Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 18 (2023) 90-103

A NOVEL FINITE DIFFERENCE APPROACH TO DISCRETIZE THE SYMPLECTIC DIRAC OPERATOR

RASHMIREKHA PATRA[1*] AND NIHAR RANJAN SATAPATHY[2]

Abstract. Symplectic Dirac operator is an intertwining differential operator. Discretising symplectic Dirac operator gives a new direction to study the quantum space. The construction of discrete symplectic Dirac operator requires the theory of discrete symplectic Clifford analysis or the concept of discrete symplectic connections, which are not explained in literature. In this work, a discretization approach for symplectic Dirac operator is suggested by considering the forward and backward basis vectors on symplectic Clifford algebra. The suggested discrete symplectic Dirac operator is $\mathfrak{D}_s = \mathfrak{D}_s^+ + \mathfrak{D}_s^-$, where the $\mathfrak{D}_s^+$ and $\mathfrak{D}_s^-$ are the forward and backward discrete symplectic Dirac operators, respectively. The new discrete symplectic Dirac operator gives the factorization of discrete Laplacian on symplectic spaces. Further, we establish commutation relations involving forward and backward discrete symplectic Dirac operators in the representation theory.

1. Introduction

Discrete symplectic Dirac operator is formulated here, using a similar scheme as followed in discrete Clifford analysis for discretizing Dirac operators. This factorizes discrete Laplacian. Symplectic Dirac operator was introduced by Katharina Habermann as a symplectic analogue of geometric Dirac operator on a manifold [13]. The author suggested symplectic Dirac operator using symplectic spinors suggested by Kostant [17]. In [14], Habermann suggested two symplectic Dirac operators D and $\widehat{D}$, where D is due to symplectic form ω and $\widehat{D}$ is due to the ω compatible metric g. Here, g is compatible with respect to an almost complex structure J on the symplectic manifold with the compatible condition $g\left(x, y\right) = \omega\left(x, Jy\right)$. Further, the author introduced the symplectic Dirac operator on a Kaehler manifold [15]. On Kaehler manifold, the symplectic connection is equal to the Levi-Civita connection on Riemannian manifold. Although it is unique on Kaehler manifold, this is not the case on symplectic manifold. Therefore Symplectic connection is not unique on symplectic manifold [16]. In [13], symplectic Dirac operator is defined on a Hilbert space bundle $P \times_L L^2\left(\mathbb{R}^n\right)$,

Date: Received: Sep 3, 2023; Accepted: Sep 25, 2023.

* Corresponding author.

2010 *Mathematics Subject Classification.* 53D05, 53C27, 39A12, 81S10.

Key words and phrases. Symplectic Dirac Operator, Discrete Dirac operator, Symplectic Clifford algebra, Discrete Laplacian.

where P is a metaplectic principal bundle. Here, the Hilbert space bundle is defined via Segal-Shale-Weil representation. Discretization of Dirac operators play a key role in digital geometry processing, shape analysis etc. Further, the discretization of operators and the underlying spaces gives a better approach to understand relativistic quantum mechanics in a discrete set up. However, the existing problems for discretizing symplectic Dirac operators are: 1) Symplectic connection is not unique on a symplectic manifold. A generalized discrete connection is not available yet. Without a discrete symplectic connection, a discrete analogue of symplectic Dirac operator is not feasible in geometric set up. 2) The symplectic spinor bundle is infinite dimensional. So, many discrete function theories will not work in this case. 3) A discrete approach for symplectic Clifford analysis is not available in literature. Therefore, we suggest a discretizing mechanism for solving the problem in hand. In this work, a discrete symplectic Clifford algebra is used for discretizing symplectic Dirac operator.

The Riemannian Dirac operator is similar to the symplectic Dirac operator. However, there are significant differences between them. The first and most basic one is in their Clifford set up. Symplectic Dirac operator is defined in terms of symplectic Clifford algebra. Kostant [17], introduced the idea on super manifold structure based on the fact that every geometric structure on super differential geometry has a $\mathbb{Z}^2$-graded symmetry. This means, it has an even as well as an odd part, where the even part deals with symmetric objects and the odd part deals with skew symmetric objects. In this process, the classical Riemannian spin geometry is used to explain the symmetric objects and the symplectic spin geometry is used to explain the skew symmetric objects. In case of Riemannian Dirac operators, the respective spinor bundles are finite dimensional, while symplectic spinor bundles are infinite dimensional. Spin group is a double cover of the special orthonormal group. In the case of Riemannian Dirac operators, the representation of Spin group is a matrix representation. The symplectic group is double covered by the metaplectic group. The representation of metaplectic group is not a matrix representation. This is a unitary representation on the Hilbert space $L^2(\mathbb{R}^n)$ and it is unique upto isomorphism. In local coordinates, Riemannian Dirac operators have bounded coefficient operators and the symplectic Dirac operators have unbounded coefficient operators. Riemannian Dirac operator is elliptic while symplectic Dirac operators $D, \widehat{D}$ are not elliptic. But the Lie bracket of D and $\widehat{D}$ is elliptic.

Geometric Dirac operators and Dirac type operators are being discretized in different contexts. Gürlebeck and Sprößig [11] introduced a finite difference Dirac operator for developing the discrete function theory. These operators do not factorize the finite difference Laplacian and do not preserve many algebraic properties. In [12], Gürlebeck and Hommel suggested a finite difference Dirac operator in $\mathbb{R}^4$ for factorizing the finite difference Laplacian. In [18], Oliveira and Prado suggested an one dimensional Bernoulli discrete Dirac operator by Dirac equation. They used the shift operators on the space $l^2(\mathbb{Z}, \mathbb{C}^2)$ and analyzed their dynamical delocalization. The use of shift operators for discretization helped in defining discrete Dirac operators. However, there is no proper developing themes to define a discrete Dirac operator. In [9], the authors suggested the Fisher decomposition of discrete Laplacian for discretizing Dirac operators. This is the way by which the discrete monogenic function theory is analysed more closely. The quaternionic difference Dirac operator developed here factorizes the discrete Laplacian. Further, the authors suggested discrete Dirac operator in Clifford analysis [8]. This is also the representation of discrete Dirac operator on quad graph. However, the outcomes of these approaches seem to be challenging from symplectic point of view for quantum discretization. Further, no general discretization scheme is available in literature in this case.

In [4], Brackx et al. suggested discrete Clifford analysis in discrete function theory for higher dimensions. They also developed discrete analogue of monogenic functions, Stoke's theorem, Cauchy's theorem and Cauchy's integral formula on the rectangular $\mathbb{Z}^m$ grid. Each element of the basis of Clifford algebra is decomposed into forward and backward basis vectors. Zakharov suggested a form of discrete Dirac operator while discretizing the modified Veselov-Novikov hierarchy [24]. In [19], Prado and Oliveira suggested a self adjoint discrete Dirac operator using shift operators. They defined the operator with a sparse potential added to it in $l^2(\mathbb{N}, \mathbb{C}^2)$. Further, they analyzed the role of a particle mass in spectral mesh analysis using discrete Dirac operator. In [5], Cimasoni suggested discrete Dirac operator on a flat surface with finite genus. This method concludes that the Kasteleyn matrices of bipartite graphs on the flat surface with finite genus is the discrete Dirac operator.

Crane in [6], introduced quaternionic Dirac operator. Quaternionic Dirac operator is a generalized gradient operator. This method is used for different applications in the area of shape analysis in image processing. In [10], Golé nia and Haugomat analyzed the spectrum of one dimensional discrete Dirac operator defined on the space $l^2(\mathbb{Z}, \mathbb{C}^2)$. The recent studies also suggest different forms of discrete Dirac operator such as spinor space in discrete Clifford analysis [21], spectral mesh analysis, etc.

Discretisation of symplectic Dirac operator is not yet studied in the literature. In this context, a local form of symplectic Dirac operator is suggested by K. Habbermann on $\mathbb{R}^2$ [13]. A discrete symplectic Dirac operator factorizing the discrete Laplacian is physically significant. The phase space in quantum mechanical system is a symplectic space and the observables are self adjoint operators. Symplectic Dirac operators on symplectic spinors give a better understanding of geometric quantization. Therefore, discretization in quantization gives the idea of discretizing each symplectic object. However, to find out a discrete form of symplectic Dirac operator in geometric case requires the discretization of infinite dimensional spinor bundle and symplectic connections. The general form of discrete symplectic connection are not found in the existing literature. There is no suitable approach to discretize an infinite dimensional spinor bundle. This has motivated us to formulate a new technique for discretizing symplectic Dirac operator as in the case of discrete Dirac operators. In this paper, we suggest a mechanism for discretizing of symplectic Dirac operator in symplectic Clifford analysis [2].

The symplectic Dirac operator obtained by De Bie et al. [1, 2] is a intertwining differential operator. This is a combination of basis elements of symplectic Clifford algebra with symplectic coordinate vectors. By discretizing these basis elements and symplectic coordinate vectors, we obtain a discrete symplectic Dirac operator which factorizes the discrete Laplacian on symplectic space.

After a brief review of the related works in section 1, the background of symplectic Dirac operator is discussed in section 2. The rules and assumptions for formulating the discrete symplectic Dirac operators are given in section 3. Some results are given in section 4 followed by a brief conclusion.

2. Symplectic Dirac operator

De Bie et al. [2] suggested the following algebraic symplectic Dirac operator in terms of symplectic coordinates with the coordinate vector fields and the basis elements of symplectic Clifford algebra. Let $(\mathbb{R}^{2n}, \omega)$ be the symplectic vector space with the coordinates $z_1, z_2, z_3, ...z_n, w_1, w_2, ..., w_n$ and the coordinate vector fields be $\partial_{z_1}, \partial_{z_2}, ..., \partial_{z_n}, \partial_{w_1}, \partial_{w_2}, ..., \partial_{w_n}$.

Let $m_1, m_2, m_3, ..., m_{2n}$ be the symplectic basis satisfying

$$\omega\left(m_i, m_{i+n}\right) = 1, \omega\left(m_{n+i}, m_i\right) = -1, \omega\left(m_i, m_j\right) = \omega\left(m_{n+i}, m_{n+j}\right) = 0, \quad (2.1)$$

where $i \neq j$ and $i, j = 1, 2, ..., n$. The matrix realization of sympletic Lie algebra $sp\left(2n, \mathbb{C}\right)$ is the span of the followings:

$$P_{jk} = M_{j,k} - M_{n+k,n+j},\ Q_{jj} = M_{j,n+j},\ Q_{jk} = M_{j,n+k} + M_{k,n+j} \quad (2.2)$$

$$R_{jj} = M_{n+j,j},\ R_{jk} = M_{n+j,k} + M_{n+k,j}, \quad (2.3)$$

for $j \neq k$, where $j, k = 1, 2, ..., n$ and $M_{j,k}$ is the $2n \times 2n$ matrix with 1 at the intersection of j^{th} row and k^{th} column and zero otherwise. The representation of symplectic Lie algebra on the symmetric algebra is given in [2]. Crumeyrolle introduced symplectic Clifford algebra [7]. De Bie suggested symplectic Dirac operator as an intertwining differential operator using symplectic Clifford algebra.

Definition 2.1. [2] The symplectic Clifford algebra $Cl_s\left(\mathbb{R}^{2n}, \omega\right)$ on the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ with a basis $m_1, m_2, m_3....., m_{2n}$ is an associative unital algebra over $\mathbb{C}$. This is the quotient of the tensor algebra $T\left(m_1, m_2, m_3....., m_{2n}\right)$ by a two sided ideal K generated by $u.v - v.u = -i\omega\left(u, v\right)$ for all $u, v \in \mathbb{R}^{2n}$ and i is the imaginary unit.

The symplectic Clifford algebra is isomorphic to the algebra of complex valued algebraic differential operators on $\mathbb{R}^n$ called the Weyl algebra $\mathbb{W}_{2n}$. The symplectic Lie algebra acts as a sub algebra of $\mathbb{W}_{2n}$. Weyl algebra $\mathbb{W}_{2n}$ is an associative algebra generated by $p_1, p_2, ..., p_n, \partial_{p_1}, \partial_{p_2}, ..., \partial_{p_n}$, where p_i is the multiplication operator and ∂_{p_i} is the differential operator for $i = 1, 2, 3, ..., n$. The representation of symplectic Lie algebra as a sub algebra of Weyl algebra is given by

$$P_{jk} = p_k\partial_{p_j} + \frac{1}{2}\delta_{j,k},\ Q_{jj} = -\frac{i}{2}\partial_{p_j}^2, \quad (2.4)$$

$$Q_{jk} = i\partial_{p_j}\partial_{p_k},\ R_{jj} = -\frac{i}{2}{p_j}^2, R_{jk} = ip_jp_k, \quad (2.5)$$

for $j \neq k$, and $j, k = 1, 2, ..., n$ and where $\delta_{j,k}$ is the Kronecker delta. Let $S(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$. The representation of the symplectic Lie algebra on the space of polynomial symplectic spinors $Pol(\mathbb{R}^{2n}, \mathbb{C}) \otimes S(\mathbb{R}^n)$ is expressed as

$$P_{jk} = -z_j \partial_{z_k} + w_k \partial_{w_j} + p_k \partial_{p_j} + \frac{1}{2}\delta_{j,k} \tag{2.6}$$

$$Q_{jj} = -z_j \partial_{w_j} - \frac{i}{2}{p_j}^2 \tag{2.7}$$

$$Q_{jk} = z_k \partial_{w_j} + z_j \partial_{w_k} + i\partial_{p_j}\partial_{p_k} \tag{2.8}$$

$$R_{jj} = -w_j \partial_{v_j} - \frac{i}{2}{p_j}^2 \tag{2.9}$$

$$R_{jk} = w_k \partial_{z_j} + w_j \partial_{z_k} + ip_j p_k, \text{ for} j \neq k. \tag{2.10}$$

The action of the basis elements $m_1, m_2, m_3....., m_{2n}$ on a symplectic spinor $f \in S(\mathbb{R}^n)$ is given by $m_j.f = ip_j f$, $m_{n+j}.f = \partial_{p_j} f$, for $j = 1, 2...., n$. The three differential operators defined on $S(\mathbb{R}^n)$ are expressed as:

$$D_s = \sum_{j=1}^{n} \left(ip_j \partial_{w_j} - \partial_{z_j} \partial_{p_j} \right) \tag{2.11}$$

$$X_s = \sum_{j=1}^{n} \left(w_j \partial_{p_j} + iz_j p_j \right) \tag{2.12}$$

$$E = \sum_{j=1}^{n} \left(z_j \partial_{z_j} + w_j \partial_{w_j} \right) \tag{2.13}$$

The above mentioned operators satisfy the following commutation relations: [2]

$$[E + n, D_s] = -D_s \tag{2.14}$$

$$[E + n, X_s] = X_s \tag{2.15}$$

$$[X_s, D_s] = i(E + n), \tag{2.16}$$

where, D_s is the symplectic Dirac operator. In this paper, we suggest an approach to discretize the above mentioned symplectic Dirac operator following the discrete Clifford analysis [23] formalism.

3. The construction of discrete symplectic Dirac operator

The symplectic Dirac operator D_s has the form as in (2.11). This shows the combination of coordinate vectors, differential coordinate vectors and the basis elements of Weyl algebra. In this work, we formulate a mechanism for discretizing symplectic Dirac operator expressing the coordinate vectors and the basis elements in terms of the forward and backward basis vectors as in discrete Clifford analysis. This is done without changing the basic algebraic assumption while defining the symplectic Dirac operator. Here, $\mathbb{R}^{2n}$ is replaced by $\mathbb{Z}^{2m}$ rectangular grid. This is obtained by considering the following: In moving from continuous to discrete set up, the partial derivatives are replaced by partial differences. The differences are in the directions of the coordinate axes. Further, the axes carries two senses, one is forward and the other is the backward difference. In case of symplectic Dirac operator, partial derivatives are combined with the basis elements of Weyl algebra. This also carries the chosen senses of partial differences. Here, the symplectic Clifford algebra is embedded in the double dimension $4m$. Assuming the existence of $2m$ vectors p_j^+ and p_j^-, the following discretization rules are formulated as:

Rule 1: The basis vector p_i in Weyl algebra is assumed to be the combination of forward and backward basis vectors.

$$p_j^+ + p_j^- = p_j, j = 1, 2, ..., m \tag{3.1}$$

In Weyl algebra $p_j p_k = p_k p_j$, i.e. the basis vector p_js commute with each other. Here, the forward and backward basis vectors p_j^+, p_j^- are assumed to form a free algebra satisfying the following relations:

$$\left(p_j^+ p_k^+ + p_k^+ p_j^+\right) = 0 \tag{3.2}$$
$$\left(p_j^- p_k^- + p_k^- p_j^-\right) = 0 \tag{3.3}$$
$$\left(p_j^+ p_k^- + p_k^- p_j^+\right) = \delta_{j,k} \tag{3.4}$$
$$\left(p_j^- p_k^+ + p_k^+ p_j^-\right) = -\delta_{j,k} \tag{3.5}$$

Rule 2: The basis vector ∂_{p_j} in Weyl algebra is decomposed into forward and backward difference operators.

$$\partial_{p_j}^+ + \partial_{p_j}^- = \partial_{p_j}, j = 1, 2, ..., m \tag{3.6}$$

As in case of p_j, ∂_{p_j} also satisfies commutative property i.e. $\partial_{p_j}\partial_{p_k} = \partial_{p_k}\partial_{p_j}$. Here, the forward and backward basis vectors $\partial_{p_j}^+, \partial_{p_j}^-$ are assumed to satisfy the following relations:

$$\left(\partial_{p_j}^+ \partial_{p_k}^+ + \partial_{p_k}^+ \partial_{p_j}^+\right) = 0$$
$$\left(\partial_{p_j}^- \partial_{p_k}^- + \partial_{p_k}^- \partial_{p_j}^-\right) = 0$$
$$\left(\partial_{p_j}^+ \partial_{p_k}^- + \partial_{p_k}^- \partial_{p_j}^+\right) = \delta_{j,k}$$
$$\left(\partial_{p_j}^- \partial_{p_k}^+ + \partial_{p_k}^+ \partial_{p_j}^-\right) = -\delta_{j,k}$$

Rule 3 The multiplication operators p_j and differential operators $\partial_{p_j}, j = 1, 2, 3, ..., n$, in Weyl algebra satisfy $\partial_{p_j} p_k - p_k \partial_{p_j} = \delta_{j,k}$.

$$\begin{aligned}
&\partial_{p_j} p_k - p_k \partial_{p_j} = \delta_{jk} \\
&\left(\partial_{p_j}^+ + \partial_{p_j}^-\right)\left(p_k^+ + p_k^-\right) - \left(p_k^+ + p_k^-\right)\left(\partial_{p_j}^+ + \partial_{p_j}^-\right) = \delta_{jk} \\
&\left(\partial_{p_j}^+ p_k^+ - p_k^+ \partial_{p_j}^+\right) + \left(\partial_{p_j}^- p_k^- - p_k^- \partial_{p_j}^-\right) \\
&+ \left(\partial_{p_j}^+ p_k^- - p_k^- \partial_{p_j}^+\right) + \left(\partial_{p_j}^- p_k^+ - p_k^+ \partial_{p_j}^-\right) = \delta_{jk}
\end{aligned}$$

Here, we take the help of skew Weyl relation [20] for further calculation:

$$\partial^+_{p_j} p^+_j - p^-_j \partial^-_{p_j} = 1 \tag{3.7}$$

$$\partial^-_{p_j} p^-_j - p^+_j \partial^+_{p_j} = 1 \tag{3.8}$$

Rule 4: The coordinates and the coordinate vector fields are decomposed into forward and backward coordinate basis vectors and coordinate vector fields respectively. This is defined as:

$$z_j = z^+_j + z^-_j, \omega_j = \omega^+_j + \omega^-_j, j = 1, 2, ..., n \tag{3.9}$$

$$\partial_{z_j} = \partial^+_{z_j} + \partial^-_{z_j}, \partial_{\omega_j} = \partial^+_{\omega_j} + \partial^-_{\omega_j}, j = 1, 2, ..., n. \tag{3.10}$$

Considering the above rules, we assume certain notations for a simplified representation.

Assumption-1:

$$\partial^+_{p_j} p^+_k - p^+_k \partial^+_{p_j} = -\mathfrak{g}^+_{jk} \tag{3.11}$$

$$\partial^-_{p_j} p^-_k - p^-_k \partial^-_{p_j} = -\mathfrak{g}^-_{jk} \tag{3.12}$$

$$\partial^+_{p_j} p^-_k - p^-_k \partial^+_{p_j} = -\mathfrak{n}^+_{jk} \tag{3.13}$$

$$\partial^-_{p_j} p^+_k - p^+_k \partial^-_{p_j} = -\mathfrak{n}^-_{jk}, \tag{3.14}$$

where, $\mathfrak{g}^+_{jk}$, $\mathfrak{g}^-_{jk}$,$\mathfrak{n}^+_{jk}$ and $\mathfrak{n}^-_{jk}$ are the skew symmetric tensors. Applying Assumption-1 in Rule 3 we formulate the following statement.

$$\mathfrak{g}^+_{jk} + \mathfrak{g}^-_{jk} + \mathfrak{n}^+_{jk} + \mathfrak{n}^-_{jk} = -\delta_{j,k}, j, k = 1, 2, ..., n \tag{3.15}$$

Assumption-2:

$$p^+_j p^-_k = p^-_k p^+_j = -\beta^+_{jk} \tag{3.16}$$

$$p^-_j p^+_k = p^+_k p^-_j = -\beta^+_{jk}, \tag{3.17}$$

$$\partial^+_{p_j} \partial^-_{p_k} = \partial^-_{p_k} \partial^+_{p_j} = -\eta^+_{jk} \tag{3.18}$$

$$\partial^-_{p_j} \partial^+_{p_k} = \partial^+_{p_k} \partial^-_{p_j} = -\eta^-_{jk}, \tag{3.19}$$

$$p^+_j \partial^+_{p_j} = \alpha^+_{jk},\ p^-_j \partial^-_{p_j} = \alpha^-_{jk} \tag{3.20}$$

where, $\alpha^+_{jk}, \alpha^-_{jk}$, $\beta^+_{jk}, \beta^-_{jk}$ and η^+_{jk}, η^-_{jk} are symmetric tensors.

Assumption-3: The Cartesian directions for both the forward and backward vectors described above play same role in the metric.

Therefore according to the Assumption-1,

$$\mathfrak{g}^+_{11} = \mathfrak{g}^+_{22} = ... = \mathfrak{g}^+_{mm} = \nu^+ \tag{3.21}$$

$$\mathfrak{g}^-_{11} = \mathfrak{g}^-_{22} = ... = \mathfrak{g}^-_{mm} = \nu^-, \tag{3.22}$$

where, $-\mathfrak{g}^\pm_{jj} = \partial^\pm_{p_j} p^\pm_j - p^\pm_j \partial^\pm_{p_j}$ and

$$\mathfrak{n}^+_{11} = \mathfrak{n}^+_{22} = ... = \mathfrak{n}^+_{mm} = \xi^+ \tag{3.23}$$

$$\mathfrak{n}^-_{11} = \mathfrak{n}^-_{22} = ... = \mathfrak{n}^-_{mm} = \xi^-,$$

where, $-\mathfrak{n}^+_{jj} = \partial^+_{p_j} p^-_j - p^-_j \partial^+_{p_j}$ and $-\mathfrak{n}^-_{jj} = \partial^-_{p_j} p^+_j - p^+_j \partial^-_{p_j}$. For $j \neq k$, $\mathfrak{g}_{jk}$, $\mathfrak{n}_{jk}$ are written as

$$\mathfrak{g}^{\pm}_{jk} = \mathfrak{g}^{\pm}, j, k = 1, 2, ..., m, j \neq k \tag{3.24}$$

$$\mathfrak{n}^{\pm}_{jk} = \mathfrak{n}^{\pm}, j, k = 1, 2, ..., m, j \neq k \tag{3.25}$$

Combining the equation (3.15) with (3.21),(3.23) and (3.24), we get

$$\nu^+ + \nu^- + \xi_+ + \xi_- = 1 \tag{3.26}$$

$$\mathfrak{g}^+ + \mathfrak{g}^- + \mathfrak{n}^+ + \mathfrak{n}^- = 0.$$

Assumption-2 and Assumption-3 also satisfy Assumption-4.

$$\beta^{\pm}_{jk} = \beta^{\pm} \tag{3.27}$$

$$\eta^{\pm}_{jk} = \eta^{\pm}$$

Assumption-4: The positive and negative orientations of the Cartesian directions are assumed to be equivalent. This means the forward and backward difference operators are reflection invariant.

Hence, we have

$$\mathfrak{g}^+ = \mathfrak{g}^- = \mathfrak{g}, \mathfrak{n}^+ = \mathfrak{n}^- = \mathfrak{n} \tag{3.28}$$

$$\nu^+ = \nu^- = \nu, \xi_+ = \xi_- = \xi. \tag{3.29}$$

So

$$\nu + \xi = \frac{1}{2}, \mathfrak{g} + \mathfrak{n} = 0. \tag{3.30}$$

Considering all the rules and assumptions, the forward and backward multiplication operators p^+_j, p^-_j and the difference operators $\partial^+_{p_j}, \partial^-_{p_j}$ satisfy the following multiplication rules in this discrete set up. Here $j \neq k$

$$\partial^+_{p_j} p^+_k - p^+_k \partial^+_{p_j} = -\mathfrak{g} \tag{3.31}$$

$$\partial^-_{p_j} p^-_k - p^-_k \partial^-_{p_j} = -\mathfrak{g} \tag{3.32}$$

$$\partial^+_{p_j} p^-_k - p^-_k \partial^+_{p_j} = \mathfrak{g} \tag{3.33}$$

$$\partial^-_{p_j} p^+_k - p^+_k \partial^-_{p_j} = \mathfrak{g} \tag{3.34}$$

and when $j = k$, we have

$$\partial^+_{p_j} p^+_j - p^+_j \partial^+_{p_j} = \nu \tag{3.35}$$

$$\partial^-_{p_j} p^-_j - p^-_j \partial^-_{p_j} = \nu \tag{3.36}$$

$$\partial^+_{p_j} p^-_j - p^-_j \partial^+_{p_j} = \xi \tag{3.37}$$

$$\partial^-_{p_j} p^+_j - p^+_j \partial^-_{p_j} = \xi \tag{3.38}$$

According to **Assumption-4** the equation (3.27) is expressed as

$$\beta^+ = \beta^- = \beta \tag{3.39}$$

$$\eta^+ = \eta^- = \eta$$

Combining equations (3.16) with (3.27) and (3.39), we have

$$p_j^+ p_j^- + p_j^- p_j^+ = 2\beta \tag{3.40}$$

$$\partial_{p_j}^+ \partial_{p_j}^- + \partial_{p_j}^- \partial_{p_j}^+ = 2\eta \tag{3.41}$$

Definition 3.1. The discrete symplectic Dirac operator is the first order symplectic Clifford vector valued differential operator defined as

$$\mathfrak{D}_s = \mathfrak{D}_s^+ + \mathfrak{D}_s^-, \tag{3.42}$$

where, the $\mathfrak{D}_s^+$ and $\mathfrak{D}_s^-$ are the forward and backward discrete symplectic Dirac operators respectively and are given by

$$\mathfrak{D}_s^+ = \sum_{j=1}^m \left(i p_j^+ \partial_{\omega_j}^+ - \partial_{z_j}^+ \partial_{p_j}^+ \right) \tag{3.43}$$

and

$$\mathfrak{D}_s^- = \sum_{j=1}^m \left(i p_j^- \partial_{\omega_j}^- - \partial_{z_j}^- \partial_{p_j}^- \right) \tag{3.44}$$

4. Results

Theorem 4.1. *The discrete symplectic Dirac operator factorizes the Laplacian on Z^{2m} grid. i.e.*

$$(\mathfrak{D}_s)^2 = -i \sum_{j=1}^m \partial_{\omega_j}^+ \partial_{z_j}^+ + \partial_{\omega_j}^- \partial_{z_j}^- + \partial_{\omega_j}^+ \partial_{\omega_j}^- + \partial_{z_j}^+ \partial_{z_j}^- - i \sum_{j=1}^m 2\beta \left(\partial_{\omega_j}^+ \partial_{z_j}^+ + \partial_{\omega_j}^- \partial_{z_j}^- \right),$$

where, β is the symmetric tensor given in equation (3.39).

Proof. Using the multiplication rules defined above we obtain

$$\mathfrak{D}_s^+ \mathfrak{D}_s^+ = \left(\sum_{j=1}^m \left(i p_j^+ \partial_{\omega_j}^+ - \partial_{z_j}^+ \partial_{p_j}^+ \right) \right) \left(\sum_{j=1}^m \left(i p_j^+ \partial_{\omega_j}^+ - \partial_{z_j}^+ \partial_{p_j}^+ \right) \right) \tag{4.1}$$

$$= \sum_{j=1}^m \left(i^2 \left(p_j^+\right)^2 \left(\partial_{\omega_j}^+\right)^2 + \left(\partial_{z_j}^+\right)^2 \left(\partial_{p_j}^+\right)^2 \right) - i \left(p_j^+ \partial_{p_j}^+ + \partial_{p_j}^+ p_j^+ \right) \partial_{\omega_j}^+ \partial_{z_j}^+$$

$$= -i \sum_{j=1}^m \left(p_j^+ \partial_{p_j}^+ + \partial_{p_j}^+ p_j^+ \right) \partial_{\omega_j}^+ \partial_{z_j}^+$$

$$\mathfrak{D}_s^- \mathfrak{D}_s^- = \left(\sum_{j=1}^m \left(i p_j^- \partial_{\omega_j}^- - \partial_{z_j}^- \partial_{p_j}^- \right) \right) \left(\sum_{j=1}^m \left(i p_j^- \partial_{\omega_j}^- - \partial_{z_j}^- \partial_{p_j}^- \right) \right) \tag{4.2}$$

$$= \sum_{j=1}^m \left(i^2 \left(p_j^-\right)^2 \left(\partial_{\omega_j}^-\right)^2 + \left(\partial_{z_j}^-\right)^2 \left(\partial_{p_j}^-\right)^2 \right) - i \left(p_j^- \partial_{p_j}^- + \partial_{p_j}^- p_j^- \right) \partial_{\omega_j}^- \partial_{z_j}^-$$

$$= -i \sum_{j=1}^m \left(p_j^- \partial_{p_j}^- + \partial_{p_j}^- p_j^- \right) \partial_{\omega_j}^- \partial_{z_j}^-$$

$$\mathfrak{D}_s^+\mathfrak{D}_s^- + \mathfrak{D}_s^-\mathfrak{D}_s^+ = \left(\sum_{j=1}^m \left(ip_j^+\partial_{\omega_j}^+ - \partial_{z_j}^+\partial_{p_j}^+\right)\right)\left(\sum_{j=1}^m \left(ip_j^-\partial_{\omega_j}^- - \partial_{z_j}^-\partial_{p_j}^-\right)\right) \tag{4.3}$$
$$+ \left(\sum_{j=1}^m \left(ip_j^-\partial_{\omega_j}^- - \partial_{z_j}^-\partial_{p_j}^-\right)\right)\left(\sum_{j=1}^m \left(ip_j^+\partial_{\omega_j}^+ - \partial_{z_j}^+\partial_{p_j}^+\right)\right)$$
$$= \sum_{j=1}^m - \left(p_j^+p_j^- + p_j^-p_j^+\right)\partial_{\omega_j}^+\partial_{\omega_j}^- + \left(\partial_{p_j}^+\partial_{p_j}^- + \partial_{p_j}^-\partial_{p_j}^+\right)\partial_{z_j}^+\partial_{z_j}^-$$
$$- i\left(p_j^+\partial_{p_j}^- + \partial_{p_j}^-p_j^+\right)\partial_{\omega_j}^+\partial_{z_j}^- - i\left(\partial_{p_j}^+p_j^- + p_j^-\partial_{p_j}^+\right)\partial_{z_j}^+\partial_{\omega_j}^-$$

Combining equations (4.1), (4.2), (4.3), we have

$$(\mathfrak{D}_s)^2 = \mathfrak{D}_s^+\mathfrak{D}_s^+ + \mathfrak{D}_s^-\mathfrak{D}_s^- + \left(\mathfrak{D}_s^+\mathfrak{D}_s^- + \mathfrak{D}_s^-\mathfrak{D}_s^+\right)$$
$$= -i\sum_{j=1}^m \left(p_j^+\partial_{p_j}^+ + \partial_{p_j}^+p_j^+\right)\partial_{\omega_j}^+\partial_{z_j}^+ + \sum_{j=1}^m \left(p_j^-\partial_{p_j}^- + \partial_{p_j}^-p_j^-\right)\partial_{\omega_j}^-\partial_{z_j}^-$$
$$+ i\sum_{j=1}^m - \left(p_j^+p_j^- + p_j^-p_j^+\right)\partial_{\omega_j}^+\partial_{\omega_j}^- + \left(\partial_{p_j}^+\partial_{p_j}^- + \partial_{p_j}^-\partial_{p_j}^+\right)\partial_{z_j}^+\partial_{z_j}^-$$
$$- i\left(p_j^+\partial_{p_j}^- + \partial_{p_j}^-p_j^+\right)\partial_{\omega_j}^+\partial_{z_j}^- - i\left(\partial_{p_j}^+p_j^- + p_j^-\partial_{p_j}^+\right)\partial_{z_j}^+\partial_{\omega_j}^-$$
$$= -i\sum_{j=1}^m \left(p_j^+\partial_{p_j}^+ + p_j^-\partial_{p_j}^- + 1\right).\left(\partial_{\omega_j}^+\partial_{z_j}^+ + \partial_{\omega_j}^-\partial_{z_j}^-\right) - i\sum_{j=1}^m \left(\partial_{\omega_j}^+\partial_{\omega_j}^- + \partial_{z_j}^+\partial_{z_j}^-\right)$$
$$= -i\sum_{j=1}^m \left(\partial_{\omega_j}^+\partial_{z_j}^+ + \partial_{\omega_j}^-\partial_{z_j}^-\right) - i\sum_{j=1}^m \left(\partial_{\omega_j}^+\partial_{\omega_j}^- + \partial_{z_j}^+\partial_{z_j}^-\right)$$
$$- i\sum_{j=1}^m \left(p_j^+\partial_{p_j}^+ + p_j^-\partial_{p_j}^-\right).\left(\partial_{\omega_j}^+\partial_{z_j}^+ + \partial_{\omega_j}^-\partial_{z_j}^-\right)$$
$$= -i\sum_{j=1}^m \partial_{\omega_j}^+\partial_{z_j}^+ + \partial_{\omega_j}^-\partial_{z_j}^- + \partial_{\omega_j}^+\partial_{\omega_j}^- + \partial_{z_j}^+\partial_{z_j}^-$$
$$- i\sum_{j=1}^m 2\beta\left(\partial_{\omega_j}^+\partial_{z_j}^+ + \partial_{\omega_j}^-\partial_{z_j}^-\right)$$

□

The vector variables corresponding to the discrete symplectic Dirac operator is $\mathfrak{X}_s$, which is given by

$$\mathfrak{X}_s = \mathfrak{X}_s^+ + \mathfrak{X}_s^-, \tag{4.4}$$

where

$$\mathfrak{X}_s^+ = \sum_{j=1}^{m} \left(\omega_j^+ \partial_{p_j}^+ + i z_j^+ p_j^+ \right) \tag{4.5}$$

$$\mathfrak{X}_s^- = \sum_{j=1}^{m} \left(\omega_j^- \partial_{p_j}^- + i z_j^- p_j^- \right) \tag{4.6}$$

Theorem 4.2.

$$\mathfrak{X}_s \mathfrak{D}_s + \mathfrak{D}_s \mathfrak{X}_s = 2im + i\mathfrak{E}, \tag{4.7}$$

where

$$\mathfrak{E} = \sum_{j=1}^{m} \left(\left(\omega_j^+ \partial_{\omega_j}^+ + \omega_j^- \partial_{\omega_j}^- + z_j^+ \partial_{z_j}^+ + z_j^- \partial_{z_j}^- \right) \left(p_j^+ \partial_{p_j}^+ + p_j^- \partial_{p_j}^- + 1 \right) \right)$$
$$+ \sum_{j=1}^{m} \left(- \left(z_j^+ \partial_{\omega_j}^- + z_j^- \partial_{\omega_j}^+ - \omega_j^+ \partial_{z_j}^- - \omega_j^- \partial_{z_j}^+ \right) \right).$$

Proof.

$$\mathfrak{X}_s^+ \mathfrak{D}_s^+ = \sum_{j=1}^{m} \left(i\omega_j^+ \partial_{\omega_j}^+ \partial_{p_j}^+ p_j^+ - i z_j^+ \partial_{z_j}^+ p_j^+ \partial_{p_j}^+ \right)$$

$$\mathfrak{X}_s^- \mathfrak{D}_s^- = \sum_{j=1}^{m} \left(i\omega_j^- \partial_{\omega_j}^- \partial_{p_j}^- p_j^- - i z_j^- \partial_{z_j}^- p_j^- \partial_{p_j}^- \right)$$

$$\mathfrak{X}_s^+ \mathfrak{D}_s^- = \sum_{j=1}^{m} \left(i\omega_j^+ \partial_{\omega_j}^- \partial_{p_j}^+ p_j^- - i z_j^+ \partial_{z_j}^- p_j^+ \partial_{p_j}^- - z_j^+ \partial_{\omega_j}^- p_j^+ p_j^- - \omega_j^+ \partial_{z_j}^- \partial_{p_j}^+ \partial_{p_j}^- \right)$$

$$\mathfrak{X}_s^- \mathfrak{D}_s^+ = \sum_{j=1}^{m} \left(i\omega_j^- \partial_{\omega_j}^+ \partial_{p_j}^- p_j^+ - i z_j^- \partial_{z_j}^+ p_j^- \partial_{p_j}^+ - z_j^- \partial_{\omega_j}^+ p_j^- p_j^+ - \omega_j^- \partial_{z_j}^+ \partial_{p_j}^- \partial_{p_j}^+ \right)$$

$$\mathfrak{D}_s^+ \mathfrak{X}_s^+ = \sum_{j=1}^{m} \left(i\omega_j^+ \partial_{\omega_j}^+ p_j^+ \partial_{p_j}^+ - i z_j^+ \partial_{z_j}^+ \partial_{p_j}^+ p_j^+ + i p_j^+ \partial_{p_j}^+ - i \partial_{p_j}^+ p_j^+ \right)$$

$$\mathfrak{D}_s^- \mathfrak{X}_s^- = \sum_{j=1}^{m} \left(i\omega_j^- \partial_{\omega_j}^- p_j^- \partial_{p_j}^- - i z_j^- \partial_{z_j}^- \partial_{p_j}^- p_j^- + i p_j^- \partial_{p_j}^- - i \partial_{p_j}^- p_j^- \right)$$

$$\mathfrak{D}_s^+ \mathfrak{X}_s^- = \sum_{j=1}^{m} \left(i\omega_j^- \partial_{\omega_j}^+ p_j^+ \partial_{p_j}^- - i z_j^- \partial_{z_j}^+ \partial_{p_j}^+ p_j^- - z_j^- \partial_{\omega_j}^+ p_j^+ p_j^- - \omega_j^- \partial_{z_j}^+ \partial_{p_j}^+ \partial_{p_j}^- \right)$$

$$\mathfrak{D}_s^- \mathfrak{X}_s^+ = \sum_{j=1}^{m} \left(i\omega_j^+ \partial_{\omega_j}^- p_j^- \partial_{p_j}^+ - i z_j^+ \partial_{z_j}^- \partial_{p_j}^- p_j^+ - z_j^+ \partial_{\omega_j}^- p_j^- p_j^+ - \omega_j^+ \partial_{z_j}^- \partial_{p_j}^- \partial_{p_j}^+ \right)$$

Combining the above equations we have,

$$\begin{aligned}
\mathfrak{X}_s\mathfrak{D}_s + \mathfrak{D}_s\mathfrak{X}_s &= \sum_{j=1}^{m}\left(2i + i\left(\omega_j^+\partial_{\omega_j}^+ + z_j^+\partial_{z_j}^+\right).\left(\partial_{p_j}^+ p_j^+ + p_j^+\partial_{p_j}^+\right)\right)\\
&+ i\sum_{j=1}^{m}\left(\left(\omega_j^-\partial_{\omega_j}^- + z_j^-\partial_{z_j}^-\right).\left(\partial_{p_j}^- p_j^- + p_j^-\partial_{p_j}^-\right)\right)\\
&+ i\sum_{j=1}^{m}\left(\left(\omega_j^-\partial_{\omega_j}^+ + z_j^+\partial_{z_j}^-\right).\left(\partial_{p_j}^- p_j^+ + p_j^+\partial_{p_j}^-\right)\right)\\
&+ i\sum_{j=1}^{m}\left(\left(\omega_j^+\partial_{\omega_j}^- + z_j^+\partial_{z_j}^-\right).\left(\partial_{p_j}^+ p_j^- + p_j^-\partial_{p_j}^+\right)\right)\\
&- \sum_{j=1}^{m}\left(\left(\omega_j^+\partial_{z_j}^- + \omega_j^-\partial_{z_j}^+\right).\left(\partial_{p_j}^+\partial_{p_j}^- + \partial_{p_j}^-\partial_{p_j}^+\right)\right)\\
&- \sum_{j=1}^{m}\left(\left(z_j^+\partial_{\omega_j}^- + z_j^-\partial_{\omega_j}^+\right).\left(p_j^+ p_j^- + p_j^- p_j^+\right)\right)\\
&= 2im + \sum_{j=1}^{m}\left(\left(\omega_j^+\partial_{\omega_j}^+ + \omega_j^-\partial_{\omega_j}^- + z_j^+\partial_{z_j}^+ + z_j^-\partial_{z_j}^-\right)\left(p_j^+\partial_{p_j}^+ + p_j^-\partial_{p_j}^- + 1\right)\right)\\
&+ \sum_{j=1}^{m}\left(-\left(z_j^+\partial_{\omega_j}^- + z_j^-\partial_{\omega_j}^+ - \omega_j^+\partial_{z_j}^- - \omega_j^-\partial_{z_j}^+\right)\right).
\end{aligned}$$

□

We denote

$$\begin{aligned}
\mathfrak{E} &= \sum_{j=1}^{m}\left(\left(\omega_j^+\partial_{\omega_j}^+ + \omega_j^-\partial_{\omega_j}^- + z_j^+\partial_{z_j}^+ + z_j^-\partial_{z_j}^-\right)\left(p_j^+\partial_{p_j}^+ + p_j^-\partial_{p_j}^- + 1\right)\right)\\
&+ \sum_{j=1}^{m}\left(-\left(z_j^+\partial_{\omega_j}^- + z_j^-\partial_{\omega_j}^+ - \omega_j^+\partial_{z_j}^- - \omega_j^-\partial_{z_j}^+\right)\right).
\end{aligned}$$

as the Euler operator in this case. The discrete Euler operator, Discrete symplectic Dirac operator and the corresponding vector variables satisfy the following theorem.

Theorem 4.3.

$$\mathfrak{E}\mathfrak{D} + \mathfrak{D}\mathfrak{E} = -\mathfrak{D} \tag{4.8}$$

$$\mathfrak{E}\mathfrak{X} + \mathfrak{X}\mathfrak{E} = -\mathfrak{X} \tag{4.9}$$

5. Conclusion

The suggested discretization technique on symplectic Dirac operator is formulated using forward and backward basis vectors on symplectic Clifford algebra.

This operator is the combination of forward and backward symplectic Dirac operators on the symplectic space. Square of this operator is negative of the discrete Laplacian on symplectic space. This also satisfies commutation relations on the space of polynomial symplectic spinors in representation theory. Discrete symplectic Dirac operator will be useful to study symplectic Dirac equation. This discrete symplectic Dirac operator may be studied in the context of super manifolds. The techniques like discrete symplectic monogenics, discrete symplectic Fourier transform may be used to have further study on this operator. Polynomial solutions of discerete symplectic Dirac operators in representation theory may also be studied.

5.1. **Declaration.**

(1) Competing interest: Both the authors declare that there is no conflict of interest.
(2) Funding: We don't have a funding source.
(3) Data Availability: Data are available in published research papers and textbooks.

References

[1] Bie, H.D., Somberg, P., Součekb, V., : The metaplectic Howe duality and polynomial solutions for the symplectic Dirac operator, Journal of Geometry and Physics, 75, 120-128, (2014).
[2] Bie, H. D., Holíková, M., Somberg, P., : Basic Aspects of Symplectic Clifford Analysis for the Symplectic Dirac Operator, Advances in Applied Clifford Algebra, 27, 1103–1132, (2017).
[3] Brackx, F., Schepper, H. D., Sommen, F., Van de Voorde, L., : Discrete Clifford analysis: an overview, Cubo, 11 (1), 55 - 71, (2009).
[4] Brackx, F. , Schepper, H. D., Sommen, F., Van de Voorde, L., : Discrete Clifford Analysis: A Germ of Function Theory, Hypercomplex Analysis, In: Sabadini, I. , Shapiro, M. , Sommen, F. (eds.) Trends in Mathematics, pp. 37–54, Boston: Birkhäuser, (2009).
[5] Cimasoni, D., : Discrete Dirac operators on Riemann surfaces and Kasteleyn matrices, Journal of the European Mathematical Society, 14, 4, 1209–1244, (2012).
[6] Crane, K. M. , : Conformal Geometry Processing, Dissertation (Ph. D.), California Institute of Technology, (2013), doi: 10. 7907/8V9Z-N286.
[7] Crumeyrolle, A., : Orthogonal and Symplectic Clifford Algebras: Spinor Structures. Springer Netherlands, 2009, ISBN-13: 978-9048140596.
[8] Faustino, N., Kaehler, U., Sommen, F., : Discrete Dirac Operators in Clifford Analysis. Adv. Appl. Clifford Algebras 17: 3, 451–467, (2007).
[9] Faustino, N., Kaehler, U., : Fischer Decomposition for Difference Dirac Operators, Adv. Appl. Clifford Algebras 17: 1, 37–58,(2007).
[10] Golénia, S., Haugomat, T., : On the a. c. spectrum of the 1D discrete Dirac operator, Methods Funct. Anal. Topology, 20, 3, 252-273, (2014).
[11] Gürlebeck, K., Sprößig, W. , : Quaternionic and Clifford calculus for Engineers and Physicists. John Wiley and Sons, Chichester, 1997.
[12] Gürlebeck, K., Hommel, A., : On the finite difference Dirac Operators and their fundamental solution. Advances in Applied Clifford Algebras. 11, 89–106, (2001).
[13] Habermann, K., : The Dirac Operator on Symplectic Spinors, Ann. Global Anal. Geom., 13, 155-168, (1995).
[14] Habermann, K., : Basic Properties of Symplectic Dirac Operators, Commun. Math. Phys., 184, 629-652, (1997).

[15] Habermann, K., : Symplectic Dirac Operators on Kaehler Manifolds, Math. Nachr. 211, 37-62, (2000).

[16] Habermann, K., Habermann, L., : Introduction to Symplectic Dirac Operators, Lecture Notes in Mathematics 1887. Springer-Verlag, (2006).

[17] Kostant, B., : Symplectic Spinors, Rome Symposia, XIV, 139-152, (1974).

[18] Oliveria, C. R de. , Prado, R. A. , : Dynamical delocalisation for 1D Bernoulli discrete Dirac Operator, Journal of Physics A: Mathematical and General. 38, L115-L119, (2005).

[19] Prado, R. A., Olivera, C. R. D., : Sparse 1d discrete Dirac operators i: quantum transport, Journal of Mathematical Analysis and Applications, 85, 2, 947-960, (2012).

[20] Ridder, H. D., Schepper, H. D., Kähler, U., Sommen, F., : Discrete function theory based on skew Weyl relations. Proc. Amer. Math. Soc., 138(9): 3241 - 3256, 2010.

[21] Ridder, H. D., Raeymaekers, T., : Spinor Spaces in Discrete Clifford Analysis, Complex Analysis and Operator Theory. 11, 1113–1137, (2017).

[22] Schepper, H. D., Faustino, N., Kaehler, U. , Sommen, F., : Discrete Dirac Operators in Clifford Analysis, Adv. appl. Clifford alg. 17, 451-467, (2007).

[23] Schepper, H. D., Sommen, F., Van de Voode, L., : A basic framework for discrete Clifford analysis, Experimental Mathematics, 18(4), 385-395, (2009).

[24] Zakharov, D. V., : Discrete analogue of Dirac operator and discrete modified Veselov and Novikov heirarchy, International Mathematics Research Notices, 18, 3463-3488, (2010).

[1] Department of Mathematics, Sambalpur University, Sambalpur, Odisha-768019, India

Email address: rashmath12@gmail.com

[1] Department of Mathematics, Sambalpur University, Sambalpur, Odisha-768019, India

Email address: niharmath@suniv.com