

TRANSPORTATION INEQUALITIES FOR MEAN-FIELD NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATION DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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ABSTRACT. In this paper we prove an existence and uniqueness result of mild solution for a mean-field neutral stochastic differential equation with finite delay, driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ in a Hilbert space and we establish the transportation inequalities, with respect to the uniform distance, for the law of the mild solution.

1. INTRODUCTION AND PRELIMINARIES

Let (E, d) be a metric space equipped with σ -field $\mathcal{B}$ such that $d(., .)$ is $\mathcal{B} \times \mathcal{B}$ -measurable. Given $p \geq 1$ and two probability measures μ and ν on E , we define the Wasserstein distance of order p between μ and ν by

$$W_p^d(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{E \times E} d(x, y)^p d\pi(x, y) \right)^{1/p}$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on the product space $E \times E$ with marginal distributions μ and ν . The relative entropy of ν with respect to μ is defined by

$$H(\nu/\mu) = \begin{cases} \int_E \log \frac{d\nu}{d\mu} d\nu, & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise} \end{cases}$$

We say that a probability measure μ satisfies the L^p transportation inequality on (E, d) if there exists a non-negative constant C such that for any other probability measure ν defined on E , $W_p^d(\mu, \nu) \leq \sqrt{2CH(\nu/\mu)}$. We denote this relation as $\mu \in T_p(C)$.

The cases $p = 1$ and $p = 2$ are particularly significant. The concept of measure concentration is closely tied to the $T_1(C)$ relation. Based on the work of Bobkov

Date: Received: Aug 20, 2023; Accepted: Sep 16, 2023.

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2010 *Mathematics Subject Classification.* Primary 60H15; Secondary 60G15, 60J65.

Key words and phrases. Mild solution, Fractional Brownian motion, Wiener integral, Girsanov transformation, Transportation inequality.

and Götze [2], Djellout *et al* proved in [4] that

$$\mu \in T_1(C) \text{ iff } \int_{E^2} e^{\delta d^2(x,y)} \mu(dx) \mu(dy) < +\infty \text{ for some } \delta > 0.$$

The property $T_2(C)$ is stronger than $T_1(C)$, but it is not so well characterized. It was first established for Gaussian measures in [15], and later extended to abstract Wiener spaces in [5, 6].

Numerous approaches have been developed to establish transportation inequalities. One common method is the Girsanov transformation, which was introduced in [15] and has been effectively utilized in various contexts, such as infinite-dimensional dynamical systems [17], time-inhomogeneous diffusions [11], multi-valued SDEs and singular SDEs [16], neutral functional SDEs [1], SDEs driven by a fractional Brownian motion [13], and stochastic delay evolution equations driven by fractional Brownian motion [8].

As far as we are aware, there has been no research that explores both the existence of solutions and the property $T_2(C)$ for mean-field neutral stochastic differential equation with delays driven by a fractional Brownian motion.

In order to fill this gap, this paper examines the existence and uniqueness of solutions, as well as the $T_2(C)$ property for the law of the mild solution of the following mean-field neutral stochastic differential equation with finite delay:

$$\begin{aligned} d[x(t) + g(t, x(t - r(t)), \mathbb{P}_{x(t)})] &= [Ax(t) + f(t, x(t - \rho(t)), \mathbb{P}_{x(t)})]dt \\ &\quad + \sigma(t)dB^H(t), \quad 0 \leq t \leq T, \\ x(t) &= \varphi(t), \quad -\tau \leq t \leq 0. \end{aligned} \tag{1.1}$$

The given equation involves a fractional Brownian motion B^H on a real and separable Hilbert space Y and an infinitesimal generator A of an analytic semi-group of bounded linear operators, denoted by $(S(t))_{t \geq 0}$, in a Hilbert space X , $\sigma : [0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$, $f, g : [0, +\infty) \times X \times \mathcal{P}_2(X) \rightarrow X$ are appropriate functions and $\rho, r : [0, +\infty) \rightarrow [0, \tau]$ ($\tau > 0$) are continuous. The notation $\mathcal{P}_2(X)$ refers to the space of probability measures with finite second order moments, where for any $\mu \in \mathcal{P}_2(X)$, it holds that $\int |x|^2 \mu(dx) < \infty$. The notation $\mathcal{L}_2^0(Y, X)$ represents the space of all Q -Hilbert-Schmidt operators from Y into X , which will be further discussed in section 2 below.

The theory of mean-field stochastic differential equations, also known as McKean-Vlasov stochastic differential equations, was initiated by McKean in [9] who was inspired by Kac's program in kinetic theory. Since then, there have been numerous results in this area (c.f. [14]).

The subsequent sections of this paper are structured as follows: Section 2 introduces the necessary notations, concepts, and fundamental results concerning fractional Brownian motion, Wiener integral over Hilbert spaces, as well as a review of preliminary findings on analytic semi-groups and the fractional power associated with its generator. The discussion on the existence and uniqueness of mild solutions is presented in Section 3 by using Banach fixed point theorem. In Section 4, we investigate the property $T_2(C)$ for the law of the mild solution of equation (1.1) with respect to the uniform distance on path space.

2. PRELIMINARIES

In this section, we will introduce the necessary concepts, notions, and lemmas on Wiener integrals with respect to an infinite-dimensional fractional Brownian motion. Additionally, we will provide a review of some basic results concerning analytical semi-groups and fractional powers of their infinitesimal generators that will be utilized throughout the entirety of this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ be the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) d\beta(s) \quad (2.1)$$

where $\beta = \{\beta(t), t \in [0, T]\}$ is a Wiener process, and $K_H(t; s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ denotes the Beta function. We put $K_H(t, s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

To provide some background, let us note that the mapping $1_{[0,t]} \rightarrow \beta^H(t)$ can be extended to an isometry between the Hilbert space $\mathcal{H}$ and the first Wiener chaos. We denote by $\int_0^T \varphi(s) d\beta^H(s)$ the image of φ under this isometry.

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr$$

For the proof that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0, T])$, we refer to [10]. Additionally, for any $\varphi \in \mathcal{H}$, we have the following expression:

$$\int_0^T \varphi(t) d\beta^H(t) = \int_0^T (K_H^* \varphi)(t) d\beta(t) \quad (2.2)$$

Consider two real, separable Hilbert spaces X and Y , and let $\mathcal{L}(Y, X)$ denote the space of bounded linear operators from Y to X . For the sake of convenience, we will use the same notation to denote the norms in X , Y , and $\mathcal{L}(Y, X)$. Let $Q \in \mathcal{L}(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with a finite trace $\text{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ ($n = 1, 2, \dots$) are non-negative real numbers and

e_n ($n = 1, 2, \dots$) is a complete orthonormal basis in Y . We define the infinite-dimensional fractional Brownian motion on Y with covariance Q as:

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t).$$

Here, β_n^H are real, independent fBm's.

In order to define Wiener integrals with respect to the Q -fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q -Hilbert-Schmidt operators $\psi : Y \rightarrow X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q -Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Now, let $\phi(s); s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$, The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^*(\phi e_n)(s) d\beta_n(s) \quad (2.3)$$

where β_n is the standard Brownian motion used to present β_n^H as in (2.1) and we have the following result (see [3])

Lemma 2.1. *If $\psi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$ then the above sum in (2.3) is well defined as a X -valued random variable and we have*

$$\mathbb{E} \left\| \int_0^t \psi(s) dB^H(s) \right\|^2 \leq 2Ht^{-1+2H} \int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds$$

Let us now introduce some notations and basic facts about the theory of fractional power operators and semi-groups.

Consider the infinitesimal generator $A : D(A) \rightarrow X$ of an analytic semigroup $(S(t))_{t \geq 0}$ of bounded linear operators on X . For the theory of strongly continuous semigroups, we refer to [7] and [12]. In this work, we will use some notations and properties that are well-known in this theory. It is known that there exist $M \geq 1$ and $\lambda \in \mathbb{R}$ such that $|S(t)| \leq Me^{\lambda t}$ for every $t \geq 0$.

If $(S(t))_{t \geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A , then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X , and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known (cf. [12], p. 74).

Lemma 2.2. *Suppose that the preceeding conditions are satisfied.*

(1) *For every $0 < \alpha \leq 1$ there exists $M_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

(2) *If $0 < \beta \leq \alpha$ then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.*

(3) *Let $0 < \alpha \leq 1$. Then X_α is a Banach space.*

3. EXISTENCE AND UNIQUENESS OF A SOLUTION

The existence and uniqueness of mild solutions to Equation (1.1) are investigated in this section. The following conditions are assumed to be true for this equation.

(H.1) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X . Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$, and that, see Lemma 2.2,

$$\|S(t)\| \leq M \quad \text{and} \quad \|(-A)^{1-\beta} S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}$$

for some constants M , $M_{1-\beta}$ and every $t \in [0, T]$.

(H.2) The function $f : [0, +\infty) \times X \times \mathcal{P}_2(X) \rightarrow X$ satisfies the following Lipschitz conditions: that is, there exist positive constants $C_i := C_i(T)$, $i = 1, 2$ such that, for all $t \in [0, T]$, $x, y \in X$ and $\mu, \nu \in \mathcal{P}_2(X)$

$$- \|f(t, x, \mu) - f(t, y, \nu)\| \leq C_1 (\|x - y\| + W_2(\mu, \nu)).$$

$$- \|f(t, x, \mu)\|^2 \leq C_2 (1 + \|x\|^2 + W_2^2(\mu, \delta_0)).$$

(H.3) There exist constants $\frac{1}{2} < \beta < 1$, $C_i := C_i(T)$, $i = 3, 4$, such that the function g is X_β -valued and satisfies for all $t \in [0, T]$, $x, y \in X$ and $\mu, \nu \in \mathcal{P}_2(X)$

$$(i) \quad \|(-A)^\beta g(t, x, \mu) - (-A)^\beta g(t, y, \nu)\| \leq C_3 (\|x - y\| + W_2(\mu, \nu)).$$

$$(ii) \quad \|(-A)^\beta g(t, x, \mu)\|^2 \leq C_4 (1 + \|x\|^2 + W_2^2(\mu, \delta_0)).$$

(iii) The constants C_3 and β satisfy the following inequality

$$2\|(-A)^{-\beta}\|C_3 < 1.$$

(H.4) The function $(-A)^\beta g$ is continuous in the quadratic mean sense: For all $x \in \mathcal{C}([0, T], \mathbb{L}^2(\Omega, X))$ and $\mu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$

$$\lim_{t \rightarrow s} \mathbb{E} \|(-A)^\beta g(t, x(t), \mu(t)) - (-A)^\beta g(s, x(s), \mu(s))\|^2 = 0.$$

(H.5) The function $\sigma : [0, \infty) \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds < \infty, \quad \forall T > 0$$

Moreover, we assume that $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$.

Next, we introduce the concept of a mild solution of equation (1.1).

Definition 3.1. A X -valued process $\{x(t), t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if

$$i) \quad x(\cdot) \in \mathcal{C}([-\tau, T], \mathbb{L}^2(\Omega, X)),$$

- ii) $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$.
- iii) For arbitrary $t \in [0, T]$, we have

$$\begin{aligned}
x(t) &= S(t)(\varphi(0) + g(0, \varphi(-r(0)), \mathbb{P}_{\varphi(0)})) - g(t, x(t-r(t)), \mathbb{P}_{x(t)}) \\
&\quad - \int_0^t AS(t-s)g(s, x(s-r(s)), \mathbb{P}_{x(s)})ds \\
&\quad + \int_0^t S(t-s)f(s, x(s-\rho(s)), \mathbb{P}_{x(s)})ds + \int_0^t S(t-s)\sigma(s)dB^H(s) \quad \mathbb{P} - a.s
\end{aligned}$$

We can now state the main result of this section.

Theorem 3.2. *Suppose that (H.1)–(H.5) hold. Then, for all $T > 0$, the equation (1.1) has a unique mild solution on $[-\tau, T]$.*

Proof. Let $\mu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$ be temporarily fixed. The classical existence result for neutral stochastic functional differential equations guarantees the existence and uniqueness of a mild solution of the equation

$$\begin{aligned}
d[x(t) + g(t, x(t-r(t)), \mu(t))] &= [Ax(t) + f(t, x(t-\rho(t)), \mu(t))]dt \\
&\quad + \sigma(t)dB^H(t), \quad 0 \leq t \leq T, \\
x(t) &= \varphi(t), \quad -\tau \leq t \leq 0.
\end{aligned} \tag{3.1}$$

We denote its solution by x_μ . Let $\mathcal{L}(x_\mu) = \{\mathcal{L}(x_\mu(t)) : t \in [0, T]\}$ denote the law of x_μ and define an operator ψ on $\mathcal{C}([0, T], \mathcal{P}_2(X))$ by $\psi(\mu) = \mathcal{L}(x_\mu)$.

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator ψ . Next we will show by using Banach fixed point theorem that ψ has a unique fixed point. We divide the subsequent proof into two steps.

Step 1.

For arbitrary $\mu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$, let us prove that $t \rightarrow \psi(\mu)(t)$ is continuous on the interval $[0, T]$.

Let $0 < t < T$ and $|h|$ be sufficiently small. Then, we have

$$\begin{aligned}
W_2^2(\psi(\mu)(t+h), \psi(\mu)(t)) &= W_2^2(\mathbb{P}_{x_\mu(t+h)}, \mathbb{P}_{x_\mu(t)}) \\
&\leq E \|x_\mu(t+h) - x_\mu(t)\|^2 \rightarrow 0 \quad \text{as } h \rightarrow 0
\end{aligned}$$

Hence, we conclude that the function $t \rightarrow \psi(\mu)(t)$ is continuous on $[0, T]$.

Step 2.

Now, our aim is to demonstrate that ψ is a contraction mapping in $\mathcal{C}([0, T_1], \mathcal{P}_2(X))$, where $T_1 \leq T$ will be determined later.

Let $\mu, \nu \in \mathcal{C}([0, T], \mathcal{P}_2(X))$ by using the inequality $(a+b+c)^2 \leq \frac{1}{k}a^2 + \frac{2}{1-k}b^2 + \frac{2}{1-k}c^2$, where $k := 2C_3\|(-A)^{-\beta}\| < 1$, we obtain for any fixed $t \in [0, T]$

$$\begin{aligned}
& \|x_\mu(t) - x_\nu(t)\|^2 \\
& \leq \frac{1}{k} \|g(t, x_\mu(t-r(t)), \mu(t)) - g(t, x_\nu(t-r(t)), \nu(t))\|^2 \\
& \quad + \frac{2}{1-k} \left\| \int_0^t AS(t-s)(g(s, x_\mu(s-r(s)), \mu(s)) - g(s, x_\nu(s-r(s)), \nu(s))) ds \right\|^2 \\
& \quad + \frac{2}{1-k} \left\| \int_0^t S(t-s)(f(s, x_\mu(s-\rho(s)), \mu(s)) - f(s, x_\nu(s-\rho(s)), \nu(s))) ds \right\|^2 \\
& \leq \frac{1}{k} \|(-A)^{-\beta}\|^2 \|(-A)^\beta g(t, x_\mu(t-r(t)), \mu(t)) - (-A)^\beta g(t, x_\nu(t-r(t)), \nu(t))\|^2 \\
& \quad + \frac{2}{1-k} \left\| \int_0^t (-A)^{1-\beta} S(t-s)(-A)^\beta (g(s, x_\mu(s-r(s)), \mu(s)) \right. \\
& \quad \quad \left. - g(s, x_\nu(s-r(s)), \nu(s))) ds \right\|^2 \\
& \quad + \frac{2}{1-k} \left\| \int_0^t S(t-s)(f(s, x_\mu(s-\rho(s)), \mu(s)) - f(s, x_\nu(s-\rho(s)), \nu(s))) ds \right\|^2.
\end{aligned}$$

By Lipschitz property of $(-A)^\beta g$ and f combined with Hölder's inequality, we obtain

$$\begin{aligned}
& \mathbb{E} \|x_\mu(t) - x_\nu(t)\|^2 \\
& \leq \frac{k}{2} \{ \mathbb{E} \|x_\mu(t-r) - x_\nu(t-r)\|^2 + W_2^2(\mu(t), \nu(t)) \} \\
& \quad + \frac{4}{1-k} C_3^2 M_{1-\beta}^2 \left(\frac{t^{2\beta-1}}{2\beta-1} \right) \int_0^t \{ \mathbb{E} \|x_\mu(s-r) - x_\nu(s-r)\|^2 + W_2^2(\mu(s), \nu(s)) \} ds \\
& \quad + \frac{4}{1-k} t M^2 C_1^2 \int_0^t \{ \mathbb{E} \|x_\mu(s-\rho(s)) - x_\nu(s-\rho(s))\|^2 + W_2^2(\mu(s), \nu(s)) \} ds.
\end{aligned}$$

Then, Gronwall's lemma implies that

$$\sup_{s \in [0, t]} \mathbb{E} \|x_\mu(s) - x_\nu(s)\|^2 \leq \gamma(t) \sup_{s \in [0, t]} W_2^2(\mu(s), \nu(s)).$$

where

$$\begin{aligned}
\gamma(t) &= \left(\frac{k/2}{1-k/2} + \frac{4}{(1-k)(1-k/2)} \left\{ \frac{C_3^2 M_{1-\beta}^2}{2\beta-1} t^{2\beta} + M^2 C_1^2 t^2 \right\} \right) \\
&\quad \times \exp \left(\frac{4}{(1-k)(1-k/2)} \left\{ \frac{C_3^2 M_{1-\beta}^2}{2\beta(2\beta-1)} t^{2\beta} + M^2 C_1^2 \frac{t^2}{2} \right\} \right)
\end{aligned}$$

Hence, it follows that

$$\sup_{s \in [0, t]} W_2^2(\mathbb{P}_{x_\mu(s)}, \mathbb{P}_{x_\nu(s)}) \leq \gamma(t) \sup_{s \in [0, t]} W_2^2(\mu(s), \nu(s)).$$

According to condition (iii) in $(\mathcal{H}.3)$, we have $\gamma(0) = \frac{k}{2-k} < 1$. Hence, there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$. Consequently, ψ becomes a contraction mapping on $\mathcal{C}([0, T_1], \mathcal{P}2(X))$, ensuring the existence of a unique fixed

point μ . As a result, $x\mu$ serves as a mild solution for equation (1.1) on the interval $[-\tau, T_1]$. This procedure can be repeated finitely many times to extend the solution to the entire interval $[-\tau, T]$. Hence, the proof is complete. $\square$

4. TRANSPORTATION INEQUALITIES FOR MEAN-FIELD NEUTRAL SDES

In this section, our focus is on investigating the property $T_2(C)$ for the law of the mild solution of equation (1.1) on the space $\mathcal{C} = \mathcal{C}([0, T], X)$ equipped with the uniform metric d_∞ . Precisely, we have the following theorem:

Theorem 4.1. *Suppose that (H.1), $-$, (H.5) hold and let $\mathbb{P}_\varphi$ be the law of $\{x(t, \varphi), t \in [0, T]\}$ where $x(\cdot, \varphi)$ is the mild solution of equation (1.1) with $x_0 = \varphi$. Assume further that $\tilde{\sigma} = \sup_{0 \leq t \leq T} \|\sigma(t)\|_{\mathcal{L}_2^0} < \infty$. Then the probability measure $\mathbb{P}_\varphi$ satisfies $T_2(C)$ on the metric space $\mathcal{C}([0, T], X)$ with the metric d_∞ .*

Proof. Let $\mathbb{P}_\varphi$ be the law of $\{x(t, \varphi), t \in [0, T]\}$ on $\mathcal{C} = \mathcal{C}([0, T], X)$ and let $\mathbb{Q}$ be any probability measure on $\mathcal{C}$ such that $\mathbb{Q} \ll \mathbb{P}_\varphi$. Define $\tilde{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}_\varphi}(x(\cdot, \varphi))\mathbb{P}$ which is a probability measure on $(\Omega, \mathcal{F})$. we have

$$\begin{aligned} H(\tilde{\mathbb{Q}}/\mathbb{P}) &= \int_{\Omega} \log \left(\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \right) . d\tilde{\mathbb{Q}} \\ &= \int_{\Omega} \log \left(\frac{d\mathbb{Q}}{d\mathbb{P}_\varphi}(x(\cdot, \varphi)) \right) \frac{d\mathbb{Q}}{d\mathbb{P}_\varphi}(x(\cdot, \varphi)) . d\mathbb{P} \\ &= \int_{\mathcal{C}} \log \left(\frac{d\mathbb{Q}}{d\mathbb{P}_\varphi} \right) \frac{d\mathbb{Q}}{d\mathbb{P}_\varphi} . d\mathbb{P}_\varphi \\ &= H(\mathbb{Q}/\mathbb{P}_\varphi) \end{aligned}$$

Following [4], there exists a predictable process $(h(t))_{0 \leq t \leq T} \in X$ with $\int_0^T \|h(s)\|^2 ds < +\infty$ $\mathbb{P}$ -a.s such that:

$$H(\mathbb{Q}/\mathbb{P}_\varphi) = H(\tilde{\mathbb{Q}}/\mathbb{P}) = \frac{1}{2} \mathbb{E}_{\tilde{\mathbb{Q}}} \int_0^T \|h(s)\|^2 ds$$

Due to the Girsanov theorem, the process $(\tilde{B}(t))_{t \in [0, T]}$ defined by

$$\tilde{B}(t) = B(t) - \int_0^t h(s) ds$$

is a Brownian motion with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{Q}})$. Let $(\tilde{B}^H(t))_{t \in [0, T]}$ the $\tilde{\mathbb{Q}}$ -fractional Brownian motion associated to $(\tilde{B}(t))_{t \in [0, T]}$ defined by

$$\begin{aligned} \tilde{B}^H(t) &= \int_0^t K_H(t, s) d\tilde{B}(s) \\ &= \int_0^t K_H(t, s) dB(s) - (K_H h)(t) \end{aligned}$$

where $K_H h$ is defined by $(K_H h)(t) = \int_0^t K_H(t, s) h(s) ds$

By Fubini's Theorem, we obtain that

$$\begin{aligned}(K_H h)(t) &= c_H \int_0^t \left(s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du \right) h(s) ds \\ &= \int_0^t g(u) du\end{aligned}$$

where $g(u) = c_H u^{H-1/2} \int_0^u s^{1/2-H} (u-s)^{H-3/2} h(s) ds$

Consequently, under the measure $\tilde{\mathbb{Q}}$, the process $\{x(t, \varphi)\}_{t \in [-\tau, T]}$ satisfies

$$\begin{aligned}x(t) &= S(t)(\varphi(0) + g(0, \varphi(-r(0)), \mu(0))) - g(t, x(t-r(t)), \mu(t)) \\ &\quad - \int_0^t AS(t-s)g(s, x(s-r(s)), \mu(s))ds + \int_0^t S(t-s)f(s, x(s-\rho(s)), \mu(s))ds \\ &\quad + \int_0^t S(t-s)\sigma(s)d\tilde{B}^H(s) + \int_0^t S(t-s)\sigma(s)d(K_H h)(s), \quad 0 \leq t \leq T.\end{aligned}$$

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0.$$

where $\mu(t) = \mathbb{P}_{x(t)}$.

We now consider the mild solution $\{y(t, \varphi), t \in [-\tau, T]\}$ (under $\tilde{\mathbb{Q}}$) of the following equation

$$\begin{aligned}d[y(t) + g(t, y(t-r(t)), \mu(t))] &= [Ay(t) + f(t, y(t-\rho(t)), \mu(t))]dt \\ &\quad + \sigma(t)d\tilde{B}^H(t), \quad 0 \leq t \leq T, \\ y(t) &= \varphi(t), \quad -\tau \leq t \leq 0.\end{aligned}$$

By Theorem 5 in [3], under $\tilde{\mathbb{Q}}$, the law of $\{y(t, \varphi), t \in [0, T]\}$ is $\mathbb{P}_\varphi$. Thus, $\{(x(t), y(t)), t \in [0, T]\}$, under $\tilde{\mathbb{Q}}$, is a coupling of $(\mathbb{Q}, \mathbb{P}_\varphi)$ and it follows that:

$$(W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_\varphi))^2 \leq \mathbb{E}_{\tilde{\mathbb{Q}}}(d_\infty(x, y))^2 = \mathbb{E}_{\tilde{\mathbb{Q}}}(\sup_{0 \leq t \leq T} \|x(t) - y(t)\|^2)$$

We now estimate the distance between x and y with respect to d_∞ .

By using the inequality $(a + b + c + d)^2 \leq \frac{1}{k}a^2 + \frac{3}{1-k}b^2 + \frac{3}{1-k}c^2 + \frac{3}{1-k}d^2$, where $k := C_3\|(-A)^{-\beta}\| < 1$, we obtain for any fixed $t \in [0, T]$

$$\begin{aligned}&\|x(t) - y(t)\|^2 \\ &\leq \frac{1}{k} \|g(t, x(t-r(t)), \mu(t)) - g(t, y(t-r(t)), \mu(t))\|^2 \\ &\quad + \frac{3}{1-k} \left\| \int_0^t AS(t-s)(g(s, x(s-\rho(s)), \mu(s)) - g(s, y(s-\rho(s)), \mu(s)))ds \right\|^2 \\ &\quad + \frac{3}{1-k} \left\| \int_0^t S(t-s)(f(s, x(s-\rho(s)), \mu(s)) - f(s, y(s-\rho(s)), \mu(s)))ds \right\|^2 \\ &\quad + \frac{3}{1-k} \left\| \int_0^t S(t-s)\sigma(s)d(K_H h)(s) \right\|^2 \\ &\leq I_1(t) + I_2(t) + I_3(t) + I_4(t)\end{aligned}\tag{4.1}$$

By Lipschitz property of $(-A)^\beta g$, we obtain

$$\begin{aligned} I_1(t) &\leq k (\|x(t - \rho(t)) - y(t - \rho(t))\|^2) \\ &\leq k \sup_{0 \leq s \leq t} \|x(s) - y(s)\|^2 \end{aligned} \quad (4.2)$$

By Lipschitz property of $(-A)^\beta g$, Lemma (2.2) and Hölder's inequality, we obtain that

$$I_2(t) \leq \frac{3}{1-k} \frac{C_3^2 M_{1-\beta}^2 t^{2\beta-1}}{2\beta-1} \int_0^t \sup_{u \in [0,s]} \|x(u) - y(u)\|^2 ds \quad (4.3)$$

By Lipschitz property of f combined with Hölder's inequality, we obtain that

$$I_3(t) \leq 2 \frac{3}{1-k} M^2 C_1^2 t \int_0^t \sup_{u \in [0,s]} \|x(u) - y(u)\|^2 ds \quad (4.4)$$

By Fubini's theorem, we obtain that

$$\begin{aligned} &I_4(t) \\ &\leq \frac{3c_H^2}{1-k} \left(\int_0^t \|S(t-s)\| \|\sigma(s)\| s^{H-1/2} \left(\int_0^s u^{1/2-H} (s-u)^{H-3/2} \|h(u)\| du \right) ds \right)^2 \\ &\leq \frac{3c_H^2}{1-k} \left(\int_0^t u^{1/2-H} \|h(u)\| \left(\int_u^t \|S(t-s)\| \|\sigma(s)\| s^{H-1/2} (s-u)^{H-3/2} ds \right) du \right)^2 \\ &\leq \frac{3c_H^2 M^2 \tilde{\sigma}^2 T^{2H-1}}{(1-k)(2H-1)} \mathcal{B}(2H, 2-2H) \int_0^t \|h(u)\|^2 du \end{aligned} \quad (4.5)$$

Inequalities (4.1), (4.2), (4.3), (4.4) and (4.5) together imply that:

$$\sup_{0 \leq s \leq t} \|x(s) - y(s)\|^2 \leq \delta_1 \int_0^t \sup_{u \in [0,s]} \|x(u) - y(u)\|^2 ds + \delta_2 \int_0^t \|h(s)\|^2 ds.$$

where

$$\delta_1 := \frac{3}{(1-k)^2} \left\{ \frac{C_3^2 M_{1-\beta}^2 T^{2\beta-1}}{2\beta-1} + M^2 C_1^2 T \right\}$$

and

$$\delta_2 := \frac{3c_H^2 M^2 \tilde{\sigma}^2 T^{2H-1}}{(1-k)^2 (2H-1)} \mathcal{B}(2H, 2-2H)$$

Then, Gronwall's lemma implies that

$$\sup_{0 \leq t \leq T} \|x(t) - y(t)\|^2 \leq \delta_2 \exp(\delta_1 T) \int_0^T \|h(s)\|^2 ds.$$

Hence, it follows that

$$\begin{aligned} (W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_\varphi))^2 &\leq \delta_2 \exp(\delta_1 T) \mathbb{E}_{\tilde{\mathbb{Q}}} \int_0^T \|h(s)\|^2 ds \\ &\leq 2CH(\mathbb{Q}/\mathbb{P}_\varphi). \end{aligned}$$

where $C := \frac{\delta_2 \exp(\delta_1 T)}{2}$. The proof is complete. $\square$

Acknowledgement. The authors would like to thank the referee and the editor for their careful comments and valuable suggestions on this work..

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