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ON FIRST WEIGHTED ZAGREB INDEX OF TOTAL TRANSFORMATION GRAPHS

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Abstract. In QSPR studies topological indices plays vital role. To enrich this field we put forward novel topological index namely first weighted Zagreb index. In this paper we give explicit formulae for first weighted Zagreb index of total transformation graphs in terms of elements of a graph G.

1. Introduction and preliminaries

All graphs considered in this paper are simple with vertex set V and edge set E. The order and size of G is denoted by $|V| = n$ and $|E| = m$ respectively. The degree of a vertex $v \in V$ is the number of edges incident to v and it is denoted by $d_G(v)$. The degree of an edge $e = uv$ is defined as $d_G(e) = d_G(u) + d_G(v) - 2$. For undefined terminology in this paper refer [6].

Topological index is simply a numeric associated with the molecular graph. So far, large number of such quantities are put forward by many researchers right from 1972 [3]. According to Prof. Gutman(Personal Communication) a useful topological index is one which has a good predicting power in QSPR studies. Therefore, topological indices can be categorized into two categories useful and not so useful TI's see [1, 4, 5, 7, 8, 14, 17]. One of the most useful topological indices are the Zagreb indices which are defined as:

$$M_1(G) = \sum_{i=1}^{n} d_G(v)^2 \tag{1.1}$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) \tag{1.2}$$

where $M_1(G)$ and $M_2(G)$ are the first and second Zagreb indices respectively.

Motivated by the Zagreb indices, here we put forward the weighted version of the first and second Zagreb indices. For this first we have to define the vertex and edge weights of a graph G as follows:

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Vertex weight: Let $w_1, w_2, w_3 \cdots, w_n$ be the weights of the vertices $v_1, v_2, v_3, \cdots, v_n$ such that $w_1 = d_G(v_1)$, $w_2 = d_G(v_2)$, $w_3 = d_G(v_3)$, $\cdots$, $w_n = d_G(v_n)$.

Edge weight: Let $e_1, e_2, e_3, \cdots, e_m$ be the edges of a graph G. Then the edge weight of $e = uv \in E(G)$ is defined as $w(e) = d_G(u) + d_G(v) - 2$.

Weighted degree of a vertex: The weighted degree of a vertex $v \in V(G)$ is defined as:

$$d_G^w(v) = \sum_{e=uv} w(e). \tag{1.3}$$

First weighted Zagreb index: The first weighted Zagreb index $Z_1^w(G)$ is defined as

$$Z_1^w(G) = \sum_{v \in V} d_G^w(v)^2.$$

2. Transformation graphs and total transformation graphs

Let $G = (V, E)$ be a graph with n vertices and m edges. In the mathematical literature, several transformation graphs have been considered, constructed so that their vertex set is equal to $V(G) \cup E(G)$. In what follows we define the best known representatives of such graphs.

The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$, such that two vertices are adjacent if and only if they are either adjacent or incident in G.

Sampathkumar and Chikkodimath introduced the concepts of semitotal-point graph and semitotal-line graph [16] which are stated as follows:

The semitotal-point graph $T_1(G)$ is a graph with vertex set $V(T_1(G)) = V(G) \cup E(G)$, such that any two vertices $u, v \in V(T_1(G))$ are adjacent if and only if (i) u and v are adjacent vertices in G or (ii) one is a vertex of G and other is an edge of G incident to it. Thus, $T_1(G)$ has $m + n$ vertices, whereas the number of its edges is $3m$.

The semitotal-line graph $T_2(G)$ is a graph with vertex set $V(T_2(G)) = V(G) \cup E(G)$, such that any two vertices $u, v \in V(T_2(G))$ are adjacent if and only if (i) u and v are adjacent edges in G and (ii) one is a vertex of G and other is an edge of G incident to it. Thus, $T_2(G)$ has $m + n$ vertices, whereas the number of its edges is $\frac{1}{2}M_1(G) - m$ by (i) plus 2m by (ii), i.e., a total of $\frac{1}{2}M_1(G) + m$. Recall that $\frac{1}{2}M_1(G) - m$ is just the number of edges of the line graph of the graph G.

Note that the middle graph" considered is identical to the above defined semitotal-line graph.

Let $G = (V, E)$ be a graph and x, y, z be three variables taking values + or −. The total transformation graph G^{xyz} is a graph whose vertex set is $V(G) \cup E(G)$, and for $\alpha, \beta \in V(G^{xyz})$, α and β are adjacent in G^{xyz} if and only if

- $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α and β are not adjacent in G if $x = -$
- $\alpha, \beta \in E(G)$, α, β are adjacent in G if $y = +$ and α and β are not adjacent in G if $y = -$

• $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $z = +$ and α and β are not incident in G if $z = -$

The basic properties of total transformation graphs can be found in [18]. For instance, G^{+++} has $m+n$ vertices and $\frac{1}{2}M_1(G) + 2m$ edges.

Since there are eight distinct 3-permutations of {+,-}, there are eight different transformations of a given graph G. It is interesting to see that G^{+++} is just the total graph $T(G)$ of G whereas G^{---} is the complement of $T(G)$. In addition, the pairs G^{++-} and G^{--+}, G^{+-+} and G^{-+-}, as well as G^{+--} and G^{-++} are mutually complementary, i.e., $G^{--+} \cong \overline{G^{++-}}$, $G^{-+-} \cong \overline{G^{+-+}}$, and $G^{-++} \cong \overline{G^{+--}}$. (As usual, $\cong$ indicates that the respective two graphs are isomorphic.)

In this section, we give explicit formulae for first weighted Zagreb index of semitotal-point graph, semitotal-line graph and total transformation graphs in terms of elements of a graph G.

Theorem 2.1. *For any graph G,*

$$M_1^w(T_1(G)) \;=\; 4[F(G) + 2M_2(G) - 2M_1(G) + n] + 4m + 4M_1(G).$$

Proof: Degree of every edge in $T_1(G)$

a) If $u, v \in V(T_1(G)) \bigcap V(G)$ then

$$\begin{aligned}
deg_{T_1(G)}(e) &= \sum_{u,v \in V(T_1(G)) \bigcap V(G)} [2deg_G(u) + 2deg_G(v) - 2]^2 \\
&= 4 \sum_{u,v \in V(T_1(G)) \bigcap V(G)} [deg(u) + deg(v) - 1]^2 \\
&= 4\Bigg\{ \sum_{u,v \in V(T_1(G)) \bigcap V(G)} [deg(u) + deg(v)]^2 + \sum_{u,v \in V(T_1(G)) \bigcap V(G)} \{1\} \\
&- 2 \sum_{u,v \in V(T_1(G)) \bigcap V(G)} (deg(u) + deg(v)) \Bigg\} \\
&= 4\Bigg\{ \sum_{u,v \in V(T_1(G)) \bigcap V(G)} [deg(u)^2 + deg(v)^2] \\
&+ 2 \sum_{u,v \in V(T_1(G)) \bigcap V(G)} (deg(u)deg(v)) + n - 2M_1(G) \Bigg\} \\
&= 4[F(G) + 2M_2(G) - 2M_1(G) + n].
\end{aligned}$$

b) If $u, v \in V(T_1(G)) \bigcap E(G)$ then

$$\begin{aligned}
deg_{T_1(G)}(e) &= \sum_{u,v \in V(T_1(G)) \bigcap E(G)} (2)^2 \\
&= 4 \sum_{u,v \in V(T_1(G)) \bigcap E(G)} (1) \\
&= 4m.
\end{aligned}$$

c) If $u \in V(T_1(G)) \bigcap V(G) and u \in V(T_1(G)) \bigcap E(G)$ then

$$\begin{aligned}
deg_{T_1(G)}(e) &= \sum_{u \in V(T_1(G)) \bigcap V(G), v \in V(T_1(G)) \bigcap E(G)} (2deg_G(u) + 2 - 2)^2 \\
&= 4 \sum_{u \in V(T_1(G)) \bigcap V(G), v \in V(T_1(G)) \bigcap E(G)} (deg_G(u))^2 \\
&= 4M_1(G).
\end{aligned}$$

Theorem 2.2. *For any graph G,*

$$\begin{aligned}
M_1^w(T_2(G)) &= 4[F(G) + 2M_2(G) - 2M_1(G) + n] \\
&+ \{F(G) + 2M_2(G) - 4M_1(G) + 4m\} \\
&+ \{F(G) - M_1(G) + 4M_2(G) - 4m\}.
\end{aligned}$$

Proof: Degree of every edge in $T_2(G)$
a) If $u, v \in V(T_2(G)) \bigcap V(G)$ then

$$\begin{aligned}
deg_{T_2(G)}(e) &= \sum_{u,v \in V(T_2(G)) \bigcap V(G)} [2deg_G(u) + 2deg_G(v) - 2]^2 \\
&= 4 \sum_{u,v \in V(T_2(G)) \bigcap V(G)} [deg(u) + deg(v) - 1]^2 \\
&= 4\Bigg\{ \sum_{u,v \in V(T_2(G)) \bigcap V(G)} [deg(u) + deg(v)]^2 + \sum_{u,v \in U(T_1(G)) \bigcap V(G)} \{1\} \\
&- 2 \sum_{u,v \in V(T_2(G)) \bigcap V(G)} (deg(u) + deg(v)) \Bigg\} \\
&= 4\Bigg\{ \sum_{u,v \in V(T_2(G)) \bigcap V(G)} [deg(u)^2 + deg(v)^2] \\
&+ 2 \sum_{u,v \in V(T_2(G)) \bigcap V(G)} [deg(u)deg(v)] + n - 2M_1(G) \Bigg\} \\
&= 4[F(G) + 2M_2(G) - 2M_1(G) + n].
\end{aligned}$$

b) If $u, v \in V(T_2(G)) \bigcap E(G)$ then

$$\begin{aligned}
deg_{T_2(G)}(e) &= \sum_{u,v \in V(T_2(G)) \bigcap E(G)} [deg_G(u) + deg_G(v) - 2]^2 \\
&= \sum_{u,v \in V(T_2(G)) \bigcap E(G)} [deg_G(u) + deg_G(v)]^2 \\
&+ \sum_{u,v \in V(T_2(G)) \bigcap E(G)} 4 - 4 \sum_{u,v \in V(T_2(G)) \bigcap E(G)} [deg_G(u) + deg_G(v)] \\
&= \sum_{u,v \in V(T_2(G)) \bigcap E(G)} [deg_G(u)^2 + deg_G(v)^2]
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{u,v\in V(T_2(G))\bigcap E(G)} 2deg_G(u)deg_G(v) + \sum_{u,v\in V(T_2(G))\bigcap E(G)} 4 \\
&- 4 \sum_{u,v\in V(T_2(G))\bigcap E(G)} (deg_G(u) + deg_G(v)) \\
&= F(G) + 2M_2(G) - 4M_1(G) + 4m.
\end{aligned}
$$

c) If $u \in V(T_2(G))\bigcap V(G)$ and $v \in V(T_2(G))\bigcap E(G)$ then

$$
\begin{aligned}
deg_{T_2(G)}(e) &= \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [2deg_G(u) + deg_G(v) - 2]^2 \\
&= \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [2deg_G(u) + deg_G(v)]^2 \\
&+ \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} 4 \\
&- 4 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [2deg_G(u) + deg_G(v)] \\
&= \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [4deg_G(u)^2 + deg_G(v)^2] \\
&+ 4 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} deg_G(u)deg_G(v) \\
&+ \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} 4 \\
&- 4 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} deg_G(u) \\
&- 4 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [deg_G(u) + deg_G(v)] \\
&= 3 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [deg_G(u)^2] \\
&+ \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [deg_G(u)^2 + deg_G(v)^2] \\
&+ 4 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} deg_G(u)deg_G(v) \\
&+ \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} 4 \\
&- 4 \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [deg_G(u)] \\
&+ \sum_{u\in V(T_2(G))\bigcap V(G), v\in V(T_2(G))\bigcap E(G)} [deg_G(u) + deg_G(v))]
\end{aligned}
$$

$$
\begin{aligned}
&= 3M_1(G) + F(G) + 4M_2(G) - 4M_1(G) + 4m - 8m \\
&= F(G)) - M_1(G) + 4M_2(G) - 4m.
\end{aligned}
$$

Theorem 2.3. *For any graph G,*

$$
\begin{aligned}
M_2^w(G^{+++}) &= 4[F(G) + 2M_2(G) - 2M_1(G) + n] \\
&+ 4[F(G) + 2M_2(G) - 2M_1(G) + m] \\
&+ F(G) + 12M_1(G) + 6M_2(G) - 12m.
\end{aligned}
$$

Proof: Degree of every edge in G^{+++}
a) If $u, v \in V(G^{+++}) \bigcap V(G)$ then

$$
\begin{aligned}
deg_{G^{+++}}(e) &= \sum_{u,v \in V(G^{+++}) \bigcap V(G)} [2deg_G(u) + 2deg_G(v) - 2]^2 \\
&= 4 \sum_{u,v \in U(T_1(G)) \bigcap V(G)} [deg(u) + deg(v) - 1]^2 \\
&= 4\Big\{ \sum_{u,v \in V(T_2(G)) \bigcap V(G)} [deg(u) + deg(v)]^2 + \sum_{u,v \in U(T_1(G)) \bigcap V(G)} \{1\} \Big\} \\
&- 2 \sum_{u,v \in U(T_2(G)) \bigcap V(G)} (deg(u) + deg(v)) \\
&= 4\Big\{ \sum_{u,v \in V(G^{+++}) \bigcap V(G)} [deg(u)^2 + deg(v)^2] \\
&+ 2 \sum_{u,v \in V(G^{+++}) \bigcap V(G)} [deg(u)deg(v)] + n - 2M_1(G) \Big\} \\
&= 4[F(G) + 2M_2(G) - 2M_1(G) + n].
\end{aligned}
$$

b) If $u, v \in V(G^{+++}) \bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{+++}}(e) &= \sum_{u,v \in V(G^{+++}) \bigcap E(G)} [2deg_G(u) + 2deg_G(v) - 2]^2 \\
&= 4 \sum_{u,v \in V(G^{+++}) \bigcap E(G)} [deg(u) + deg(v) - 1]^2 \\
&= 4\Big\{ \sum_{u,v \in V(G^{+++}) \bigcap E(G)} [deg(u) + deg(v)]^2 + \sum_{u,v \in V(G^{+++}) \bigcap E(G)} \{1\} \\
&- 2 \sum_{u,v \in V(G^{+++}) \bigcap E(G)} [deg(u) + deg(v)] \Big\} \\
&= 4\Big\{ \sum_{u,v \in V(G^{+++}) \bigcap E(G)} [deg(u)^2 + deg(v)^2] \\
&+ 2 \sum_{u,v \in V(G^{+++}) \bigcap E(G)} [deg(u)deg(v)] + m - 2M_1(G) \Big\}
\end{aligned}
$$

$$= \quad 4[F(G) + 2M_2(G) - 2M_1(G) + m].$$

c) If $u \in V(G^{+++}) \bigcap V(G) and v \in V(G^{+++}) \bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{+++}}(e) &= \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [3deg_G(u) + deg_G(v) - 2]^2 \\
&= \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [3deg_G(u) + deg_G(v)]^2 \\
&+ \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} 4 \\
&- 4 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [3deg_G(u) + deg_G(v)] \\
&= \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [9deg_G(u)^2 + deg_G(v)^2] \\
&+ 6 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [deg_G(u) deg_G(v)] \\
&+ \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} 4 \\
&- 4 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [2deg_G(u)] \\
&- 4 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [deg_G(u) + deg_G(v)] \\
&= 2 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [deg_G(u)]^2 \\
&+ \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [deg_G(u)^2 + deg_G(v)^2] \\
&+ 6 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} deg_G(u) deg_G(v) \\
&+ \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} 4 \\
&- 8 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [deg_G(u)] \\
&- 4 \sum_{u \in V(G^{+++}) \bigcap V(G), v \in V(G^{+++}) \bigcap E(G)} [deg_G(u) + deg_G(v)] \\
&= 8M_1(G) + F(G) + 6M_2(G) + 4m - 16m + 4M_1(G) \\
&= F(G) + 12M_1(G) + 6M_2(G) - 12m.
\end{aligned}$$

Theorem 2.4. *For any graph G,*

$$\begin{aligned} M_1^w(G^{---}) &= [2m+2n-4]^2 n + 4F(G) + 8M_2(G) - 8(m+n-2)M_1(G) \\ &+ 4m(m+n-2)[m+n-4] + F(G) + 3M_1(G) - 4m_2(G) \\ &+ 4m[m+n-6] + M_1(-16m-16n+40)(G) \\ &+ F(G) + M_2(G)(m+n-2). \end{aligned}$$

Proof: Degree of every edge in G^{---}
a) If $u, v \in V(G^{---}) \bigcap V(G)$ then

$$\begin{aligned} deg_{G^{---}}(e) &= \sum_{u,v \in V(G^{---}) \bigcap V(G)} [2m+2n-2deg_G(u)-2deg_G(v)-4]^2 \\ &= \sum_{u,v \in V(G^{---}) \bigcap V(G)} [2m+2n-4-2deg_G(u)-2deg_G(v)]^2 \\ &= \sum_{u,v \in V(G^{---}) \bigcap V(G)} [2m+2n-4]^2 \\ &+ \sum_{u,v \in V(G^{---}) \bigcap V(G)} [2deg_G(u)+2deg_G(v)]^2 \\ &- 2 \sum_{u,v \in V(G^{---}) \bigcap V(G)} (2m+2n-4)(2deg(u)+2deg(v)) \\ &= \sum_{u,v \in V(G^{---}) \bigcap V(G)} [2m+2n-4]^2 \\ &+ 4 \sum_{u,v \in V(G^{---}) \bigcap V(G)} [deg_G(u)^2 + deg_G(v)^2 + 2deg_G(u)deg_G(v)] \\ &- 2 \sum_{u,v \in V(G^{---}) \bigcap V(G)} (2m+2n-4)(2deg(u)+2deg(v)) \\ &= \sum_{u,v \in V(G^{---}) \bigcap V(G)} [2m+2n-4]^2 + 4 \sum_{u,v \in V(G^{---}) \bigcap V(G)} \\ & [deg_G(u)^2 + deg_G(v)^2] + 8 \sum_{u,v \in V(G^{---}) \bigcap V(G)} [deg_G(u)deg_G(v)] \\ &- 4 \sum_{u,v \in V(G^{---}) \bigcap V(G)} (2m+2n-4)(deg(u)+deg(v)) \\ &= [2m+2n-4]^2 n + 4F(G) + 8M_2(G) - 8(m+n-2)M_1(G). \end{aligned}$$

b) If $u, v \in V(G^{---}) \bigcap E(G)$ then

$$\begin{aligned} deg_{G^{---}}(e) &= \sum_{u,v \in V(G^{---}) \bigcap E(G)} [2m+2n-2deg_G(u)+deg_G(v)-4]^2 \\ &= \sum_{u,v \in V(G^{---}) \bigcap E(G)} [2m+2n-4-2deg_G(u)+deg_G(v)]^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]^2 \\
&+ \sum_{u,v\in V(G^{---})\bigcap E(G)} [deg_G(v)-2deg_G(u)]^2 \\
&+ 2\sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4][deg_G(v)-2deg_G(u)] \\
&= \sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]^2 \\
&+ \sum_{u,v\in V(G^{---})\bigcap E(G)} [deg_G(v)^2+4deg_G(u)^2] \\
&- 4\sum_{u,v\in V(G^{---})\bigcap E(G)} [deg_G(u)deg_G(v)] \\
&+ 2\sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]deg_G(v) \\
&- 4\sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]deg_G(u) \\
&= \sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]^2 \\
&+ \sum_{u,v\in V(G^{---})\bigcap E(G)} [deg_G(v)^2+deg_G(u)^2] \\
&+ 3\sum_{u,v\in V(G^{---})\bigcap E(G)} deg_G(u)^2 \\
&- 4\sum_{u,v\in V(G^{---})\bigcap E(G)} [deg_G(u)deg_G(v)] \\
&+ 2\sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]deg_G(v) \\
&- 4\sum_{u,v\in V(G^{---})\bigcap E(G)} [2m+2n-4]deg_G(u) \\
&= (2m+2n-4)^2m+F(G)+3M_1(G)-4m_2(G)-8m(m+n-2) \\
&= 4m(m+n-2)[m+n-4]+F(G)+3M_1(G)-4m_2(G).
\end{aligned}
$$

c) If $u\in V(G^{---})\bigcap V(G)$ and $v\in V(G^{---})\bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{---}}(e) &= \sum_{u\in V(G^{---})\bigcap V(G), v\in V(G^{---})\bigcap E(G)} [2m+2n-3deg_G(u)-deg_G(v)-4]^2 \\
&= \sum_{u,v\in V(G^{---})\bigcap V(G), v\in V(G^{---})\bigcap E(G)} [2m+2n-4-3deg_G(u)-deg_G(v)]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} [2m+2n-4]^2 + \sum [deg_G(v)+3deg_G(u)]^2 \\
&- 2\sum [2m+2n-4][deg_G(v)+3deg_G(u)] \\
&= \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} [2m+2n-4]^2 \\
&+ \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} [deg_G(v)^2+9deg_G(u)^2] + 6\sum deg_G(u)deg_G(v) \\
&- 4[m+n-2] \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} [deg_G(u)+deg_G(v)] \\
&- 8[m+n-2] \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} deg_G(u) \\
&= 4m[m+n-2]^2 + 8 \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} (deg_(G)(u))^2 \\
&+ \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} [deg_G(u)^2+deg_G(v)^2] \\
&+ \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G)} [deg(u)deg(v)] \\
&- 4[m+n-2] \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G} (deg(u)+deg(v)) \\
&- 8[m+n-2] \sum_{u,v\in V(G^{---})\bigcap V(G),v\in V(G^{---})\bigcap E(G} deg(u) \\
&= 4m[m+n-2] + 8M_1(G) + F(G) + M_2(G) - 4(m+n-2)M_1(G) \\
&- 16m(m+n-2) \\
&= 4m[m+n-6] + M_1(-16m-16n+40)(G) + F(G) + M_2(G)(m+n-2).
\end{aligned}
$$

Theorem 2.5. *For any graph G,*

$$
\begin{aligned}
M_1^w(G^{+-+}) &= 4[F(G)+2M_2(G)-2M_1(G)+n] \\
&+ 4m^2(m+2)+F(G)+3M_1(G)-4M_2(G) \\
&+ (m+1)^2m+F(G)-2M_2(G).
\end{aligned}
$$

Proof: Degree of every edge in G^{+-+}
a) If $u,v\in V(G^{+-+})\bigcap V(G)$ then

$$
\begin{aligned}
deg_{G^{+-+}}(e) &= \sum_{u,v\in V(G^{+-+})\bigcap V(G)} [2deg_G(u)+2deg_G(v)-2]^2 \\
&= 4\sum_{u,v\in V(G^{+-+})\bigcap V(G)} [deg_G(u)+deg_G(v)-1]^2 \\
&= 4\Bigg\{ \sum_{u,v\in V(G^{+-+})\bigcap V(G)} [deg_G(u)+deg_G(v)]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{u,v\in V(G^{+-+})\bigcap V(G)} (1)^2 \\
&- 2\sum_{u,v\in V(G^{+-+})\bigcap V(G)} [deg_G(u)+deg_G(v)]\Bigg\} \\
&= 4\Bigg\{\sum_{u,v\in V(G^{+-+})\bigcap V(G)} [deg(u)^2+deg(v)^2] \\
&+ 2\sum_{u,v\in V(G^{+-+})\bigcap V(G)} [deg(u)deg(v)]+n-2M_1(G)\Bigg\} \\
&= 4[F(G)+2M_2(G)-2M_1(G)+n].
\end{aligned}
$$

b) If $u \in V(G^{+-+})\bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{+-+}}(e) &= \sum_{u,v\in V(G^{+-+})\bigcap E(G)} [2m-2deg_G(u)+deg_G(v)+4]^2 \\
&= \sum_{u,v\in V(G^{+-+})\bigcap E(G)} [2m+4+(deg_G(v)-2deg_G(u))]^2 \\
&= \sum_{u,v\in V(G^{+-+})\bigcap E(G)} (2m+4)^2 \\
&+ \sum_{u,v\in V(G^{+-+})\bigcap E(G)} [deg_G(v)-2deg_G(u)]^2 \\
&+ 2\sum_{u,v\in V(G^{+-+})\bigcap E(G)} (2m+4)(deg_G(v)-2deg_G(u)) \\
&= \sum_{u,v\in V(G^{+-+})\bigcap E(G)} (2m+4)^2 \\
&+ \sum_{u,v\in V(G^{+-+})\bigcap E(G)} [deg_G(v)^2+4deg_G(u)^2-4deg_G(u)deg_G(v)] \\
&+ 4\sum_{u,v\in V(G^{+-+})\bigcap E(G)} (m+2)(deg_G(v)) \\
&- 8\sum_{u,v\in V(G^{+-+})\bigcap E(G)} (2m+4)(deg_G(u)) \\
&= \sum_{u,v\in V(G^{+-+})\bigcap E(G)} (2m+4)^2 \\
&+ \sum_{u,v\in V(G^{+-+})\bigcap E(G)} [deg_G(v)^2+deg_G(u)^2] \\
&+ 3\sum_{u,v\in V(G^{+-+})\bigcap E(G)} [deg_G(u)^2]-4\sum[deg_G(u)deg_G(v)] \\
&+ 4\sum_{u,v\in V(G^{+-+})\bigcap E(G)} (m+2)(deg_G(v))
\end{aligned}
$$

$$
\begin{aligned}
&- \; 8 \sum_{u,v\in V(G^{+-+})\bigcap E(G)} (m+2)(deg_G(u)) \\
&= \; 4m^2(m+2) + F(G) + 3M_1(G) - 4M_2(G).
\end{aligned}
$$

c) If $u \in V(G^{+-+})\bigcap V(G)$ and $v \in V(G^{+-+})\bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{+-+}}(e) &= \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [m + 2deg_G(u) - deg_G(v) + 1]^2 \\
&= \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [m + 1 + 2deg_G(u) - (deg_G(v)]^2 \\
&= \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)^2 \\
&+ \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [2deg_G(u) - deg_G(v)]^2 \\
&+ \; 2 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)[deg_G(u) - deg_G(v)] \\
&= \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)^2 \\
&+ \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [4deg_G(u)^2 + deg_G(v)^2] \\
&- \; 4 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [deg_G(u)deg_G(v)] \\
&+ \; 2 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)[deg_G(u)] \\
&- \; 2 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)[deg_G(v)] \\
&= \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)^2 \\
&+ \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [deg_G(u)^2 + deg_G(v)^2] \\
&+ \; 3 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [deg_G(u)^2] \\
&- \; 4 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} [deg_G(u)deg_G(v)] \\
&+ \; 2 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)[deg_G(u)] \\
&- \; 2 \sum_{u\in V(G^{+-+})\bigcap V(G), v\in V(G^{+-+})\bigcap E(G)} (m+1)[deg_G(v)]
\end{aligned}
$$

$$= (m+1)^2 m + F(G) - 2M_2(G).$$

Theorem 2.6. *For any graph G,*

$$\begin{aligned} M_1^w(G^{-+-}) &= 4(m+n-2)M_1(G) + 8(m+n-2)m \\ &+ 2m(2n-6)(n-1) + 2(2n-6)M_1(G) + 3M_1(G) + F(G) + 4M_2(G) \\ &+ 6M_2(G) + 2(m+2n-7)M_1(G) + 8(m+2n-7)m. \end{aligned}$$

Proof: Degree of every edge in G^{-+-}
a) If $u, v \in V(G^{-+-}) \bigcap V(G)$ then

$$\begin{aligned} deg_{G^{-+-}}(e) &= 2m + 2n + 2deg_G(u) + deg_G(v) - 4 \\ &= \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} [2m + 2n + 2deg_G(u) + deg_G(v) - 4]^2 \\ &= \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} [2m + 2n - 4]^2 + \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} (2deg_G(u) + deg_G(v))^2 \\ &+ \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} 2(2m + 2n - 4)(2deg_G(u) + deg_G(v)) \\ &= n[2m + 2n - 4]^2 + \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} [4deg_G(u)^2 + deg_G(v)^2 + 4deg_G(u) \cdot deg_G(v)] \\ &+ 2(2m + 2n - 4) \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} (deg_G(u) + deg_G(v)) \\ &+ 2(2m + 2n - 4) \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} deg_G(u) \\ &= n[2m + 2n - 4]^2 + \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} [deg_G(u)^2 + deg_G(v)^2] \\ &+ 3 \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} deg_G(u)^2 + 4 \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} [deg_G(u) \cdot deg_G(v)] \\ &+ 4(m + n - 2) \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} (deg_G(u) + deg_G(v)) \\ &+ 4(m + n - 2) \sum_{u,v \in V(G^{-+-}) \bigcap V(G)} deg_G(u) \\ &= n[2m + 2n - 4]^2 + F(G) + 3M_1(G) + 4M_2(G) \\ &+ 4(m + n - 2)M_1(G) + 8(m + n - 2)m. \end{aligned}$$

b) If $u, v \in V(G^{-+-}) \bigcap E(G)$ then

$$\begin{aligned} deg_{G^{-+-}}(e) &= 2n + 2deg_G(u) + deg_G(v) - 6 \\ &= \sum_{u,v \in V(G^{-+-}) \bigcap E(G)} [2n - 6 + 2deg_G(u) + deg_G(v)]^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{u,v\in V(G^{-+-})\bigcap E(G)} [2n-6]^2 + \sum_{u,v\in V(G^{-+-})\bigcap E(G)} (2deg_G(u)+deg_G(v))^2 \\
&+ 2\sum_{u,v\in V(G^{-+-})\bigcap E(G)} (2n-6)(2deg_G(u)+deg_G(v)) \\
&= m(2n-6)^2 + 2(2n-6)\sum_{u,v\in V(G^{-+-})\bigcap E(G)} deg_G(u) \\
&+ \sum_{u,v\in V(G^{-+-})\bigcap E(G)} [deg_G(u)+deg_G(v)] + \sum_{u,v\in V(G^{-+-})\bigcap E(G)} [2deg_G(u)+deg_G(v)]^2 \\
&= m(2n-6)^2 + 2(2n-6)[2m+M_1(G)] + 4\sum_{u,v\in V(G^{-+-})\bigcap E(G)} deg_G(u)^2 \\
&+ \sum_{u,v\in V(G^{-+-})\bigcap E(G)} deg_G(v)^2 + 4\sum_{u,v\in V(G^{-+-})\bigcap E(G)} [deg_G(u)\cdot deg_G(v)] \\
&= m(2n-6)^2 + 4m(2n-6) + 2(2n-6)M_1(G) + 3M_1(G) + F(G) + 4M_2(G) \\
&= 2m(2n-6)(n-1) + 2(2n-6)M_1(G) + 3M_1(G) + F(G) + 4M_2(G).
\end{aligned}$$

c) If $u \in V(G^{-+-})\bigcap V(G)$ and $v \in V(G^{-+-})\bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{-+-}}(e) &= m + 2n + 3deg_G(u) + deg_G(v) - 7 \\
&= \sum_{u\in V(G^{-+-})\bigcap V(G), v\in V(G^{-+-})\bigcap E(G)} [m+2n-7+3deg_G(u)+deg_G(v)]^2 \\
&= \sum_{u\in V(G^{-+-})\bigcap V(G), v\in V(G^{-+-})\bigcap E(G)} [(m+2n-7)^2 + (3deg_G(u)+deg_G(v))^2 \\
&+ 2(m+2n-7)(3deg_G(u)+deg_G(v))] \\
&= m(m+2n-7)^2 + \sum_{u\in V(G^{-+-})\bigcap V(G), v\in V(G^{-+-})\bigcap E(G)} (3deg_G(u)+deg_G(v))^2 \\
&+ 2(m+2n-7)\sum_{u\in V(G^{-+-})\bigcap V(G), v\in V(G^{-+-})\bigcap E(G)} (deg_G(u)+deg_G(v)) \\
&+ 4(m+2n-7)\sum_{u\in V(G^{-+-})\bigcap V(G), v\in V(G^{-+-})\bigcap E(G)} [deg_G(u)] \\
&= m(m+2n-7)^2 + 8M_1(G) + F(G) \\
&+ 6M_2(G) + 2(m+2n-7)M_1(G) + 8(m+2n-7)m.
\end{aligned}$$

Theorem 2.7. *For any graph G,*

$$\begin{aligned}
M_1^w(G^{++-}) &= n(2m-2)^2 \\
&+ 4\{m(n-5)^2 + 2(n-5)^2M_1(G) + F(G) + 2M_2(G)\} \\
&+ 2(m+n-6)M_1(G) + 2M_2(G) + F(G) + m(m+n-6)^2.
\end{aligned}$$

Proof: Degree of every edge in G^{++-}
a) If $u, v \in V(G^{++-}) \bigcap V(G)$ then

$$\begin{aligned}
deg_{G^{++-}}(e) &= 2m-2 \\
&= \sum_{u,v \in V(G^{++-}) \bigcap V(G)} [2m-2]^2 \\
&= n(2m-2)^2.
\end{aligned}$$

b) If $u, v \in V(G^{++-}) \bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{++-}}(e) &= 2deg_G(u) + 2deg_G(v) + 2n - 10 \\
&= \sum_{u,v \in V(G^{++-}) \bigcap E(G)} [2deg_G(u) + 2deg_G(v) + 2n - 10]^2 \\
&= 4\{ \sum_{u,v \in V(G^{++-}) \bigcap E(G)} (deg_G(u) + deg_G(v))^2 + \sum_{u,v \in V(G^{++-}) \bigcap E(G)} [n-5]^2 \\
&+ 2 \sum_{u,v \in V(G^{++-}) \bigcap E(G)} (n-5)(deg_G(u) + deg_G(v))\} \\
&= 4\{m(n-5)^2 + 2(n-5)^2 M_1(G) + \sum_{u,v \in V(G^{++-}) \bigcap E(G)} (deg_G(u) + deg_G(v))^2\} \\
&= 4\{m(n-5)^2 + 2(n-5)^2 M_1(G) + \sum_{u,v \in V(G^{++-}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 2 \sum_{u,v \in V(G^{++-}) \bigcap E(G)} (deg_G(u) \cdot deg_G(v))\} \\
&= 4\{m(n-5)^2 + 2(n-5)^2 M_1(G) + F(G) + 2M_2(G)\}.
\end{aligned}$$

c) If $u \in V(G^{++-}) \bigcap V(G)$ and $v \in V(G^{++-}) \bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{++-}}(e) &= m + n - 6 + deg_G(u) + deg_G(v) \\
&= \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} [m + n - 6 + deg_G(u) + deg_G(v)]^2 \\
&= \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} (m+n-6)^2 \\
&+ \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} (deg_G(u) + deg_G(v))^2 \\
&+ 2 \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} (m+n-6)(deg_G(u) + deg_G(v)) \\
&= m(m+n-6)^2 + \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 2 \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} (deg_G(u) \cdot deg_G(v))
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2(m+n-6) \sum_{u \in V(G^{++-}) \bigcap V(G), v \in V(G^{++-}) \bigcap E(G)} (deg_G(u) + deg_G(v)) \\
&= \quad m(m+n-6)^2 + F(G) + 2M_2(G) + 2(m+n-6)M_1(G) \\
&= \quad 2(m+n-6)M_1(G) + 2M_2(G) + F(G) + m(m+n-6)^2.
\end{aligned}
$$

Theorem 2.8. *For any graph G,*

$$
\begin{aligned}
M_1^w(G^{--+}) &= 4n^3 + F(G) + 2M_2(G) - 4nM_1(G) \\
&+ m(2n+4)^2 + F(G) + 3M_1(G) - 4M_2(G) - 4m(2n+4) \\
&+ m(m+n+2)(m+n-2) + F(G) - M_1(G)[2m+2n+1] + 4M_2(G).
\end{aligned}
$$

Proof: Degree of every edge in G^{--+}
a) If $u, v \in V(G^{--+}) \bigcap V(G)$ then

$$
\begin{aligned}
deg_{G^{--+}}(e) &= 2n - deg_G(u) - deg_G(v) \\
&= \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [2n - deg_G(u) - deg_G(v)]^2 \\
&= \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [4n^2] + \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [deg_G(u) + deg_G(v)]^2 \\
&- 4 \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [n(deg_G(u) + deg_G(v))] \\
&= 4 \sum_{u,v \in V(G^{--+}) \bigcap V(G)} n^2 + \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [deg_G(u)^2 + deg_G(v)^2 \\
&+ 2deg_G(u)deg_G(v)] - 4 \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [n(deg_G(u) + deg_G(v))] \\
&= 4 \sum_{u,v \in V(G^{--+}) \bigcap V(G)} n^2 + \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [deg_G(u)^2 + deg_G(v)^2] \\
&+ 2 \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [deg_G(u) \cdot deg_G(v)] \\
&- 4n \sum_{u,v \in V(G^{--+}) \bigcap V(G)} [deg_G(u) + deg_G(v)] \\
&= 4n^3 + F(G) + 2M_2(G) - 4nM_1(G).
\end{aligned}
$$

b) If $u, v \in V(G^{--+}) \bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{--+}}(e) &= 2n - 2deg_G(u) + deg_G(v) + 4 \\
&= \sum_{u,v \in V(G^{--+}) \bigcap E(G)} [2n + 4 - 2deg_G(u) + deg_G(v)]^2 \\
&= \sum_{u,v \in V(G^{--+}) \bigcap E(G)} [2n+4]^2 + \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(v) - 2deg_G(u))^2 \\
&+ 2 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} [2n+4](deg_G(v) - 2deg_G(u))
\end{aligned}
$$

$$
\begin{aligned}
&= m(2n+4)^2 + \sum_{u,v \in V(G^{--+}) \bigcap E(G)} [deg_G(v)^2 + 4deg_G(u)^2 - 4deg_G(u) \cdot deg_G(v)] \\
&+ 2(2n+4) \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(v) - 2deg_G(u)) \\
&= m(2n+4)^2 + \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 3 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} deg_G(u)^2 + 4 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(u) \cdot deg_G(v)) \\
&+ 2(2n+4) \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(v) - 2deg_G(u)) \\
&= m(2n+4)^2 + \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 3 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} deg_G(u)^2 + 4 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(u) \cdot deg_G(v)) \\
&+ 2(2n+4)[\sum_{u,v \in V(G^{--+}) \bigcap E(G)} deg_G(v) - 2 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} deg_G(u)] \\
&= m(2n+4)^2 + \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 3 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} deg_G(u)^2 + 4 \sum_{u,v \in V(G^{--+}) \bigcap E(G)} (deg_G(u) \cdot deg_G(v)) \\
&+ 2(2n+4)[2m - 4m] \\
&= m(2n+4)^2 + F(G) + 3M_1(G) - 4M_2(G) - 4m(2n+4).
\end{aligned}
$$

c) If $u \in V(G^{--+}) \bigcap V(G)$ and $v \in V(G^{--+}) \bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{--+}}(e) &= m + n - 2deg_G(u) - deg_G(v) + 2 \\
&= \sum_{u \in V(G^{--+}) \bigcap V(G), v \in V(G^{--+}) \bigcap E(G)} [m+n+2 - 2deg_G(u) - deg_G(v)]^2 \\
&= \sum_{u \in V(G^{--+}) \bigcap V(G), v \in V(G^{--+}) \bigcap E(G)} [m+n+2]^2 \\
&+ \sum_{u \in V(G^{--+}) \bigcap V(G), v \in V(G^{--+}) \bigcap E(G)} [2deg_G(u) + deg_G(v)]^2 \\
&- 2 \sum_{u \in V(G^{--+}) \bigcap V(G), v \in V(G^{--+}) \bigcap E(G)} (m+n+2)(2deg_G(u) + deg_G(v)) \\
&= m(m+n+2)^2 + \sum_{u \in V(G^{--+}) \bigcap V(G), v \in V(G^{--+}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 3 \sum_{u \in V(G^{--+}) \bigcap V(G), v \in V(G^{--+}) \bigcap E(G)} (deg_G(u)^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 4 \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (deg_G(u)\cdot deg_G(v))\\
&- \quad 2(m+n+2) \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (2deg_G(u)+deg_G(v))\\
&= \quad m(m+n+2)^2 + \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (deg_G(u)^2+deg_G(v)^2)\\
&+ \quad 3 \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (deg_G(u)^2)\\
&+ \quad 4 \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (deg_G(u)\cdot deg_G(v))\\
&- \quad 2(m+n+2) \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (deg_G(u)+deg_G(v))\\
&- \quad 2(m+n+2) \sum_{u\in V(G^{--+})\bigcap V(G),v\in V(G^{--+})\bigcap E(G)} (deg_G(u))\\
&= \quad m(m+n+2)^2+3M_1(G)+F(G)+4M_2(G)\\
&- \quad 2(m+n+2)M_1(G)-4(m+n+2)m\\
&= \quad m(m+n+2)(m+n-2)+F(G)-M_1(G)[2m+2n+1]+4M_2(G).
\end{aligned}
$$

Theorem 2.9. *For any graph G,*

$$
\begin{aligned}
M_1^w(G^{+--}) &= n(2n-2)^2\\
&+ 4m(m+n-2)(m+n-4)+F(G)+3M_1(G)-4M_2(G)\\
&+ m[2m+n-3]^2+F(G)+2M_2(G)-2(2m+n-3)M_1(G).
\end{aligned}
$$

Proof: Degree of every edge in G^{+--}

a) If $u,v\in V(G^{+--})\bigcap V(G)$ then

$$
\begin{aligned}
deg_{G^{+--}}(e) &= 2n-2\\
&= \sum_{u,v\in V(G^{+--})\bigcap V(G)} [2n-2]^2\\
&= n(2n-2)^2.
\end{aligned}
$$

b) If $u,v\in V(G^{+--})\bigcap E(G)$ then

$$
\begin{aligned}
deg_{G^{+--}}(e) &= \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-2deg_G(u)+deg_G(v)-4]^2\\
&= \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4-2deg_G(u)+deg_G(v)]^2\\
&= \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4]^2 + \sum_{u,v\in V(G^{+--})\bigcap E(G)} [deg_G(v)-2deg_G(u)]^2\\
&+ 2\sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4][deg_G(v)-2deg_G(u)
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4]^2 + \sum_{u,v\in V(G^{+--})\bigcap E(G)} [deg_G(v)^2 + 4deg_G(u)^2] \\
&- 4 \sum_{u,v\in V(G^{+--})\bigcap E(G)} [deg_G(u)\cdot deg_G(v)] \\
&+ 2 \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4](deg_G(v) - 2deg_G(u)) \\
&= \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4]^2 + \sum_{u,v\in V(G^{+--})\bigcap E(G)} [deg_G(v)^2 + deg_G(u)^2] \\
&+ 3 \sum_{u,v\in V(G^{+--})\bigcap E(G)} (deg_G(u))^2 - 4 \sum_{u,v\in V(G^{+--})\bigcap E(G)} [deg_G(u)\cdot deg_G(v)] \\
&+ 2 \sum_{u,v\in V(G^{+--})\bigcap E(G)} [2m+2n-4](deg_G(v) - 2deg_G(u)) \\
&= (2m+2n-4)^2 m + F(G) + 3M_1(G) - 4M_2(G) - 8m(m+n-2) \\
&= 4m(m+n-2)^2 + F(G) + 3M_1(G) - 4M_2(G) - 8m(m+n-2) \\
&= 4m(m+n-2)(m+n-4) + F(G) + 3M_1(G) - 4M_2(G).
\end{aligned}$$

c) If $u \in V(G^{+--})\bigcap V(G)$ and $v \in V(G^{+--})\bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{+--}}(e) &= 2m+n-(deg_G(u)+deg_G(v)) - 3 \\
&= \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} [2m+n-(deg_G(u)+deg_G(v))-3]^2 \\
&= \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} [2m+n-3-(deg_G(u)+deg_G(v))]^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} [2m+n-3]^2 \\
&+ \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} (deg_G(u)+deg_G(v))^2 \\
&- 2(2m+n-3) \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} (deg_G(u)+deg_G(v)) \\
&= m[2m+n-3]^2 + \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} deg_G(u)^2 \\
&+ \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} deg_G(v)^2 \\
&+ 2 \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} (deg_G(u)\cdot deg_G(v)) \\
&- 2(2m+n-3) \sum_{u\in V(G^{+--})\bigcap V(G), v\in V(G^{+--})\bigcap E(G)} (deg_G(u)+deg_G(v)) \\
&= m[2m+n-3]^2 + F(G) + 2M_2(G) - 2(2m+n-3)M_1(G).
\end{aligned}$$

Theorem 2.10. *For any graph G,*

$$\begin{aligned}
M_1^w(G^{-++}) &= 4n(n^2-4n+4) \\
&+ F(G)-M_1(G)+4M_2(G)+4n-8m \\
&+ 2(n-3)M_1(G)+2M_2(G)+F(G)+n(n-3)^2
\end{aligned}$$

Proof: Degree of every edge in G^{-++}
a) If $u,v \in V(G^{-++})\bigcap V(G)$ then

$$\begin{aligned}
deg_{G^{-++}}(e) &= 2n-4 \\
&= \sum_{u,v\in V(G^{-++})\bigcap V(G)} [2n-4]^2 \\
&= n(2n-4)^2 \\
&= 4n(n^2-4n+4).
\end{aligned}$$

b) If $u,v \in V(G^{-++})\bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{-++}(G)}(e) &= \sum_{u,v\in V(G^{-++})\bigcap E(G)} [2deg_G(u)+deg_G(v)-2]^2 \\
&= \sum_{u,v\in V(G^{-++})\bigcap E(G)} [2deg_G(u)+deg_G(v)]^2 + \sum_{u,v\in V(G^{-++})\bigcap E(G)} 4 \\
&- 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} (2deg_G(u)+deg_G(v)) \\
&= \sum_{u,v\in V(G^{-++})\bigcap E(G)} [4deg_G(u)^2+deg_G(v)^2] \\
&+ 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} (deg_G(u)\cdot deg_G(v)) \\
&+ \sum_{u,v\in V(G^{-++})\bigcap E(G)} 4 - 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} (deg_G(u)) \\
&- 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} (deg_G(u)+deg_G(v)) \\
&= 3\sum_{u,v\in V(G^{-++})\bigcap E(G)} [deg_G(u)]^2 + \sum_{u,v\in V(G^{-++})\bigcap E(G)} [deg_G(u)^2+deg_G(v)^2] \\
&+ 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} (deg_G(u)\cdot deg_G(v)) + \sum_{u,v\in V(G^{-++})\bigcap E(G)} 4 \\
&- 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} (deg_G(u)) - 4\sum_{u,v\in V(G^{-++})\bigcap E(G)} [deg_G(u)+deg_G(v)] \\
&= 3M_1(G)+F(G)+4M_2(G)-4M_1(G)+4n-8m \\
&= F(G)-M_1(G)+4M_2(G)+4n-8m.
\end{aligned}$$

c) If $u \in V(G^{-++}) \bigcap V(G)$ and $v \in V(G^{-++}) \bigcap E(G)$ then

$$\begin{aligned}
deg_{G^{-++}}(e) &= deg_G(u) + deg_G(v) + n - 3 \\
&= \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} [deg_G(u) + deg_G(v) + n - 3]^2 \\
&= \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} (deg_G(u) + deg_G(v))^2 \\
&+ \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} [n - 3]^2 \\
&+ 2 \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} (deg_G(u) + deg_G(v))(n - 3) \\
&= \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} (deg_G(u)^2 + deg_G(v)^2) \\
&+ 2 \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} (deg_G(u) \cdot deg_G(v)) \\
&+ n[n - 3]^2 + 2(n - 3) \sum_{u \in V(G^{-++}) \bigcap V(G), v \in V(G^{-++}) \bigcap E(G)} (deg_G(u) + deg_G(v)) \\
&= F(G) + 2M_2(G) + n(n - 3)^2 + 2(n - 3)M_1(G) \\
&= 2(n - 3)M_1(G) + 2M_2(G) + F(G) + n(n - 3)^2.
\end{aligned}$$

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