Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 19 (2023) 42-52 https://doi.org/10.56947/amcs.v19.217

BI-CIRCULAR MODEL WITH TEST PARTICLE VARIABLE MASS

ABDULLAH A. ANSARI[1,*] AND SABA BANO[2]

Abstract. This paper presents the effects of the perturbations on the motion of the test particle variable mass in the bi-circular Sun perturbed oblate Earth-Moon system where primary is taken as oblate Earth, secondary is taken as Moon and the third body is assumed as Sun. These three bodies are moving in the same plane around barycenter. The fourth and smallest body (test particle is assumed to be having variable mass with time) is moving in the space under the perturbations such as gravitational forces of the primaries, the solar radiation pressure, the coriolis and centrifugal forces. Then we determine the equations of motion, the locations of critical points, their stability and periodic orbits.

1. Introduction

The classical circular restricted 3-body problem is application based and most studied problem with various aspects in the celestial mechanics and dynamical astronomy. In this problem, two massive bodies are revolving in circular or elliptical orbits around their common center of mass (in general it is taken as origin) and the third body which is smaller than the other two massive bodies, is moving in the space under the gravitational forces of the massive bodies. Some researchers are as follow: [38], [21], [6], [10], [7], [9], [15], [24], [26].

Bi-circular problem and four bodies problems are the extensions of the restricted three-body problem. These problems are investigated by many scientists with various types of perturbations. Some of them are as [23], [20], [33], [31], [28], [34], [16], [19], [35], [27], [1], [18], [37], [8], [29]. Due to the variable mass, we can not reveal the motion properties by ordinary methods therefore many researchers have studied the effects of variable mass by using the Jeans law and Meshcherskii transformations. Some of them are as follows: [30], [39], [2, 4, 17, 3, 5], [36], [22], [13], [12], [14].

This paper is organised as: Section 1 presents the literature review while section 2 illustrates the model description with equations of motion. Section 3 represents the locations of critical points while section 4 examine the stability of the critical points. Sections 5 performs periodic orbits. Finally, the conclusion is drawn in section 6.

Date: Received: Sep 29, 2023; Accepted: Oct 20, 2023.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 70F05; 70F15; 37N05.

Key words and phrases. Bi-circular model, variable mass, critical points, oblate body.

2. Description of the problem with Equations of motion

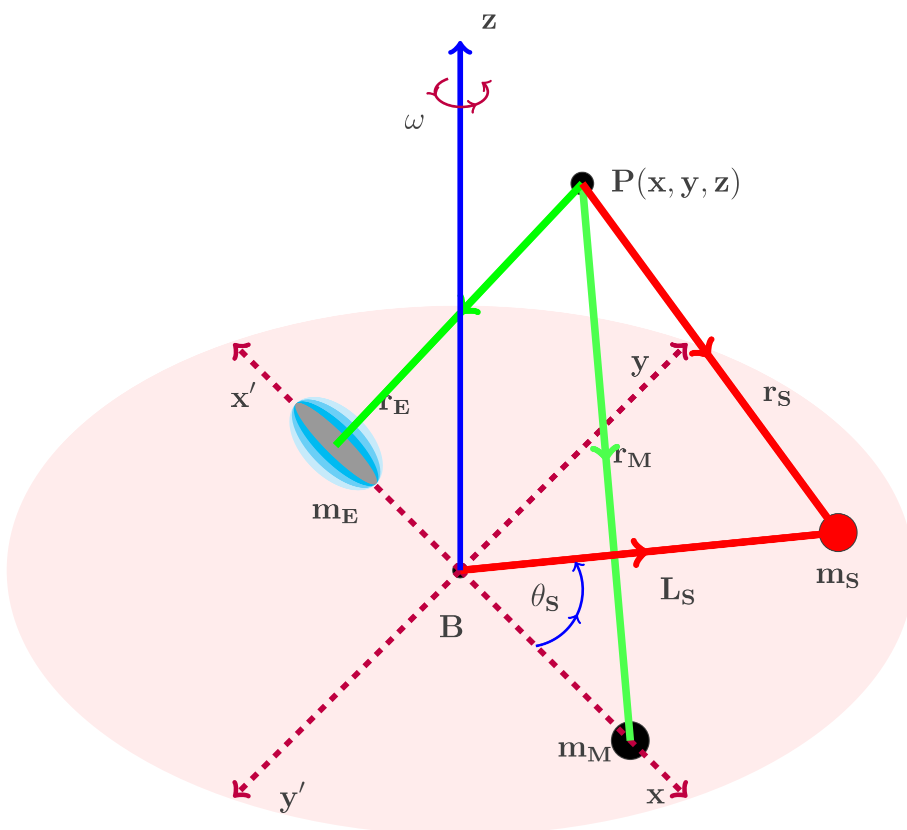

Figure 1. Geometric view of the bi-circular model with test particle variable mass

In this bi-circular model, we suppose the Earth (of mass m_E) as primary, the Moon (of mass m_M) as secondary and the Sun (of mass m_S) as the third massive body. Where the Earth is taken as oblate in shape while the Moon as well as the Sun are taken as spherical in shapes and mass of the test particle is supposed as variable with time. Also the effects of solar radiation pressure, the coriolis and centrifugal forces are assumed. We also assumed that the Earth and Moon are revolving around the common center of mass (also known as barycenter) which is taken here the origin. In this model for the motion properties of the test particle, we suppose the Sun is revolving around barycenter of the Sun-Earth-Moon system. We fix the unit here as the distance between the Earth and Moon, the sum of the masses of both bodies are separately unity. Hence, let $\mu = m_M/(m_E + m_M) = \mu_M$ be the normalized mass of the Moon, $(1 - \mu) = \mu_E$ be the normalized mass of the Earth and μ_S be the normalized mass of the Sun. Complete view is given in figure 1. Now utilizing the method given by [11], we can write the equations of motion of the variable mass test particle as

$$
\begin{aligned}
\frac{\dot{m}}{m}\,(\dot{x}\,-\,\nu\,\varphi_1\,y) \;&+\; \ddot{x}\,-\,2\,\nu\,\varphi_1\,\dot{y} \;=\; \frac{\partial\,\Pi}{\partial\,x},\\
\frac{\dot{m}}{m}\,(\dot{y}\,+\,\nu\,\varphi_1\,x) \;&+\; \ddot{y}\,+\,2\,\nu\,\varphi_1\,\dot{x} \;=\; \frac{\partial\,\Pi}{\partial\,y},\\
\frac{\dot{m}}{m}\,\dot{z} \;&+\; \ddot{z} \;=\; \frac{\partial\,\Pi}{\partial\,z}.
\end{aligned}
\tag{2.1}
$$

with

$$
\begin{aligned}
\Pi \;&=\; \frac{\nu^2\,\varphi_2}{2}\,(x^2\,+\,y^2) + \frac{\mu_E}{r_E} + \frac{\mu_E\,\sigma}{2\,r_E^3} + \frac{\mu_M}{r_M} + \frac{\alpha\,\mu_S}{r_S} - \frac{\mu_S}{L_S^2}\,(x\;\mathrm{Cos}\,\theta_S + y\,\mathrm{Sin}\,\theta_S),\\
\nu^2 \;&=\; 1\,+\,\frac{3}{2}\,\sigma,\\
r_E^2 \;&=\; (x\,+\,\mu)^2 + y^2 + z^2,\\
r_M^2 \;&=\; (x\,+\,\mu\,-\,1)^2 + y^2 + z^2,\\
r_S^2 \;&=\; (x\,-\,x_S)^2 + (y - y_S)^2 + z^2.
\end{aligned}
$$

Due to mass variation of the test particle, we will utilize Jean's law [25] and Meshcherskii space-time transformations [32] as:

$$
m \;=\; m_0\,e^{-d_1\,t} \;\text{ and }\; (x,\,y,\,z) \;=\; d_2^{-1/2}\,(X,Y,Z). \tag{2.2}
$$

where d_1 is variation constant and $d_2 = \dfrac{m}{m_0}$, where m_0 is the initial mass.
Eqs. (2.1) and (2.2) give as

$$
\begin{aligned}
\ddot{X}\,-\,2\,\nu\,\varphi_1\,\dot{Y} \;&=\; \frac{\partial\,\Upsilon}{\partial\,X},\\
\ddot{Y}\,+\,2\,\nu\,\varphi_1\,\dot{X} \;&=\; \frac{\partial\,\Upsilon}{\partial\,Y},\\
\ddot{Z} \;&=\; \frac{\partial\,\Upsilon}{\partial\,Z},
\end{aligned}
\tag{2.3}
$$

with

$$\begin{aligned}
\Upsilon &= \frac{\varphi_2\,\nu^2}{2}\,(X^2+Y^2)+\frac{d_1^2}{8}\,(X^2+Y^2+Z^2)+d_2^{3/2}\left(\frac{\mu_E}{R_E}+\frac{\mu_E\,\sigma\,d_2}{2\,R_E^3}+\frac{\mu_M}{R_M}+\frac{\alpha\,\mu_S}{R_S}\right)\\
&\quad -\frac{\mu_S\sqrt{d_2}}{L_S^2}\,(X\ \mathrm{Cos}\,\theta_S+Y\,\mathrm{Sin}\,\theta_S),\\
R_E^2 &= \left(X+\mu\sqrt{d_2}\right)^2+Y^2+Z^2,\\
R_M^2 &= \left(X+(\mu-1)\sqrt{d_2}\right)^2+Y^2+Z^2,\\
R_S^2 &= \left(X-x_S\sqrt{d_2}\right)^2+\left(Y-y_S\sqrt{d_2}\right)^2+Z^2,.
\end{aligned}$$

3. Critical points

The locations of critical points can be obtained by solving the $\partial\Upsilon/\partial X=0$ and $\partial\Upsilon/\partial Y=0$.

After numerically solving these partial derivatives for the numerical values of the parameters as $\mu=0.01215$, $q=0.8$, $d_1=0.2$, $d_2=0.4$, $\varphi_1=\varphi_2=1.2$, $\sigma=0.001$, $L_S=388.81114$, $\mu_S=328900.54$, one can illustrate the locations of the critical points. Here we got three critical points near barycenter of the Earth-Moon system while one critical point near Sun, these positions of critical points are given in figures (2) and (3) for different values of θ_S. Figure (2) contains four sub-figures a, b, c, d, where sub-figure (2)(a) is for $\theta=0$, sub-figure (2)(b) is for $\theta=\pi/6,\pi/4,\pi/3,\pi/2,2\pi/3,3\pi/4,5\pi/6$, sub-figure (2)(c) is for $\theta=\pi$ and sub-figure (2)(d) is for $\theta=4\pi/3,5\pi/4,7\pi/6,3\pi/2,5\pi/3,7\pi/4,11\pi/6$. Out of these sub-figures, from sub-figures (2)(c) and (2)(c), we observed that three critical points are lie on the X-axis while in the other two sub-figures 2(b), 2(d), these critical points are not lie on the X-axis. We also observed that with the variation of θ_S, these three critical points are changing there locations. Now, figure (3) shows the location of critical points near the third primary (i.e. near Sun) for different values of $\theta_S=0,\pi/6,\pi/4,\pi/3,\pi/2,2\pi/3,3\pi/4,5\pi/6,\pi,4\pi/3,5\pi/4,7\pi/6,3\pi/2,$ $5\pi/3,7\pi/4,11\pi/6$. We observed from here that as Sun is moving in the anti-clockwise direction, the location of critical point is also moving near the Sun.

4. Stability analysis of the fixed points

Here we will examine the stability of the critical points for which we will displace the critical point (X_{01},Y_{01},Z_{01}) as

$$\begin{aligned}
X &= X_{01}+X_{02},\\
Y &= Y_{01}+Y_{02},\\
Z &= Z_{01}+Z_{02},
\end{aligned} \tag{4.1}$$

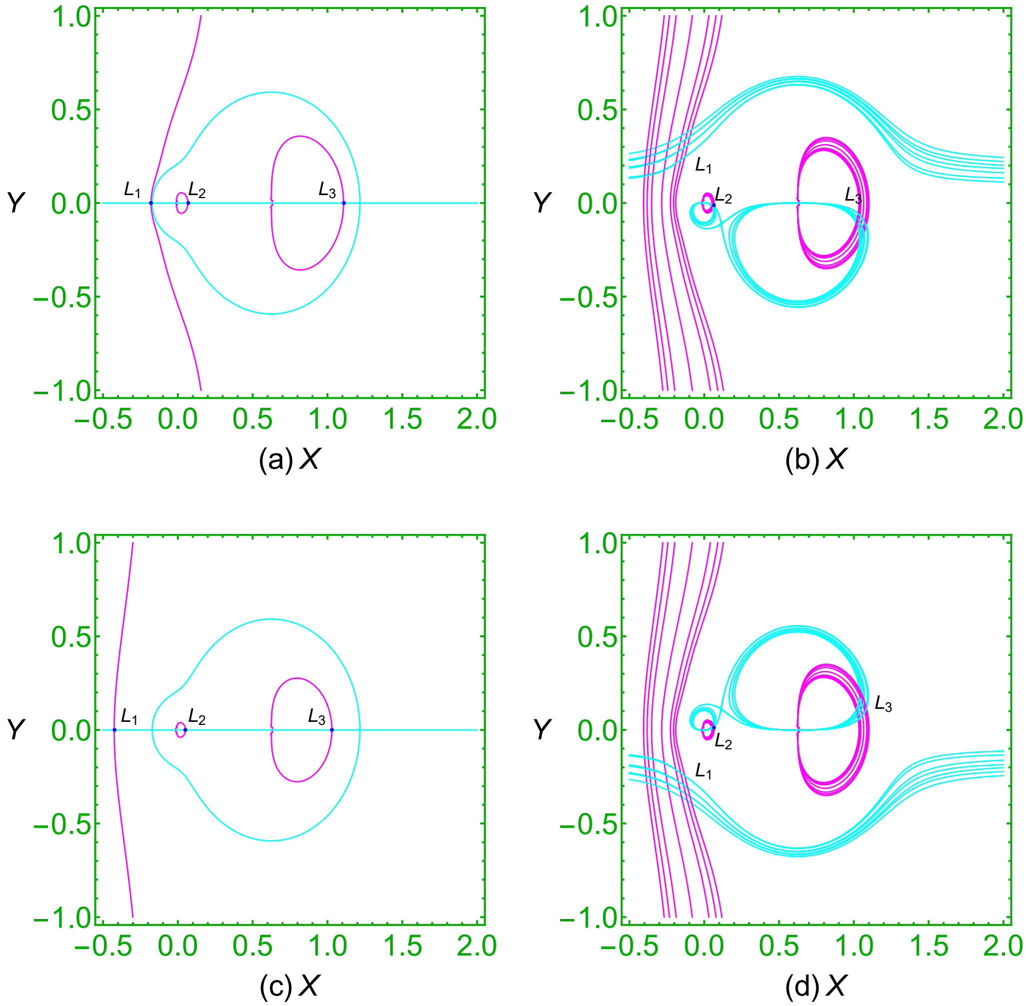

FIGURE 2. Position of critical points near the primary and secondary for various values of θ_S

where (X_{02}, Y_{02}, Z_{02}) is the displacement from the critical point.
Hence the equations of motion (2.3) becomes as:

$$
\begin{aligned}
&\ddot{X}_{02} - 2\,\nu\,\phi_1\,\dot{Y}_{02} = \Upsilon^0_{X\,X}\,X_{02} + \Upsilon^0_{X\,Y}\,Y_{02} + \Upsilon^0_{X\,Z}\,Z_{02},\\
&\ddot{Y}_{02} + 2\,\nu\,\phi_1\,\dot{X}_{02} = \Upsilon^0_{Y\,X}\,X_{02} + \Upsilon^0_{Y\,Y}\,Y_{02} + \Upsilon^0_{Y\,Z}\,Z_{02},\\
&\ddot{Z}_{02} = \Upsilon^0_{Z\,X}\,X_{02} + \Upsilon^0_{Z\,Y}\,Y_{02} + \Upsilon^0_{Z\,Z}\,Z_{02}.
\end{aligned}
\tag{4.2}
$$

where superscript zero represents the value at critical point. The subscripts of Υ in Eq. (4.2) denotes the derivatives with respect to respective variables.

The system (4.2) can be rewritten as

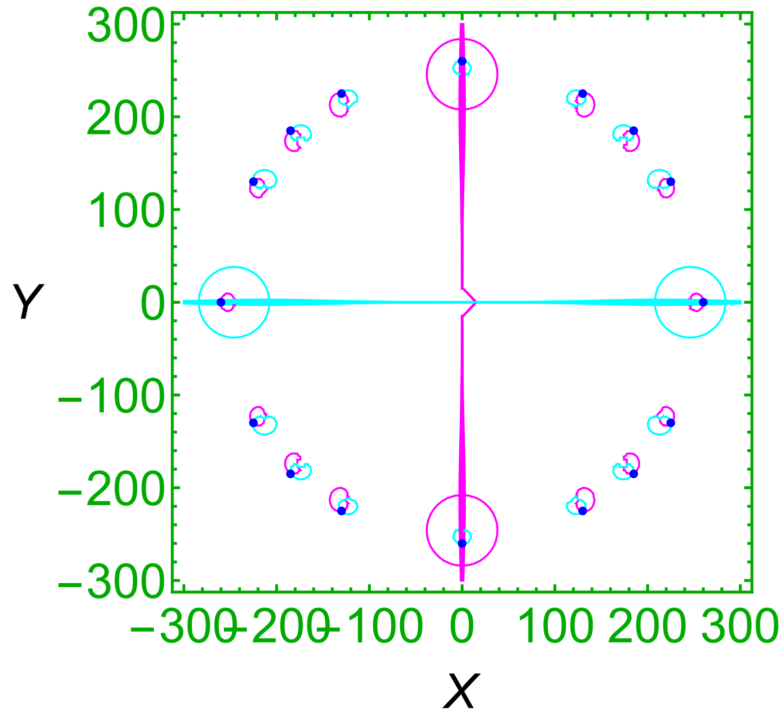

FIGURE 3. Position of critical point L_4 near the third primary (i.e. near The Sun) for various values of θ_S

$$
\begin{aligned}
\dot{X}_{02} &= X_{03},\\
\dot{Y}_{02} &= Y_{03},\\
\dot{Z}_{02} &= Z_{03},\\
\dot{X}_{03} &= \Upsilon^0_{X\,X}\,X_{02} + \Upsilon^0_{X\,Y}\,Y_{02} + \Upsilon^0_{X\,Z}\,Z_{02} + 2\,\nu\,\phi_1\,Y_{03},\\
\dot{Y}_{03} &= \Upsilon^0_{Y\,X}\,X_{02} + \Upsilon^0_{Y\,Y}\,Y_{02} + \Upsilon^0_{Y\,Z}\,Z_{02} - 2\,\nu\,\phi_1\,X_{03},\\
\dot{Z}_{03} &= \Upsilon^0_{Z\,X}\,X_{02} + \Upsilon^0_{Z\,Y}\,Y_{02} + \Upsilon^0_{Z\,Z}\,Z_{02}.
\end{aligned}
\tag{4.3}
$$

Due to mass variation of the test particle, we will use the Meshcherskii-space-time inverse transformations which are as follow:

$$
\begin{aligned}
X_{04} &= \frac{1}{\sqrt{d_2}}\,X_{02}, \quad X_{05} = \frac{1}{\sqrt{d_2}}\,X_{03},\\
Y_{04} &= \frac{1}{\sqrt{d_2}}\,Y_{02}, \quad Y_{05} = \frac{1}{\sqrt{d_2}}\,Y_{03},\\
Z_{04} &= \frac{1}{\sqrt{d_2}}\,Z_{02}, \quad Z_{05} = \frac{1}{\sqrt{d_2}}\,Z_{03}.
\end{aligned}
\tag{4.4}
$$

From Eqs. (4.3) and (4.4), we can obtain

$$\dot{\Gamma} \;=\; \mathbf{H} \times \Gamma, \tag{4.5}$$

where

$$\dot{\Gamma} \;=\; \begin{pmatrix} \dot{X}_{04} \\ \dot{Y}_{04} \\ \dot{Z}_{04} \\ \dot{X}_{05} \\ \dot{Y}_{05} \\ \dot{Z}_{05} \end{pmatrix}, \mathbf{H} \;=\; \begin{pmatrix} \frac{1}{2}\,d_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2}\,d_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\,d_1 & 0 & 0 & 1 \\ \Upsilon^0_{X\,X} & \Upsilon^0_{X\,Y} & \Upsilon^0_{X\,Z} & \frac{1}{2}\,d_1 & 2\,\nu\,\phi_1 & 0 \\ \Upsilon^0_{Y\,X} & \Upsilon^0_{Y\,Y} & \Upsilon^0_{Y\,Z} & -\,2\,\nu\,\phi_1 & \frac{1}{2}\,d_1 & 0 \\ \Upsilon^0_{Z\,X} & \Upsilon^0_{Z\,Y} & \Upsilon^0_{Z\,Z} & 0 & 0 & \frac{1}{2}\,d_1 \end{pmatrix} \;\&\, \Gamma \;=\; \begin{pmatrix} X_{04} \\ Y_{04} \\ Z_{04} \\ X_{05} \\ Y_{05} \\ Z_{05} \end{pmatrix}. \tag{4.6}$$

Now the characteristic polynomial of the matrix H will be

$$\lambda^6 \,+\, \Upsilon_5\,\lambda^5 \,+\, \Upsilon_4\,\lambda^4 \,+\, \Upsilon_3\,\lambda^3 + \,\Upsilon_2\,\lambda^2 + \,\Upsilon_1\,\lambda + \Upsilon_0, \tag{4.7}$$

where,

$$\begin{aligned}
\Upsilon_5 =&\; -\,3\,d_1 \\
\Upsilon_4 =&\; \frac{15}{4}\,d_1^2 + 4\,\phi_1^2\,\nu^2 \,-\, (\Upsilon_{X\,X} + \Upsilon_{Y\,Y} + \Upsilon_{Z\,Z}), \\
\Upsilon_3 =&\; -\,\frac{5}{2}\,d_1^3 - 8\,\phi_1^2\,d_1\,\nu^2 \,+\, 2\,d_1\,(\Upsilon_{X\,X} + \Upsilon_{Y\,Y} + \Upsilon_{Z\,Z}), \\
\Upsilon_2 =&\; \frac{1}{16}\{15\,d_1^4 + 96\,\phi_1^2\,d_1^2\,\nu^2 \,-24\,d_1^2\,(\Upsilon_{X\,X} \,+\, \Upsilon_{Y\,Y} \,+\, \Upsilon_{Z\,Z}) - 64\phi_1^2\,\nu^2\,\Upsilon_{Z\,Z} \\
&+\, 16(\Upsilon_{X\,X}\,\Upsilon_{Y\,Y} \,+\, \Upsilon_{X\,X}\,\Upsilon_{Z\,Z} \,+\, \Upsilon_{Y\,Y}\,\Upsilon_{Z\,Z} \,-\, \Upsilon^2_{X\,Y})\}, \\
\Upsilon_1 =&\; \frac{1}{16}\{-3\,d_1^5 - 32\,\phi_1^2\,d_1^3\,\nu^2 + 8\,d_1^3\,(\Upsilon_{X\,X} \,+\, \Upsilon_{Y\,Y} \,+\, \Upsilon_{Z\,Z}) + 64\,\phi_1^2\,\nu^2\,d_1\,\Upsilon_{Z\,Z} \\
&-\, 16\,d_1\,(\Upsilon_{X\,X}\,\Upsilon_{Y\,Y} \,+\, \Upsilon_{X\,X}\,\Upsilon_{Z\,Z} + \,\Upsilon_{Y\,Y}\,\Upsilon_{Z\,Z} \,+\, \Upsilon^2_{X\,Y})\}, \\
\Upsilon_0 =&\; \frac{1}{64}\{d_1^6 + 16\,\phi_1^2\,d_1^4\,\nu^2 - 4\,d_1^4\,(\Upsilon_{X\,X} \,+\, \Upsilon_{Y\,Y} \,+\, \Upsilon_{Z\,Z}) \\
&-\, 64\,\Upsilon_{Z\,Z}\,(\phi_1^2\,\nu^2\,d_1^2 \,+\, \Upsilon_{X\,X}\,\Upsilon_{Y\,Y} \,-\, \Upsilon^2_{X\,Y}) \\
&+\, 16\,d_1^2\,(\Upsilon_{X\,X}\,\Upsilon_{Y\,Y} \,+\, \Upsilon_{X\,X}\,\Upsilon_{Z\,Z} + \,\Upsilon_{Y\,Y}\,\Upsilon_{Z\,Z} \,-\, \Upsilon^2_{X\,Y})\}.
\end{aligned} \tag{4.8}$$

After numerically solving Eq. (4.7) for the various values of the parameters used corresponding to the fixed points, we got that all the roots are either positive real parts of the complex roots or at-least one root is positive real number which are given in the table 1. Therefore these critical points are unstable.

5. Periodic orbits

To visualise the orbits of test particle, it is necessary to draw periodic orbits. For this with the use of proper initial values, we solve the equations of motion by well known software Mathematica and with parametric plot programme, we plot the graphs of periodic orbits for the various values of the parameters used. For this model, the periodic orbit has time period of 5.05 units with initial conditions as X(0) = 0.50222, Y(0) = 0, $\dot{X}(0) = 0$ & $\dot{Y}(0) = 0.50222$ and $\theta_S = 0$. It is given in figure (4). Similarly we can draw the periodic orbits for other values of the parameters.

6. Conclusion

In the bi-circular model with variable mass, by the use of Jeans law and Meshcherskii space time transformations, we have evaluated the equations of motion which are clearly depend on the perturbation parameters used. Using these equations of motion, we have illustrated the critical points and periodic orbits, and also examined the stability of the critical points. We found at most four critical points out of which three are near the barycenter while fourth one is near the third massive body. We have examined the stability of these critical points with

Table 1. The nature of the critical points in $X - Y-$plane for $\theta_S = 0$.

	Critical Point		*Roots*	*Nature*
	$X - Co.$	$Y - Co.$		
L_1	-0.18000	0.00000	$0.09999 \pm 0.44077\,i$	*Unstable*
			$0.10016 \pm 1.42465\,i$	
			$0.10002 \pm 1.04250\,i$	
L_2	0.07000	0.00000	$0.09999 \pm 3.05032\,i$	*Unstable*
			0.10031 ± 2.94387	
			-3.89243	
			4.09243	
L_3	1.11000	0.00000	$0.09999 \pm 1.47790\,i$	*Unstable*
			$0.10017 \pm 1.72931\,i$	
			-1.25635	
			1.45635	
L_4	260.00000	0.000000	$0.099999 \pm 4.87492\,i$	*Unstable*
			$0.10056 \pm 4.94892\,i$	
			-6.60236	
			6.80236	

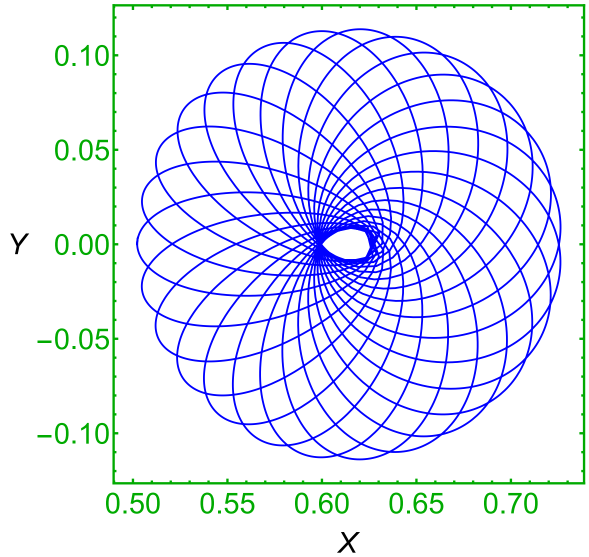

FIGURE 4. Periodic orbits for $\phi_2 = 1.2$, $d_1 = 0.2$ and $d_2 = 0.4$

the use of Meshcherskii space time inverse transformations. We found that all four critical points are unstable. Finally, we have drawn the periodic orbits with suitable initial values for this model and found periodic orbits with time period 5.05 units.

Acknowledgement. We are thankful to the International Center for Advanced Interdisciplinary Research (ICAIR), New Delhi, India.
We are also thankful to the anonymous editor and reviewer who have given us opportunity to improve the paper.

REFERENCES

1. Abdullah, *Periodic orbits around lagrangian points of the circular restricted four-body problem*, Invertis Journal of Science and Technology **7** (2014), no. 1, 29–38.
2. A. Abdullah, *Effect of albedo on the motion of the infinitesimal body in circular restricted three-body problem with variable masses*, Italian Journal Of Pure and Applied Mathematics **38** (2017), 581–600.
3. A. Abdullah and E. I. Abouelmagd, *Variable mass motion in the henon-heiles system*, Modern Physics letter A **36** (2021), no. 21, 2150150.
4. A. Abdullah, R. Kellil, and Z. A. Alhussain, *Behaviour of an infinitesimal variable mass body in crtbp; the primaries are finite straight segments*, Punjab University Journal of Mathematics **51** (2019), no. 5, 107–120.
5. A. Abdullah, R. Kellil, and S. K. Sahdev, *Numerical exploration of the motion of variable mass test particle on the perturbed circular restricted three-body configuration*, New Astronomy **97** (2022), no. 11, 101885.
6. A. Abdulraheem and J. Singh, *Combined effects of perturbations, radiation and oblateness on the periodic orbits in the restricted three-body problem*, Astrophysics and space science **317** (2008), 9–13.
7. E. I. Abouelmagd, *Periodic solution of the two–body problem by kb averaging method within frame of the modified newtonian potential*, J. Astro. Sci. **65** (2018), 291–306.

8. E. I. Abouelmagd and A. Abdullah, *The motion properties of the infinitesimal body in the framework of bicircular sun-perturbed earth-moon system*, New Astronomy **73** (2019), 101282.
9. E. I. Abouelmagd, J. I. G. Guirao, and A. K. Pal, *Periodic solution of the non-linear sitnikov restricted three-body problem*, New Astro. **75** (2020), 101319.
10. E. I. Abouelmagd, J. L. Guirao, and J. Llibre, *Periodic orbits of the planar anisotropic kepler problem*, Int. J. Bifur. Chaos **27** (2017), no. 3, 175.
11. E. I. Abouelmagd and A. Mostafa, *Out of plane equilibrium points locations and the forbidden movement regions in the restricted three-body problem with variable mass*, Astrophys. Space Sci. **357** (2015), no. 58, doi 10.1007/s10509–015–2294–7.
12. A. B. Albidah and A. Abdullah, *Halo orbits under some perturbations in the cr3b problem*, Symmetry **15** (2023), 481.
13. A. B. Albidah, A. Abdullah, and R. Kellil, *Interaction of 3-body in the circular restricted problem with variable mass*, Astronomy and Computing **42** (2023), 100688.
14. A.B. Albidah and A. Abdullah, *Shapes and mass variation effects of the bodies in the generalized elliptic restricted 3-body problem*, Astronomy reports **67** (2023), no. 4, 393–403.
15. A. Alshaery and E. I. Abouelmagd, *Analysis of the spatial quantized three–body problem*, Results in Physics **17** (2020), no. 14, 103067.
16. M. Alvarez-Ramírez and C. Vidal, *Dynamical aspects of an equilateral restricted four-body problem*, Mathematical Problems in Engineering (2009), 181360.
17. A. A. Ansari and S. N. Prasad, *Generalized elliptic restricted four-body problem with variable mass*, Astron. Lett. **46** (2020), 275–288.
18. M. Arribas, A. Abad, A. Elipe, and Palacios M., *Out-of-plane equilibria in the symmetric collinear restricted four-body problem with radiation pressure*, Astrophy. Space. Sci. **361** (2016), 270.
19. A. N. Baltagiannis and K. E. Papadakis, *Equilibrium points and their sta- bility in the restricted four-body problem*, Int. J. Bifurc. Chaos **21** (2011), 2179.
20. K. B. Bhatnagar, *Periodic orbits of collision in the plane circular problem of four bodies*, Indian J. of Pure and Applied Math. **2** (1970), no. 4, 583–596.
21. K. B. Bhatnagar and P. P. Hallan, *Effect of perturbed potentials on the stability of libration points in the restricted problem*, Celest. Mech. Dyn. Astr. **20** (1979), no. 3, 95–103.
22. F. Bouaziz and A. A. Ansari, *Perturbed hill's problem with variable mass*, Astron. Nachr. **doi: 10.1002/asna.202113870** (2021), 1–9.
23. J. Cronin, P. B. Richards, and L. H. Russell, *Some periodic solutions of a four-body problem*, ICARUS **3** (1964), no. 5, 423–428.
24. S. Ershkov, D. Leshchenko, and E. Aboeulmagd, *About influence of differential rotation in convection zone of gaseous or fluid giant planet (uranus) onto the parameters of orbits of satellites*, Eur. Phys. J. Plus **136** (2021), 387.
25. J. H. Jeans, *Astronomy and Cosmogony*, Cambridge University Press, Cambridge (1928).
26. F. Kamalov and H.H. Leung, *Fixed point theory: A review*, arxiv (2023), https://doi.org/10.48550/arXiv.2309.03226.
27. R. Kumari and B. S. Kushvah, *Stability regions of equilibrium points in restricted four-body problem with oblateness effects*, Astrophy. Space Sci. **349** (2014), no. 2, 693–704.
28. E.S.G. Leandro, *On the central configurations of the planar restricted four-body problem*, J. Differential Equations **226** (2006), 323–351.
29. J. llibre, D. Pasca, and C. Valls, *The circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem*, Celestial Mechanics and Dynamical Astronomy **133** (2021), no. 53, http://doi.org/10.1007/s10569–021–10052–6.
30. L. G. Lukyanov, *On the restricted circular conservative three-body problem with variable masses*, Astronomy Letters **35** (2009), no. 05, 349–359.

31. D. Maranhao and J. Llibre, *Ejection-collision orbits and invariant punctured tori in a restricted four-body problem*, Celestial Mechanics and Dynamical Astronomy **71** (1998), 1–14.
32. I. V. Meshcherskii, *Works on the mechanics of bodies of variable mass*, GITTL, Moscow (1949).
33. M. Michalodimitrakis, *The circular restricted four-body problem*, Astrophysics and Space Science **75** (1981), 289–305.
34. K.E. Papadakis, *Asymptotic orbits in the restricted four-body problem*, Planet. Space Sci. **55** (2007), 1368–1379.
35. J.P. Papadouris and K.E. Papadakis, *Equilibrium points in the photogravitational restricted four-body problem*, Astrophys. Space Sci. **344** (2013), no. 1, 21–38.
36. S. K. Sahdev and Abdullah A., *Generalized robe's problem having oblate heterogeneous primary containing viscous fluid inside the outer most layer and radiating spherical secondary with modified newtonian potential*, Science Int. Lahore **33** (2020), no. 2, 147–151.
37. J. Singh and A.E. Vincent, *Equilibrium points in the restricted four-body problem with radiation pressure*, Few-Body Syst. **57** (2016), 83–91.
38. V. Szebehely, *Theory of orbits*, Academic Press, New York (1967).
39. M. J. Zhang, C. Y. Zhao, and Y. Q. Xiong, *On the triangular libration points in photogravitational restricted three-body problem with variable mass*, Astrophys. Space Sci. **337** (2012), 107–113, doi 10.1007/s10509– 011–0821–8.

[1] Dyal Singh College, University of Delhi, New Delhi, India.
Email address: abdullah.maths@dsc.du.ac.in

[2] International Center for Advanced Interdisciplinary Research (ICAIR), Sangam Vihar, New Delhi, India.
Email address: icairndin@gmail.com