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ON THE PARADOXICAL BEHAVIOR OF DIVERGENT SERIES

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Abstract. This paper presents some considerations on summation method on infinite divergent series. Irrefutably divergent series don't have a sum in the traditional logic of the term. However there are extensions, where transformed definitions apportion changed values to the same divergent series and they rarely have agreeable properties. Particularly, at beginning and manipulating these instinctively, it easily comes across nasty paradoxes. In this article for any odd prime p, the divergent series $\sum\limits_{n\geqslant 1} n$ is proved to be represented by a new formula $\sum\limits_{i=1}^{(p-1)/2} i \Big/ \left(1-p^2\right)$ and on using this representation for $\sum\limits_{n\geqslant 1} n$ further leads to a paradoxical bewildering novel formula $\sum\limits_{n\geqslant 1} u_n = -\frac{1}{8}$, which evidently contradicts the basic principles of arithmetic and the definition of a divergent series identical to Ramanujan paradox $\sum\limits_{n\geqslant 1} u_n = -\frac{1}{12}$. Illustrations to support this illogicality result are discussed analytically and demonstrated graphically.

1. Introduction

Analysis of sums of infinite series are congregated into convergent, oscillating and divergent. Irrefutably divergent series don't have a sum in the traditional logic of the terminology. However there are several extensions, where transformed definitions apportion changed values to the same divergent series and they rarely have agreeable properties.

Divergent series emerges very frequently in several natural mathematical science problems. The useof such serieslegalized to do successfullyconsiderablenumerical calculations. Authors [1, 3] computed the secular perturbation of Earth orbit around the Sun due to the attraction of Jupiter by Laplace, where Laplace calculations were certified experimentally, whereas the series used were divergent.

Divergent series have some curious properties like on rearranging the terms of series $1-1+1-1+1-\ldots$ gives both $(1-1)+(1-1)+(1-1)+\ldots=0$ and $1-(1-1)-(1-1)+\ldots=1$.

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Niels Henrik Abel [4,,6,11] Gardner 1984, p. 171; Hoffman 1998, p. 218) stated that the divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever written by NielsHenrik Abe l(Gardner 1984, p. 171; Hoffman 1998, p. 218). Nevertheless, divergent series can truly be "summed" meticulously by using extensions to the usual summation rules (e.g., so-called Abel and Cesàro sums).By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes.

According to the Riemann series theorem, suitable rearrangement of series terms, a conditionally convergent series may be made to converge to any anticipated value, or to diverge.On the behaviour ofdivergent series researchers considered various problems [2,7-10].

The paradoxical outcome of divergent series led to considerable debate and discussion in mathematicsover the centuries, and there is still no universally acknowledged way to assign a value to divergent series. In general, while dealing with such series and using appropriate techniques and methods to avoid paradoxes and contradictions.

Summation of an infinite natural numbersas sequential terms is a series,plays a significant role in mathematics and other reckonable areas, such as management, statistics, finance, physical science, etc., as it supports to understand and predict performances using identified patterns.While employing series, it is vital to describe whether the series diverges or converges.

Thus working with divergent sums, technique employed and recognizing the properties of the method are explicitly important.Dutch physicist Hendrik Casimir [5] in 1948 predicted experimentally and confirmed through various measurements that the divergent series $\sum_{n\geqslant 0} n$ and $\sum_{n\geqslant 0} n^3$ are about the study of the Casimir effect which is a phenomenon in quantum field theory that manifests as a force between two closely spaced parallel uncharged conducting plates or surfaces.

The regularization techniques for divergent series are fascinating areas of study in mathematics and physics, and continue to be the subject of active research and debate. The technique involves using a regularization that allows one to assign a finite value to a divergent series.One of the most famous instances of a divergent series that can be summed using the Ramanujan summation [12] is

$$\sum_{n\geqslant 1} u_n = -\frac{1}{12} \tag{1.1}$$

If the results (1.1) and $\sum_{n\geqslant 1} u_n = -\frac{1}{8}$ are true, it is enough to quiver one's belief in mathematics, and is completely non-intuitive,firstly asthe sum of an infinite series of positive integers should not be a negative value, secondly should not be a fractional value and finally should not converge to a finite value.

In this article it is exhibited that for any odd prime$p$, the divergent series $\sum_{n\geqslant 1} n$ can be represented as $\sum_{i=1}^{(p-1)/2} i \Big/ (1-p^2)$, then expending this representation itis further proved that sum of the series leads to an another paradoxical result $\sum_{n\geqslant 1} u_n = -\frac{1}{8}$. This result also contradicts the basic principles of arithmetic and the definition of a divergent series alike to Ramanujan Paradox (1.1). Theoretical and noticeableillustrations to support this absurdity are also demonstrated and discussed.Theoretical obtainedresult is also represented graphically.re.

2. THEORETICAL RESULT

Theorem 2.1. *Let p be any odd prime, then*

$$\sum_{n\geqslant 1, n\in N} n = \sum_{i=1}^{(p-1)/2} i \Big/ \left(1-p^2\right) = -\frac{1}{8} \tag{2.1}$$

Proof. To prove (2.1), let

$$\sum_{n\geq 1, n\in N} n = 1+2+3+4+\cdots = S \tag{2.2}$$

Now consider any odd prime p and write the first $\frac{p-1}{2}$ terms of of above equation (2.2), and then rearrange the remaining terms in the sum S in groups of p terms, then sum S can be written as

$$S = 1+2+3+4+\left(\frac{p-1}{2}\right)+p^2 S = \sum_{i=1}^{(p-1)/2} i + p^2 S \tag{2.3}$$

Further the above sum S is simplified as

$$S = \frac{\sum_{i=1}^{(p-1)/2} i}{1-p^2}. \tag{2.4}$$

The above equation cane be written as

$$S = \frac{\left(\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)/2\ \right)}{(1-p^2)} = -\frac{1}{8}. \tag{2.5}$$

Hence

$$\sum_{n\geqslant 1} u_n = -\frac{1}{8}. \tag{2.6}$$

□

3. ILLUSTRATIONS

Example 3.1. consider any odd prime $p = 3$, write the the first $\frac{p-1}{2}$ terms of as its and then rearrange the remaining terms in the sum S in groups of $p(=3)$

terms, then sum S can be written as

$$\begin{aligned}\mathrm{S} &= 1+\overline{2+3+4}+\overline{5+6+7}+\overline{8+9+10}+\cdots\cdots\infty\\ &=1+9+18+27+\cdots\cdots\infty\\ &=1+3^2(1+2+3+\cdots\cdots\infty)\end{aligned}$$

Avove equation can be written as $S=1+(3^2)S \Rightarrow S-(3^2)S=1$ Hence

$$S=1+(2+3+4)+(5+6+7)+\cdots\cdots\infty=-\frac{1}{8} \tag{3.1}$$

Example 3.2. consider any odd prime $p=11$, write the the first $\frac{p-1}{2}$ terms of as its and then rearrange the remaining terms in the sum S in groups of $p(=11)$ terms, then sum S can be written as

$$\begin{aligned}\mathrm{S} &= 1+2+3+4+5+\sum_{n=6}^{16} n+\sum_{n=17}^{27} n+\sum_{n=28}^{38} n+\cdots\cdots\infty\\ &=15+121+242+363+\cdots\cdots\infty\\ &=15+11^2(1+2+3+\cdots\cdots\infty).\end{aligned}$$

Rewrite the above result as

$$\begin{aligned}&S=15+11^2S\\ &S-11^2S=15\\ &\left(1-11^2\right)S=15\\ &\Rightarrow S=\frac{\sum_{i=1}^{(11-1)/2} i}{\left(1-11^2\right)}=-\frac{1}{8}.\end{aligned}$$

Hence paradoxical bewildering formula in ths case is $\sum_{n\geq 1} u_n=-\frac{1}{8}$, for odd prime $p=11$ an odd prime.

4. GRAPHICAL REPRESENTATION

Paradoxical formula $\sum_{i=1}^{(p-1)/2} i/(1-p^2)$ obtained in Theorem 2.1 is tested and verified for any odd prime p, however large or small, it is exhibited graphically that it converges to $-\frac{1}{8}$.

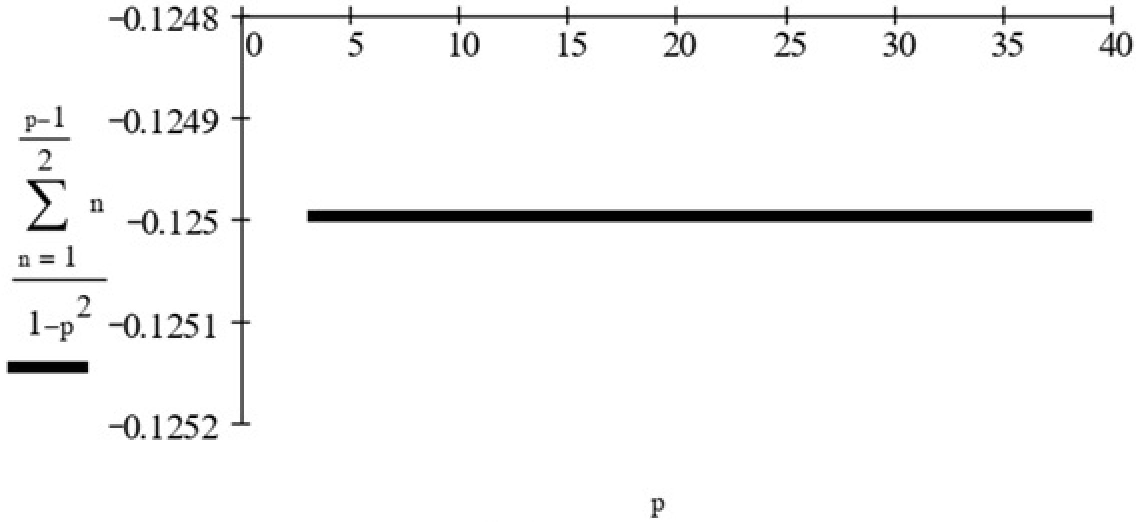

Figure: $\sum n = \sum^{(p-1)/2} i \Big/ \left(1-p^2\right)$ Vs any odd prime p.

5. Conclusion

Conclusively a divergent series don't have a sum in the logic of the terminology. On the other hand, they can be manipulated to have a number of mathematically interesting results. Several extensions, where transformed definitions apportion changed values to the same divergent series and they rarely have agreeable properties. In this article, similar to Ramanujan Paradox, another paradoxical result is obtained by manipulating the terms of the series.

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