}(r,0;g|\geq d)$. Also since F and G share $(1,m)$, we have

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &\leq \overline{N}(r,1;F|\geq m+1)\leq \frac{1}{m}\{N(r,1;F)-\overline{N}(r,1;F)\}\\
&\leq \frac{1}{m}\sum_{j=1}^{n}\{N(r,\omega_j;f)-\overline{N}(r,\omega_j;f)\}\leq \frac{1}{m}N(r,0;f'|f\neq 0)\\
&= \frac{1}{m}\overline{N}\left(r,\infty;\frac{f}{f'}\right)\leq \frac{1}{m}\overline{N}\left(r,\infty;\frac{f'}{f}\right)+S(r,f)\\
&= \frac{1}{m}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)\\
&\leq \frac{1}{m}\{\overline{N}(r,0;f|\geq d)+\overline{N}(r,\infty;f)\}+S(r,f).
\end{aligned}$$

Using Lemma 2.6 we get

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &\leq \frac{1}{m\{(n-1)d-1\}}\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)\right\}\\
&+ \frac{1}{m}\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}$$

Therefore

$$\overline{N}_*(r,1;F,G) \leq \frac{(n-1)d\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)}{m(n-1)d-(m+1)}+S(r,f).$$

Similarly,

$$\overline{N}_*(r,1;F,G) \leq \frac{(n-1)d\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)}{m(n-1)d-(m+1)}+S(r,g).$$

Therefore

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &\leq \frac{(n-1)d+1}{2\{m(n-1)d-(m+1)\}}\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]\\
&+ S(r,f)+S(r,g).
\end{aligned}$$

This proves the lemma. □

Lemma 2.8. *Let F and G be given by (2.1). Then $F=G$ implies $f=g$ for $n\geq 2$ and $d\geq 2$.*

Proof. Let $F=G$. Then

$$f^{n-1}(f+a)=g^{n-1}(g+a). \tag{2.8}$$

Putting $h=\frac{g}{f}$ in (2.8) we get

$$f^{n}(1-h^{n})+af^{n-1}(1-h^{n-1})=0. \tag{2.9}$$

If h is nonconstant, then from (2.9) we get

$$f=-a\frac{h^{n-1}-1}{h^{n}-1}=-a\frac{\sum_{i=1}^{n-2}(h-\alpha_i)}{\sum_{j=1}^{n-1}(h-\beta_j)},$$

where α_i's are $(n-1)^{th}$ roots of unity and β_j's are n^{th} roots of unity with $\alpha_i \neq 1$ and $\beta_j \neq 1$. Clearly $\alpha_i \neq \beta_j$. Again since $f \in M^d(\mathbb{C})$, each α_i and β_j points of h are of multiplicity at least d. Also from (2.8) we see that $f,\ g$ share $(0,\infty)$ and (∞,∞). Thus h has no zero and pole. Therefore from Nevanlinna's second fundamental theorem we get

$$\begin{aligned}(2n-3)T(r,h) &\leq \sum_{i=1}^{n-2}\overline{N}(r,\alpha_i;h)+\sum_{j=1}^{n-1}\overline{N}(r,\beta_j;h)\\ &+ \overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h)\\ &\leq \frac{1}{d}\sum_{i=1}^{n-2}N(r,\alpha_i;h)+\frac{1}{d}\sum_{j=1}^{n-1}N(r,\beta_j;h)+S(r,h)\\ &\leq \frac{(2n-3)}{d}T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction for $n \geq 2$ and $d \geq 2$. Therefore h is a constant and then from (2.8) we get $h = 1$. Thus $f = g$. □

Lemma 2.9. *Let $f,\ g \in E^d(\mathbb{C})$ and $F,\ G$ be defined by (2.1). Then $F = G$ implies $f = g$ for $n \geq 1$ and $d \geq 1$.*

Proof. Let $F = G$. Then

$$f^{n-1}(f+a) = g^{n-1}(g+a). \tag{2.10}$$

Putting $h = \frac{g}{f}$ in (2.10) we get

$$f^n(1-h^n) + af^{n-1}(1-h^{n-1}) = 0. \tag{2.11}$$

If h is nonconstant, then from (2.11) we get

$$f = -a\frac{h^{n-1}-1}{h^n-1} = -a\frac{\sum_{i=1}^{n-2}(h-\alpha_i)}{\sum_{j=1}^{n-1}(h-\beta_j)},$$

where α_i's are $(n-1)^{th}$ roots of unity and β_j's are n^{th} roots of unity with $\alpha_i \neq 1$ and $\beta_j \neq 1$. Clearly $\alpha_i \neq \beta_j$. Again since $f \in E^d(\mathbb{C})$, each α_i and β_j points of h are of multiplicity at least d. Also from (2.10) we see that $f,\ g$ share $(0,\infty)$ and (∞,∞). Thus h has no zero and pole, that means h is an entire function. Now since f is entire, h can not take the values β_j, $j = 1, 2, \cdots, n-1$. Also h being entire, can skip at most one value. Thus $n \leq 2$. If $n = 1$, then from (2.10) we get $f = g$. If $n = 2$, h being nonconstant, from (2.11) we get $f = -\frac{a}{h+1}$. Therefore from the second fundamental theorem of Nevanlinna we get

$$T(r,h) \leq \overline{N}(r,-1;h)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h) \leq S(r,h),$$

a contradiction.

Thus for $n \geq 2$, h is constant and hence $h = 1$, i.e., $f = g$. This completes the proof of the lemma. □

Lemma 2.10. *Let F and G be given by (2.1). Then $FG \neq 1$ for $n > 2$.*

Proof. If possible, let $FG = 1$. Then

$$f^{n-1}(f+a)\cdot g^{n-1}(g+a) = b^2 \;\; i.e., \;\; f^{n-1}(f+a) = \frac{b^2}{g^{n-1}(g+a)}. \tag{2.12}$$

Applying Nevanlinna's first fundamental theorem we get

$$T(r,f) = T(r,g) + O(1).$$

Since $f,\ g$ share $(0,l)$, it is clear from (2.12) that $f,\ g$ has no zero. Then from (2.12) we see that $\overline{N}(r,\infty;f) = \overline{N}(r,-a;g)$, $\overline{N}(r,\infty;g) = \overline{N}(r,-a;f)$ and each $a-$points of f and g are of multiplicity at least n. Hence by second fundamental theorem we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,-a;f) + S(r,f) \\ &\leq \frac{2}{n}T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction for $n > 2$. This proves the lemma. □

3. Proof of the Theorems

Proof of Theorem 1.15. Let F, G and H be given by (2.1) and (2.2). Then F, G share $(1,m)$.

Case 1: $H \not\equiv 0$. Since $f,\ g$ share $(0,0)$, we have $\overline{N}_*(r,0;f,g) \leq \overline{N}(r,0;f) = \overline{N}(r,0;g)$. Also since $f,\ g \in M^d(\mathbb{C})$, we have $\overline{N}(r,0;f) = \overline{N}(r,0;f| \geq d)$ and $\overline{N}(r,0;g) = \overline{N}(r,0;g| \geq d)$. From Lemma 2.5, we get

$$\begin{aligned} \left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}_*(r,0;f,g) \\ &+ 2[\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)] \\ &- \left(m - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\ &\leq 3\overline{N}(r,0;f) + 2[\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)] \\ &- \left(m - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g). \end{aligned}$$

Using Lemma 2.6 we get

$$\begin{aligned} \left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq \frac{2(n-1)d+1}{(n-1)d-1}[\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)] \\ &- \frac{(2m-3)(n-1)d-(2m+3)}{2\{(n-1)d-1\}}\overline{N}_*(r,1;F,G) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

Using Lemma 2.7 we get

$$\begin{aligned} \left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq \left[\frac{\{(2m+3)-(2m-3)(n-1)d\}\{(n-1)d+1\}}{4\{(n-1)d-1\}\{m(n-1)d-(m+1)\}}\right. \\ &+ \left.\frac{2(n-1)d+1}{(n-1)d-1}\right]\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} \\ &+ S(r,f) + S(r,g). \end{aligned}$$

Putting $m=1$, we get

$$\begin{aligned}\left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq \frac{3\{3(n-1)d+1\}}{4\{(n-1)d-2\}}[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]\\ &+ S(r,f)+S(r,g)\\ &\leq \frac{3\{3(n-1)d+1\}}{4d\{(n-1)d-2\}}[T(r,f)+T(r,g)]\\ &+ S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction for $n>\frac{3}{2}+\frac{36}{5d}$.
Similarly, putting $m=2$, we get

$$\begin{aligned}\left(\frac{n}{2}-1\right)[T(r,f)+T(r,g)] &\leq \frac{5\{3(n-1)d+1\}}{4d\{2(n-1)d-3\}}[T(r,f)+T(r,g)]\\ &+ S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction for $n>\frac{3}{2}+\frac{6}{d}$.
Case 2: $H\equiv 0$. Then

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}$$

Integrating twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B.$$

Subcase 2.1 $B=0$.
Then $AF=G+(A-1)$. Therefore $T(r,f)=T(r,g)+O(1)$.
Subcase 2.1.1 $A=1$.
Then $F\equiv G$. Therefore from Lemma 2.8 we get $f=g$ for $n\geq 2,\ d\geq 2$.
Subcase 2.1.2 $A\neq 1$.
Now if $A=0$, then $G\equiv 1$, a contradiction. Thus, $A\neq 0$ and hence $F=\frac{1}{A}\{G+(A-1)\}$ or $G=A\{F-\frac{A-1}{A}\}$. Therefore $\overline{N}(r,0;F)=\overline{N}(r,1-A;G)$. Also since f and g share $(0,0)$, if f or g has a zero, say z_0, then z_0 is also a zero of F and G contradicting that $A\neq 1$. Thus, for $A\neq 1$, f and g do not have zeros i.e.,

$$\overline{N}(r,0;f)=0=\overline{N}(r,0;g).$$

Using second fundamental theorem, we get

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}(r,1-A;G)+\overline{N}(r,\infty;G)+S(r,G)\\ &\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)\\ &+ \overline{N}(r,-a;f)+\overline{N}(r,-a;g)+S(r,G)\\ &\leq \frac{1}{d}N(r,\infty;g)+2T(r,g)+S(r,g)\\ &\leq \frac{1}{n}\left(2+\frac{1}{d}\right)T(r,G)+S(r,G).\end{aligned}$$

This is a contradiction for $n>2+\frac{1}{d}$.

Subcase 2.2 $B \neq 0$.

Then $F = \frac{(B+1)G+(A-B-1)}{BG+(A-B)}$.

Subcase 2.2.1 $B = -1$.

Then $F = -\frac{A}{G-(A+1)}$.

Subcase 2.2.1.1 $A = -1$.

Then $FG = 1$, which is a contradiction, by Lemma 2.10, for $n > 2$.

Subcase 2.2.1.2 $A \neq -1$.

Then $\overline{N}(r, A+1; G) = \overline{N}(r, \infty; F)$. Using second fundamental theorem we get

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}(r,A+1;G) + \overline{N}(r,\infty;G) + S(r,G) \\
&\leq \overline{N}(r,0;g) + \overline{N}(r,-a;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,g) \\
&\leq \frac{1}{d}\{N(r,0;g) + N(r,\infty;f) + N(r,\infty;g)\} + T(r,g) + S(r,g) \\
&\leq \frac{1}{n}\left(1 + \frac{3}{d}\right) T(r,G) + S(r,G).
\end{aligned}$$

This is a contradiction for $n > 1 + \frac{3}{d}$.

Subcase 2.2.2 $B \neq -1$.

Then $b\left(F - \frac{B+1}{B}\right) = \frac{\frac{A}{B}}{G - \frac{B-A}{B}}$ and hence $\overline{N}(r, \frac{B-A}{B}; G) = \overline{N}(r, \infty; F)$. Using second fundamental theorem we get

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{B-A}{B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\
&\leq \overline{N}(r,0;g) + \overline{N}(r,-a;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,g) \\
&\leq \frac{1}{d}\{N(r,0;g) + N(r,\infty;f) + N(r,\infty;g)\} + T(r,g) + S(r,g) \\
&\leq \frac{1}{n}\left(1 + \frac{3}{d}\right) T(r,G) + S(r,G).
\end{aligned}$$

This is a contradiction for $n > 1+\frac{3}{d}$. This completes the proof of the theorem. □

Proof of Theorem 1.18. Let F, G, H be given by (2.1) and (2.2). We note that $\overline{N}(r,\infty;f) = 0 = \overline{N}(r,\infty;g)$ for entire functions f and g.

Case 1. $H \not\equiv 0$.

Proceeding as Case 1 in Theorem 1.15 and using Lemmas 2.6, 2.7, we get

$$\begin{aligned}
(n-2)\left[T(r,f) + T(r,g)\right] &\leq 6\overline{N}(r,0;f) - (2m-3)\overline{N}_*(r,1;F,G) \\
&+ S(r,f) + S(r,g) \\
&\leq \left\{\frac{6}{n-2} - (2m-3)\right\} \overline{N}_*(r,1;F,G) \\
&+ S(r,f) + S(r,g) \\
&\leq \left\{\frac{n\{6-(2m-3)(n-2)\}}{2(n-2)\{m(n-2)-1\}}\right\} \left[\overline{N}(r,\infty;f)\right. \\
&+ \left.\overline{N}(r,\infty;g)\right] + S(r,f) + S(r,g).
\end{aligned}$$

For $m = 1$ and $n \geq 4$,

$$\begin{aligned} 2(n-3)(n-2)^2[T(r,f)+T(r,g)] &\leq n(n+4)[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)] \\ &+ S(r,f)+S(r,g) \\ &\leq S(r,f)+S(r,g), \end{aligned}$$

a contradiction.
For $m = 2$ and $n \geq 3$,

$$\begin{aligned} 2(2n-5)(n-2)^2[T(r,f)+T(r,g)] &\leq n(8-n)[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)] \\ &+ S(r,f)+S(r,g) \\ &\leq S(r,f)+S(r,g), \end{aligned}$$

again a contradiction.
Case 2. $H \equiv 0$.
Subcase 2.1. $B = 0$.
Proceeding as in Subcase 2.1.1 of Theorem 1.15 and using Lemma 2.9 we get $f = g$ for $n \geq 2$. For Subcase 2.1.2 we have $AF = G + (A-1)$. Clearly $A \neq 0$. Also since f, g share $(0,0)$, if z_0 be a zero of f or g, then z_0 is also a zero of F and G, which contradict that $A \neq 1$. Therefore for $A \neq 1$, f and g do not have any zero. Then using second fundamental theorem we get $T(r,G) \leq \frac{2}{n}T(r,G)+S(r,G)$, a contradiction for $n \geq 3$.
Subcase 2.2. $B \neq 0$.
Proceeding as in Subcase 2.2 of Theorem 1.15 we reach at a contradiction for $n \geq 3$. This completes the proof of the theorem. □

Acknowledgement. The author is grateful to the referee for his/her valuable suggestions and comments towards the improvement of the paper.

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