Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 18 (2023) 40-55

COMMON FIXED POINT THEOREMS IN S-METRIC SPACES INVOLVING CONTROL FUNCTION

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ABSTRACT. The aim of this paper is to prove some common fixed point theorems for the pair of weakly compatible self-mappings involving control function in the framework of S-metric spaces. Furthermore, we provide some consequences as corollaries of the main results. Moreover, we give illustrative examples in support of the results. The results presented in this paper generalize, extend, unify and enrich several previously published results from the existing literature.

1. INTRODUCTION

The Banach Contraction Principle ([3]) (or in short BCP) is a very popular tool in solving existence problems in many branches of mathematical analysis and its applications. The statement of this famous result is as follows.

Theorem 1.1. *Let (Ξ, d) be a complete metric space and let $\mathcal{T} \colon \Xi \to \Xi$ be a self-mapping. If there exists $h \in [0, 1)$ such that*

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq h\, d(x, y), \tag{1.1}$$

for all $x, y \in \Xi$, then $\mathcal{T}$ has a unique fixed point $u \in \Xi$.

Moreover, for any $e_0 \in \Xi$, the sequence $\{e_n\} \subset \Xi$ defined by $e_{n+1} = \mathcal{T} e_n$, $n \in \mathbb{N}$, is convergent to the fixed point u. Inequality (1.1) also implies the continuity of $\mathcal{T}$.

Most of the works after this was basically a generalization of the work of Banach. These generalizations include more general metric spaces, or more general contractions etc. Over last few decades, a number of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, b-metric spaces, partial metric spaces, cone metric spaces, S_b-metric space and cone S_b-metric space. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [2, 4, 5, 9, 20, 21, 24, 25, 26] (see,

Date: Received: Aug 10, 2023; Accepted: Sep 17, 2023.
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2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Common fixed point, weakly compatible mapping, control function, S-metric space.

also [6], [12], [14], [15], [17], [18], [27]). One of such generalizations is an S-metric space.

In 2012, Sedghi *et al.* [20] introduced the concept of an S-metric space and proved that this concept is a generalization of a G-metric space and D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

In 2016, Sedghi *et al.* [22] proved some common fixed point theorems for the pair of weakly compatible self mappings satisfying some Φ-type contractive conditions in the setting of S-metric spaces. The results presented in this paper extend, generalize and enrich several results from the exiting literature.

Recently, Sedghi *et al.* [23] proved some common fixed point theorems for four mappings satisfying generalized contractive condition in the framework of S-metric spaces. The results obtained in this paper extend and generalize several previously published works.

Inspired and motivate by the works of [20, 22, 23] and some others, the purpose of this paper is to prove some coincidence and common fixed point theorems using control function and weakly compatible condition. Our results extend and generalize several results from the existing literature.

2. Preliminaries

We need the following definitions and lemmas in the sequel.

Definition 2.1. ([20]) Let $\Xi \neq \emptyset$ be a set and let $S\colon \Xi^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $t \in \Xi$:

$(S1)$ $S(x,y,z) = 0$ if and only if $x = y = z$;

$(S2)$ $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called an S-metric on Ξ and the pair (Ξ, S) is called an S-metric space (in short SMS).

Some examples of such S-metric spaces are as follows.

Example 2.2. ([20])

(1) Let $\Xi = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Ξ, then $S(u,v,z) = \|v+z-2u\| + \|v-z\|$ is an S-metric on Ξ.

(2) Let $\Xi = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Ξ, then $S(u,v,z) = \|u-z\| + \|v-z\|$ is an S-metric on Ξ.

Example 2.3. ([21]) Let $\Xi = \mathbb{R}$ be the real line. Then $S(u,v,z) = |u-z|+|v-z|$ for all $u,v,z \in \mathbb{R}$ is an S-metric on Ξ. This S-metric on Ξ is called the usual S-metric on Ξ.

Example 2.4. ([13]) Let $\Xi \neq \emptyset$ be a set and d be an ordinary metric on Ξ. Then $S(u,v,z) = d(u,z) + d(v,z)$ for all $u,v,z \in \mathbb{R}$ is an S-metric on Ξ.

Example 2.5. ([23]) Let $\Xi \neq \emptyset$ be a set and d_1, d_2 be two ordinary metrics on Ξ. Then $S(u,v,z) = d_1(u,z) + d_2(v,z)$ for all $u,v,z \in \Xi$ is an S-metric on Ξ.

Definition 2.6. Let (Ξ, S) be an S-metric space. For $r > 0$ and $e \in \Xi$ we define the open ball $B_S(e,r)$ and closed ball $B_S[e,r]$ with center e and radius r

as follows, respectively:

$$B_S(e,r) = \{e' \in \Xi : S(e',e',e) < r\},$$

$$B_S[e,r] = \{e' \in \Xi : S(e',e',e) \le r\}.$$

Example 2.7. ([21]) Let $\Xi = \mathbb{R}$. Denote $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Then

$$\begin{aligned} B_S(1,2) &= \{e \in \mathbb{R} : S(e,e,1) < 2\} = \{e \in \mathbb{R} : |e-1| < 1\} \\ &= \{e \in \mathbb{R} : 0 < e < 2\} = (0,2), \end{aligned}$$

and

$$\begin{aligned} B_S[2,4] &= \{e \in \mathbb{R} : S(e,e,2) \le 4\} = \{e \in \mathbb{R} : |e-2| \le 2\} \\ &= \{e \in \mathbb{R} : 0 \le e \le 4\} = [0,4]. \end{aligned}$$

Definition 2.8. ([20], [21]) Let (Ξ, S) be an S-metric space and $A \subset \Xi$.

(1) The subset A is said to be an open subset of Ξ, if for every $e \in A$ there exists $r > 0$ such that $B_S(e,r) \subset A$.

(2) A sequence $\{e_n\}$ in Ξ converges to $e \in \Xi$ if $S(e_n, e_n, e) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $S(e_n, e_n, e) < \varepsilon$. We denote this by $\lim_{n\to\infty} e_n = e$ or $e_n \to e$ as $n \to \infty$.

(3) A sequence $\{e_n\}$ in Ξ is called a Cauchy sequence if $S(e_n, e_n, e_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$ we have $S(e_n, e_n, e_m) < \varepsilon$.

(4) The S-metric space (Ξ, S) is called complete if every Cauchy sequence in Ξ is convergent in Ξ.

(5) Let τ be the set of all $A \subset \Xi$ with $e \in A$ and there exists $r > 0$ such that $B_S(e,r) \subset A$. Then τ is a topology on Ξ (induced by the S-metric space).

(6) A nonempty subset A of Ξ is S-closed if closure of A coincides with A.

Definition 2.9. ([20]) Let (Ξ, S) be an S-metric space. A mapping $\mathcal{T}\colon \Xi \to \Xi$ is said to be a contraction if there exists a constant $0 \le L < 1$ such that

$$S(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \le L\, S(x,y,z) \tag{2.1}$$

for all $x, y, z \in \Xi$.

Note: If the S-metric space (Ξ, S) is complete then the mapping defined as above has a unique fixed point (see, [20]).

Definition 2.10. ([20]) Let (Ξ, S) and (Ξ', S') be two S-metric spaces. A function $f\colon \Xi \to \Xi'$ is said to be continuous at a point $e_0 \in \Xi$ if for every sequence $\{e_n\}$ in Ξ with $S(e_n, e_n, e_0) \to 0$, $S'(f(e_n), f(e_n), f(e_0)) \to 0$ as $n \to \infty$. We say that f is continuous on Ξ if f is continuous at every point $e_0 \in \Xi$.

Definition 2.11. Let $\Xi \ne \emptyset$ be a set and let $\mathcal{T}, g\colon \Xi \to \Xi$ be two self mappings of Ξ. Then a point $e \in \Xi$ is called a

(1) fixed point of operator $\mathcal{T}$ if $\mathcal{T}(e) = e$;

(2) common fixed point of $\mathcal{T}$ and g if $\mathcal{T}(e) = g(e) = e$.

Definition 2.12. ([1]) Let f and g be single valued self-mappings on a set Ξ. If $e' = fe = ge$ for some $e \in \Xi$, then e is called a coincidence point of f and g, and e' is called a point of coincidence of f and g.

Definition 2.13. ([10]) Let f and g be single valued self-mappings on a set Ξ. The mappings f and g are said to be commuting if $fge = gfe$ for all $e \in \Xi$.

Definition 2.14. ([11]) Let f and g be single valued self-mappings on a set Ξ. The mappings f and g are said to be weakly compatible if they commute at their coincidence points, i.e., if $fe = ge$ for some $e \in \Xi$ implies $fge = gfe$.

Example 2.15. Let $\Xi = [0, 3]$. Define the mappings $f, g\colon \Xi \to \Xi$ by

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ 3, & \text{if } x \in [1,3], \end{cases} \quad , \quad g(x) = \begin{cases} 3 - x, & \text{if } x \in [0,1), \\ 3, & \text{if } x \in [1,3]. \end{cases}$$

Then for any $x \in [1, 3]$, we have $fgx = gfx$, showing that f and g are weakly compatible maps on Ξ.

Every S-metric on Ξ defines a metric d_S on Ξ by

$$d_S = S(x, x, y) + S(y, y, x) \ \ \forall\, x, y \in \Xi. \tag{2.2}$$

Lemma 2.16. *([20], Lemma 2.5) Let (Ξ, S) be an S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in \Xi$.*

Lemma 2.17. *([20], Lemma 2.12) Let (Ξ, S) be an S-metric space. If $e_n \to e$ and $e'_n \to e'$ as $n \to \infty$ then $S(e_n, e_n, e'_n) \to S(e, e, e')$ as $n \to \infty$.*

Remark 2.18. ([21]) Every S-metric space is topologically equivalent to a B-metric space.

Lemma 2.19. *([7], Lemma 8) Let (Ξ, S) be an S-metric space and A be a nonempty subset of Ξ. Then A is said to be S-closed if and only if for any sequence $\{e_n\}$ in A such that $e_n \to e$ as $n \to \infty$, then $e \in A$.*

Lemma 2.20. *([20]) Let (Ξ, S) be an S-metric space. If $r > 0$ and $e \in \Xi$, then the ball $B_S(e, r)$ is an open subset of Ξ.*

Lemma 2.21. *([21]) The limit of the sequence $\{e_n\}$ in an S-metric space (Ξ, S) is unique.*

Lemma 2.22. *([20]) Let (Ξ, S) be an S-metric space. Then the convergent sequence $\{e_n\}$ in Ξ is Cauchy.*

Lemma 2.23. *([19],[22]) Let (Ξ, S) be an S-metric space and let $\{e_n\}$ be a sequence in it such that*

$$\lim_{n\to\infty} S(e_{n+1}, e_{n+1}, e_n) = 0.$$

If $\{e_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(e_{m_k}, e_{m_k}, e_{n_k}), S(e_{m_k}, e_{m_k}, e_{m_k+1}), S(e_{m_k-1}, e_{m_k-1}, e_{n_k}),$$
$$S(e_{m_k-1}, e_{m_k-1}, e_{n_k+1}), S(e_{m_k+1}, e_{m_k+1}, e_{n_k+1}), \ldots.$$

In the following lemma we see the relationship between a metric and S-metric.

Lemma 2.24. *([8])*
Let (Ξ, d) be a metric space. Then the following properties are satisfied:
(1) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in \Xi$ is an S-metric on Ξ.
(2) $e_n \to e$ in (Ξ, d) if and only if $e_n \to e$ in (Ξ, S_d).
(3) $\{e_n\}$ is Cauchy in (Ξ, d) if and only if $\{e_n\}$ is Cauchy in (Ξ, S_d).
(4) (Ξ, d) is complete if and only if (Ξ, S_d) is complete.

We call the function S_d defined in Lemma 2.24 (1) as the S-metric generated by the metric d. It can be found an example of an S-metric which is not generated by any metric in [8, 16].

Proposition 2.25. *([1]) Let f and g be weakly compatible self mappings on a set Ξ. If f and g have a unique point of coincidence $e' = fe = ge$, then e' is the unique common fixed point of f and g.*

Corollary 2.26. *([21]) Let $\mathcal{T}\colon \Xi \to \Xi'$ be a map from an S-metric space Ξ to an S-metric space Ξ'. Then $\mathcal{T}$ is continuous at $e \in \Xi$ if and only if $\mathcal{T}e_n \to \mathcal{T}e$ whenever $e_n \to e$ as $n \to \infty$.*

3. Common fixed point theorems

In this section, we shall prove some common fixed point theorems involving control function in the framework of S-metric spaces. Now, to prove our main result, we use the following:

Let Ψ denote the class of all functions $\psi\colon [0, \infty) \to [0, \infty)$ such that:
(Ψ_1) ψ is nondecreasing, continuous;
(Ψ_2) $\sum_{n=1}^{\infty} \psi^n(t)$ is convergent for each $t > 0$.

It is clear that $\psi^n(t) \to 0$ as $n \to \infty$ for each $t > 0$ and hence, we have $\psi(t) < t$ for each $t > 0$. Furthermore, if $\psi \in \Psi$, then $\psi(0) = 0$.

Theorem 3.1. *Let (Ξ, S) be an S-metric space and let $f, g\colon \Xi \to \Xi$ be two mappings satisfying the inequality:*

$$\begin{aligned} S(fx, fy, fz) \ \leq \ \psi\Big(\max\Big\{ & S(fx, fx, gz), S(gx, gx, fx), S(gy, gy, fy), \\ & \frac{1}{2}[S(gx, gx, fz) + S(gz, gz, fx)] \Big\}\Big) \end{aligned} \tag{3.1}$$

for all $x, y, z \in \Xi$, where $\psi \in \Psi$. If the range of g contains the range of f, and one of $f(\Xi)$ or $g(\Xi)$ is a complete subspace of Ξ, then f and g have a unique point of coincidence in Ξ. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in Ξ.

Proof. First, we assume that f and g satisfy the condition (3.1). Let $e_0 \in \Xi$ be an arbitrary point. Since the range of g contains the range of f, there exists $e_1 \in \Xi$ such that $ge_1 = fe_0$. By continuing the process as defined before, we can construct a sequence $\{ge_n\}$ such that $ge_{n+1} = fe_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $ge_n = ge_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $ge_n \neq ge_{n+1}$ for all $n \in \mathbb{N}$. Set $\zeta_n = S(ge_n, ge_n, ge_{n+1})$. Therefore, for each $n \in \mathbb{N}$, using (3.1), $(S1)$, $(S2)$ and Lemma 2.18, we obtain

$$\begin{aligned}
\zeta_n = S(ge_n, ge_n, ge_{n+1}) &= S(fe_{n-1}, fe_{n-1}, fe_n) \\
\leq &\psi\Big(\max\Big\{S(fe_{n-1}, fe_{n-1}, ge_n), S(ge_{n-1}, ge_{n-1}, fe_{n-1}), \\
&\quad S(ge_{n-1}, ge_{n-1}, fe_{n-1}), \\
&\quad \frac{1}{2}[S(ge_{n-1}, ge_{n-1}, fe_n) + S(ge_n, ge_n, fe_{n-1})]\Big\}\Big) \\
= &\psi\Big(\max\Big\{S(ge_n, ge_n, ge_n), S(ge_{n-1}, ge_{n-1}, ge_n), \\
&\quad S(ge_{n-1}, ge_{n-1}, ge_n), \\
&\quad \frac{1}{2}[S(ge_{n-1}, ge_{n-1}, ge_{n+1}) + S(ge_n, ge_n, ge_n)]\Big\}\Big) \\
\leq &\psi\Big(\max\Big\{S(ge_{n-1}, ge_{n-1}, ge_n), S(ge_{n-1}, ge_{n-1}, ge_n), \\
&\quad \frac{1}{2}[2S(ge_{n-1}, ge_{n-1}, ge_n) + S(ge_n, ge_n, ge_{n+1})]\Big\}\Big) \\
\leq &\psi\Big(\max\Big\{S(ge_{n-1}, ge_{n-1}, ge_n), S(ge_n, ge_n, ge_{n+1})\Big\}\Big) \\
= &\psi\Big(\max\{\zeta_{n-1}, \zeta_n\}\Big). \qquad (3.2)
\end{aligned}$$

If $\max\{\zeta_{n-1}, \zeta_n\} = \zeta_n$, then from equation (3.2) and using the property of ψ, we have

$$\zeta_n \leq \psi(\zeta_n)) < \zeta_n,$$

which leads to a contradiction, since $\zeta_n = S(ge_n, ge_n, ge_{n+1}) > 0$. Thus, we conclude that $\max\{\zeta_{n-1}, \zeta_n\} = \zeta_{n-1}$ for all n. This implies that

$$\zeta_n \leq \psi(\zeta_{n-1}). \qquad (3.3)$$

Thus, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\zeta_n &= S(ge_n, ge_n, ge_{n+1}) = S(fe_{n-1}, fe_{n-1}, fe_n) \\
&\leq \psi\Big(S(ge_{n-1}, ge_{n-1}, ge_n)\Big) \\
&\leq \psi^2\Big(S(ge_{n-2}, ge_{n-2}, ge_{n-1})\Big) \\
&\vdots \\
&\leq \psi^n\Big(S(ge_0, ge_0, ge_1)\Big) = \psi^n(\zeta_0). \qquad (3.4)
\end{aligned}$$

So, we have $\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} S(ge_n, ge_n, ge_{n+1}) = 0$. If $\{ge_n\} = \{fe_{n-1}\}$ is not Cauchy sequence in S-metric space (Ξ, S), then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$ by Lemma 2.23:

$$S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \quad \text{and} \quad S(ge_{m_k}, ge_{m_k}, ge_{n_k}).$$

Putting $x = y = e_{m_k}$, $z = e_{n_k}$ in (3.1), we obtain

$$
\begin{aligned}
S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) &= S(fe_{m_k}, fe_{m_k}, fe_{n_k}) \\
\leq \psi\Big(\max \Big\{ & S(fe_{m_k}, fe_{m_k}, ge_{n_k}), S(ge_{m_k}, ge_{m_k}, fe_{m_k}), \\
& S(ge_{m_k}, ge_{m_k}, fe_{m_k}), \\
& \frac{1}{2}[S(ge_{m_k}, ge_{m_k}, fe_{n_k}) + S(ge_{n_k}, ge_{n_k}, fe_{m_k})] \Big\} \Big) \\
\leq \psi\Big(\max \Big\{ & S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k}), S(ge_{m_k}, ge_{m_k}, ge_{m_k+1}), \\
& S(ge_{m_k}, ge_{m_k}, ge_{m_k+1}), \\
& \frac{1}{2}[S(ge_{m_k}, ge_{m_k}, ge_{n_k+1}) + S(ge_{n_k}, ge_{n_k}, ge_{m_k+1})] \Big\} \Big) \\
\leq \psi\Big(\max \Big\{ & S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k}), S(ge_{m_k}, ge_{m_k}, ge_{n_k}), \\
& S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \Big\} \Big).
\end{aligned} \tag{3.5}
$$

If

$$
\begin{aligned}
\max \Big\{ S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k}), S(ge_{m_k}, ge_{m_k}, ge_{n_k}), S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \Big\} \\
= S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k})
\end{aligned}
$$

and since $S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k}) > 0$, using the property of ψ, we have

$$S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \leq \psi\Big(S(ge_{m_k}, ge_{m_k}, ge_{m_k+1})\Big) < S(ge_{m_k}, ge_{m_k}, ge_{m_k+1}).$$

Letting $k \to \infty$, we obtain

$$\varepsilon \leq \lim_{k\to\infty} \psi\Big(S(ge_{m_k}, ge_{m_k}, ge_{m_k+1})\Big) \leq 0,$$

which is a contradiction. Again if

$$
\begin{aligned}
\max \Big\{ S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k}), S(ge_{m_k}, ge_{m_k}, ge_{n_k}), S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \Big\} \\
= S(ge_{m_k}, ge_{m_k}, ge_{n_k})
\end{aligned}
$$

and since $S(ge_{m_k}, ge_{m_k}, ge_{n_k}) > 0$, using the property of ψ, we have

$$S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \leq \psi\Big(S(ge_{m_k}, ge_{m_k}, ge_{n_k})\Big) < S(ge_{m_k}, ge_{m_k}, ge_{n_k}).$$

Letting $k \to \infty$, we obtain

$$\varepsilon \leq \lim_{k\to\infty} \psi\Big(S(ge_{m_k}, ge_{m_k}, ge_{n_k})\Big) < \varepsilon,$$

a contradiction. Similarly, if

$$
\begin{aligned}
\max \Big\{ S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k}), S(ge_{m_k}, ge_{m_k}, ge_{n_k}), S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \Big\} \\
= S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1})
\end{aligned}
$$

and since $S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) > 0$, again we get a contradiction.

Thus, it follows that $\{ge_n\} = \{fe_{n-1}\}$ is a Cauchy sequence in Ξ. By the completeness of $g(\Xi)$ (or $f(\Xi)$), we obtain that $\{ge_n\}$ is convergent to some point, say $u \in g(\Xi)$. So there exists $v \in \Xi$ such that $gv = u$. We will show that $gv = fv$. Suppose that $gv \neq fv$. By equation (3.1), we have

$$S(ge_n, ge_n, fv) = S(fe_{n-1}, fe_{n-1}, fv)$$

$$\begin{aligned}
\leq & \psi\Big(\max \Big\{ S(fe_{n-1}, fe_{n-1}, gv), S(ge_{n-1}, ge_{n-1}, fe_{n-1}), \\
& S(ge_{n-1}, ge_{n-1}, fe_{n-1}), \\
& \frac{1}{2}[S(ge_{n-1}, ge_{n-1}, fv) + S(gv, gv, fe_{n-1})] \Big\} \Big) \\
= & \psi\Big(\max \Big\{ S(ge_n, ge_n, gv), S(ge_{n-1}, ge_{n-1}, ge_n), \\
& S(ge_{n-1}, ge_{n-1}, ge_n), \\
& \frac{1}{2}[S(ge_{n-1}, ge_{n-1}, fv) + S(gv, gv, ge_n)] \Big\} \Big) \\
\leq & \psi\Big(\max \Big\{ S(ge_n, ge_n, gv), S(ge_{n-1}, ge_{n-1}, ge_n), \\
& S(ge_n, ge_n, fv) \Big\} \Big). \qquad (3.6)
\end{aligned}$$

The following cases arise:

Case 1. If $\max\Big\{S(ge_n, ge_n, gv), S(ge_{n-1}, ge_{n-1}, ge_n), S(ge_n, ge_n, fv)\Big\}$ $= S(ge_{n-1}, ge_{n-1}, ge_n)$, using the property of ψ, we obtain that

$$S(ge_n, ge_n, fv) \leq \psi\Big(S(ge_{n-1}, ge_{n-1}, ge_n)\Big) < S(ge_{n-1}, ge_{n-1}, ge_n).$$

By taking the limit as $n \to \infty$, we have $S(gv, gv, fv) = 0$ and so $gv = fv$.

Case 2. If $\max\Big\{S(ge_n, ge_n, gv), S(ge_{n-1}, ge_{n-1}, ge_n), S(ge_n, ge_n, fv)\Big\}$ $= S(ge_n, ge_n, gv)$, we obtain that

$$S(ge_n, ge_n, fv) \leq \psi\Big(S(ge_n, ge_n, gv)\Big).$$

By taking the limit as $n \to \infty$, using the property of ψ and $(S1)$, we have

$$S(gv, gv, fv) \leq \psi\Big(S(gv, gv, gv)\Big) = \psi(0) = 0,$$

that is $S(gv, gv, fv) = 0$ and so $gv = fv$.

Case 3. If $\max\Big\{S(ge_n, ge_n, gv), S(ge_{n-1}, ge_{n-1}, ge_n), S(ge_n, ge_n, fv)\Big\}$ $= S(ge_n, ge_n, fv)$, we obtain that

$$S(ge_n, ge_n, fv) \leq \psi\Big(S(ge_n, ge_n, fv)\Big).$$

By taking the limit as $n \to \infty$ and using the property of ψ, we have

$$S(gv, gv, fv) \leq \psi\Big(S(gv, gv, fv)\Big) < S(gv, gv, fv),$$

which leads to a contradiction. Therefore, $S(gv, gv, fv) = 0$ and so $gv = fv$.

Now, we show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in \Xi$. By (3.1) and using the property of ψ, it follows that

$$\begin{aligned} S(gv, gv, gq) &= S(fv, fv, fq) \\ &\leq \psi\Big(\max\Big\{S(fv, fv, gq), S(gv, gv, fv), S(gv, gv, fv), \\ &\qquad \frac{1}{2}[S(gv, gv, fq) + S(gq, gq, fv)]\Big\}\Big) \\ &\leq \psi\Big(S(gv, gv, gq)\Big) < S(gv, gv, gq), \end{aligned}$$

a contradiction. Therefore, $gv = gq$. This implies that f and g have a unique point of coincidence. By Proposition 2.25, we can conclude that f and g have a unique common fixed point. □

Corollary 3.2. *Let (Ξ, S) be an S-metric space and let $f, g\colon \Xi \to \Xi$ be two mappings satisfying the inequality:*

$$\begin{aligned} S(fx, fy, fz) &\leq k \max\Big\{S(fx, fx, gz), S(gx, gx, fx), S(gy, gy, fy), \\ &\qquad \frac{1}{2}[S(gx, gx, fz) + S(gz, gz, fx)]\Big\}, \end{aligned}$$

for all $x, y, z \in \Xi$, where $0 \leq k < 1$ is a constant. If the range of g contains the range of f, and one of $f(\Xi)$ or $g(\Xi)$ is complete subspace of Ξ, then f and g have a unique point of coincidence in Ξ. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in Ξ.

Proof. Putting $\psi(t) = k\,t$ for all $t > 0$, where $0 \leq k < 1$ in inequality (3.1), the result follows. □

Example 3.3. Let $\Xi = [0, 2]$ and $S(x, y, z) = \max\{|x-y|, |y-z|, |z-x|\}$ for all $x, y, z \in \Xi$ and $\psi \in \Psi$. Define $f, g\colon \Xi \to \Xi$ by $f(x) = 1$ and $g(x) = 2 - x$ for all $x \in \Xi$. We obtain that f and g satisfy condition (3.1) in Theorem 3.1. Indeed, we have

$$S(fx, fy, fz) = 0,$$

and

$$\begin{aligned} &\psi\Big(\max\Big\{S(fx, fx, gz), S(gx, gx, fx), S(gy, gy, fy), \\ &\qquad \frac{1}{2}[S(gx, gx, fz) + S(gz, gz, fx)]\Big\}\Big) \\ &= \psi\Big(\max\Big\{|1-z|, |1-x|, |1-y|, \frac{1}{2}[|1-x| + |1-z|]\Big\}\Big). \end{aligned}$$

That is,

$$\begin{aligned} S(fx, fy, fz) &\leq \psi\Big(\max\Big\{S(fx, fx, gz), S(gx, gx, fx), S(gy, gy, fy), \\ &\qquad \frac{1}{2}[S(gx, gx, fz) + S(gz, gz, fx)]\Big\}\Big). \end{aligned}$$

It is obvious that the range of g contains the range of f and $g(\Xi)$ is a complete subspace of (Ξ, S). Also, f and g are weakly compatible maps. Thus all the assumptions in Theorem 3.1 are satisfied. Hence f and g have a unique common fixed point, say $x = 1$ in Ξ.

Theorem 3.4. *Let (Ξ, S) be an S-metric space and let $f, g\colon \Xi \to \Xi$ be two mappings satisfying the inequality:*

$$\begin{aligned} S(fx, fy, fz) \leq \max\Big\{ & \psi(S(fx, fx, gz)), \psi(S(gx, gx, fx)), \psi(S(gy, gy, fy)), \\ & \psi\Big(\frac{1}{2}[S(gx, gx, fz) + S(gz, gz, fx)]\Big), \\ & \psi\Big(\frac{S(gy, gy, fx)[1 + S(gy, gy, fy)]}{1 + S(fy, fy, gx)}\Big)\Big\}, \end{aligned} \tag{3.7}$$

for all $x, y, z \in \Xi$, where $\psi \in \Psi$. If the range of g contains the range of f, and one of $f(\Xi)$ or $g(\Xi)$ is complete subspace of Ξ, then f and g have a unique point of coincidence in Ξ. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in Ξ.

Proof. First, we suppose that f and g satisfy the condition (3.7). Let $e_0 \in \Xi$ be an arbitrary point. Since the range of g contains the range of f, there exists $e_1 \in \Xi$ such that $ge_1 = fe_0$. By continuing the process as defined before, we can construct a sequence $\{ge_n\}$ such that $ge_{n+1} = fe_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $ge_n = ge_{n+1}$, then f and g have a point of coincidence. Thus we assume that $ge_n \neq ge_{n+1}$ for all $n \in \mathbb{N}$. Set $\zeta_n = S(ge_n, ge_n, ge_{n+1})$. Therefore, for each $n \in \mathbb{N}$, using (3.7), $(S1)$, $(S2)$ and Lemma 2.18, we obtain

$$\begin{aligned} \zeta_n = S(ge_n, ge_n, ge_{n+1}) &= S(fe_{n-1}, fe_{n-1}, fe_n) \\ \leq \max\Big\{ & \psi(S(fe_{n-1}, fe_{n-1}, ge_n)), \psi(S(ge_{n-1}, ge_{n-1}, fe_{n-1})), \\ & \psi(S(ge_{n-1}, ge_{n-1}, fe_{n-1})), \\ & \psi\Big(\frac{1}{2}[S(ge_{n-1}, ge_{n-1}, fe_n) + S(ge_n, ge_n, fe_{n-1})]\Big), \\ & \psi\Big(\frac{S(ge_{n-1}, ge_{n-1}, fe_{n-1})[1 + S(ge_{n-1}, ge_{n-1}, fe_{n-1})]}{1 + S(fe_{n-1}, fe_{n-1}, ge_{n-1})}\Big)\Big\} \\ = \max\Big\{ & \psi(S(ge_n, ge_n, ge_n)), \psi(S(ge_{n-1}, ge_{n-1}, ge_n)), \\ & \psi(S(ge_{n-1}, ge_{n-1}, ge_n)), \\ & \psi\Big(\frac{1}{2}[S(ge_{n-1}, ge_{n-1}, ge_{n+1}) + S(ge_n, ge_n, ge_n)]\Big), \\ & \psi\Big(\frac{S(ge_{n-1}, ge_{n-1}, ge_n)[1 + S(ge_{n-1}, ge_{n-1}, ge_n)]}{1 + S(ge_n, ge_n, ge_{n-1})}\Big)\Big\} \\ \leq \max\Big\{ & \psi(S(ge_n, ge_n, ge_{n+1})), \psi(S(ge_{n-1}, ge_{n-1}, ge_n))\Big\} \\ = \max & \{\psi(\zeta_n), \psi(\zeta_{n-1})\}. \end{aligned} \tag{3.8}$$

If $\max\{\psi(\zeta_n), \psi(\zeta_{n-1})\} = \psi(\zeta_n)$, then from equation (3.8) and using the property of ψ, we have

$$\zeta_n \leq \psi(\zeta_n) < \zeta_n,$$

which leads to a contradiction, since $\zeta_n = S(ge_n, ge_n, ge_{n+1}) > 0$. Thus, we conclude that $\max\{\psi(\zeta_n), \psi(\zeta_{n-1})\} = \psi(\zeta_{n-1})$ for all n. This implies that

$$\zeta_n \leq \psi(\zeta_{n-1}). \tag{3.9}$$

Thus, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\zeta_n &= S(ge_n, ge_n, ge_{n+1}) = S(fe_{n-1}, fe_{n-1}, fe_n) \\
&\leq \psi\Big(S(ge_{n-1}, ge_{n-1}, ge_n)\Big) \\
&\leq \psi^2\Big(S(ge_{n-2}, ge_{n-2}, ge_{n-1})\Big) \\
&\;\;\vdots \\
&\leq \psi^n\Big(S(ge_0, ge_0, ge_1)\Big) = \psi^n(\zeta_0).
\end{aligned} \tag{3.10}$$

So, we have $\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} S(ge_n, ge_n, ge_{n+1}) = 0$. Now, we have to show that $\{ge_n\} = \{fe_{n-1}\}$ is a Cauchy sequence in S-metric space (Ξ, S). The rest of the proof is similar to that of Theorem 3.1, so we omit it. This completes the proof. □

Corollary 3.5. *Let (Ξ, S) be a complete S-metric space and let $f\colon \Xi \to \Xi$ be a mapping satisfies the following condition: there exist $a_1, a_2 \in [0, 1)$ with $a_1+a_2 < 1$ and some $\psi \in \Psi$ such that*

$$\begin{aligned}
S(fx, fy, fz) \;\leq\; & a_1\, \psi\Big(\frac{S(y, y, fx)[1 + S(y, y, fy)]}{[1 + S(x, x, fy)]}\Big) \\
& + a_2 \max\Big\{\psi(S(fx, fx, z)), \psi(S(x, x, fx)), \\
& \psi(S(y, y, fy)), \psi\Big(\frac{1}{2}[S(x, x, fz) + S(z, z, fx)]\Big)\Big\},
\end{aligned} \tag{3.11}$$

for all $x, y, z \in \Xi$. Then f has a unique fixed point in Ξ.

Corollary 3.6. *Let (Ξ, S) be a complete S-metric space and let $f\colon \Xi \to \Xi$ be a mapping satisfies the following condition: there exist $a_1, a_2 \in [0, 1)$ with $a_1+a_2 < 1$ and some $\psi \in \Psi$ such that*

$$\begin{aligned}
S(fx, fy, fz) \;\leq\; & a_1\, \psi\Big(\frac{S(y, y, fx)[1 + S(y, y, fy)]}{[1 + S(x, x, fy)]}\Big) \\
& + a_2\, \psi\Big(\max\Big\{S(fx, fx, z), S(x, x, fx), \\
& S(y, y, fy), \frac{1}{2}[S(x, x, fz) + S(z, z, fx)]\Big\}\Big),
\end{aligned} \tag{3.12}$$

for all $x, y, z \in \Xi$. Then f has a unique fixed point in Ξ.

Remark 3.7. It is clear that the conclusions of the Corollary 3.6 remain valid if in condition (3.12), the second term of the right-hand side is replaced by one of the following terms:

$$a_2\,\psi(S(fx,fx,z));\quad a_2\,\psi\Big(\frac{1}{2}[S(x,x,fz)+S(z,z,fx)]\Big);$$

$$a_2\,\max\Big\{\psi(S(x,x,fx)),\psi(S(y,y,fy))\Big\};$$

$$\text{or}\quad a_2\,\max\Big\{\psi(S(fx,fx,z)),\psi(S(x,x,fx)),\psi(S(y,y,fy))\Big\}.$$

Theorem 3.8. *Let (Ξ,S) be an S-metric space and let $f,g\colon\Xi\to\Xi$ be two mappings satisfying the inequality:*

$$\begin{aligned} S(fx,fy,fz) \;\leq\;& h_1\,\psi(S(gx,gx,fx))+h_2\,\psi(S(gy,gy,fy))\\ &+h_3\,\psi(S(gz,gz,fz))\\ &+h_4\,\psi\Big(\frac{S(gy,gy,fx)[1+S(gy,gy,fy)]}{[1+S(fy,fy,gx)]}\Big), \end{aligned}\tag{3.13}$$

for all $x,y,z\in\Xi$, where $\psi\in\Psi$ and h_1,h_2,h_3,h_4 are nonnegative constants with $h_1+h_2+h_3+h_4<1$. If the range of g contains the range of f, and one of $f(\Xi)$ or $g(\Xi)$ is complete subspace of Ξ, then f and g have a unique point of coincidence in Ξ. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in Ξ.

Proof. First, we assume that f and g satisfy the condition (3.13). Let $x_0\in X$ be an arbitrary point. Since the range of g contains the range of f, there exists $e_1\in\Xi$ such that $ge_1=fe_0$. By continuing the process as defined before, we can construct a sequence $\{ge_n\}$ such that $ge_{n+1}=fe_n$ for all $n\in\mathbb{N}$. If there is $n\in\mathbb{N}$ such that $ge_n=ge_{n+1}$, then f and g have a point of coincidence. Thus we assume that $ge_n\neq ge_{n+1}$ for all $n\in\mathbb{N}$. Set $\zeta_n=S(ge_n,ge_n,ge_{n+1})$. Therefore, for each $n\in\mathbb{N}$, using (3.13), $(S1)$, $(S2)$, property of ψ and Lemma 2.18, we obtain

$$\zeta_n=S(ge_n,ge_n,ge_{n+1})=S(fe_{n-1},fe_{n-1},fe_n)$$

$$\begin{aligned} \leq& h_1\,\psi(S(ge_{n-1},ge_{n-1},fe_{n-1}))+h_2\,\psi(S(ge_{n-1},ge_{n-1},fe_{n-1}))\\ &+h_3\,\psi(S(ge_n,ge_n,fe_n))\\ &+h_4\,\psi\Big(\frac{S(ge_{n-1},ge_{n-1},fe_{n-1})[1+S(ge_{n-1},ge_{n-1},fe_{n-1})]}{[1+S(fe_{n-1},fe_{n-1},ge_{n-1})]}\Big)\\ =& h_1\,\psi(S(ge_{n-1},ge_{n-1},ge_n))+h_2\,\psi(S(ge_{n-1},ge_{n-1},ge_n))\\ &+h_3\,\psi(S(ge_n,ge_n,ge_{n+1}))\\ &+h_4\,\psi\Big(\frac{S(ge_{n-1},ge_{n-1},ge_n)[1+S(ge_{n-1},ge_{n-1},ge_n)]}{[1+S(ge_n,ge_n,ge_{n-1})]}\Big)\\ =&(h_1+h_2+h_4)\,\psi(S(ge_{n-1},ge_{n-1},ge_n))+h_3\,\psi(S(ge_n,ge_n,ge_{n+1}))\\ \leq&(h_1+h_2+h_4)\,\psi(S(ge_{n-1},ge_{n-1},ge_n))+h_3\,\psi(S(ge_n,ge_n,ge_{n+1})). \end{aligned}\tag{3.14}$$

This implies

$$\zeta_n\leq\Big(\frac{h_1+h_2+h_4}{1-h_3}\Big)\,\psi(\zeta_{n-1}).\tag{3.15}$$

Let $\mu = \left(\frac{h_1+h_2+h_4}{1-h_3}\right) < 1$, since $h_1+h_2+h_3+h_4 < 1$. Then from equation (3.15), we obtain

$$\zeta_n \leq \mu\, \psi(\zeta_{n-1}) < \psi(\zeta_{n-1}) < \cdots < \psi^n(\zeta_0).$$

This implies that $\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} S(ge_n, ge_n, ge_{n+1}) = 0$. If $\{ge_n\} = \{fe_{n-1}\}$ is not Cauchy sequence in S-metric space (Ξ, S), then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$, by Lemma 2.23:

$$S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) \quad \text{and} \quad S(ge_{m_k}, ge_{m_k}, ge_{n_k}).$$

Putting $x = y = e_{m_k}$, $z = e_{n_k}$ in (3.13), we obtain

$$S(ge_{m_k+1}, ge_{m_k+1}, ge_{n_k+1}) = S(fe_{m_k}, fe_{m_k}, fe_{n_k})$$

$$\begin{aligned}
\leq & h_1\, \psi(S(ge_{m_k}, ge_{m_k}, fe_{m_k})) + h_2\, \psi(S(ge_{m_k}, ge_{m_k}, fe_{m_k})) \\
& + h_3\, \psi(S(ge_{n_k}, ge_{n_k}, fe_{n_k})) \\
& + h_4\, \psi\Big(\frac{S(ge_{m_k}, ge_{m_k}, fe_{m_k})[1 + S(ge_{m_k}, ge_{m_k}, fe_{m_k})]}{[1 + S(fe_{m_k}, fe_{m_k}, ge_{m_k})]}\Big) \\
= & h_1\, \psi(S(ge_{m_k}, ge_{m_k}, ge_{m_k+1})) + h_2\, \psi(S(ge_{m_k}, ge_{m_k}, ge_{m_k+1})) \\
& + h_3\, \psi(S(ge_{n_k}, ge_{n_k}, ge_{n_k+1})) \\
& + h_4\, \psi\Big(\frac{S(ge_{m_k}, ge_{m_k}, ge_{m_k+1}))[1 + S(ge_{m_k}, ge_{m_k}, ge_{m_k+1})]}{[1 + S(ge_{m_k+1}, ge_{m_k+1}, ge_{m_k})]}\Big).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain $\varepsilon \leq 0$.

A contradiction. Thus, the sequence $\{ge_n\} = \{fe_{n-1}\}$ is a Cauchy sequence in Ξ. By the completeness of $g(\Xi)$ (or $f(\Xi)$), we obtain that $\{ge_n\}$ is convergent to some point, say $u \in g(\Xi)$. So there exists $v \in \Xi$ such that $gv = u$. We will show that $gv = fv$. Suppose that $gv \neq fv$. By equation (3.13), we have

$$S(ge_n, ge_n, fv) = S(fe_{n-1}, fe_{n-1}, fv)$$

$$\begin{aligned}
\leq & h_1\, \psi(S(ge_{n-1}, ge_{n-1}, fe_{n-1})) + h_2\, \psi(S(ge_{n-1}, ge_{n-1}, fe_{n-1})) \\
& + h_3\, \psi(S(gv, gv, fv)) \\
& + h_4\, \psi\Big(\frac{S(ge_{n-1}, ge_{n-1}, fe_{n-1})[1 + S(ge_{n-1}, ge_{n-1}, fe_{n-1})]}{[1 + S(fe_{n-1}, fe_{n-1}, ge_{n-1})]}\Big) \\
= & h_1\, \psi(S(ge_{n-1}, ge_{n-1}, ge_n)) + h_2\, \psi(S(ge_{n-1}, ge_{n-1}, ge_n)) \\
& + h_3\, \psi(S(gv, gv, fv)) \\
& + h_4\, \psi\Big(\frac{S(ge_{n-1}, ge_{n-1}, ge_n)[1 + S(ge_{n-1}, ge_{n-1}, ge_n)]}{[1 + S(ge_n, ge_n, ge_{n-1})]}\Big).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the property of ψ and $(S1)$, we obtain

$$\begin{aligned} S(gv, gv, fv) \leq & h_3\, \psi(S(gv, gv, fv)) < h_3\, S(gv, gv, fv) \\ & < (h_1 + h_2 + h_3 + h_4)\, S(gv, gv, fv) \\ & < S(gv, gv, fv), \end{aligned}$$

which is a contradiction, since $h_1 + h_2 + h_3 + h_4 < 1$. So, $gv = fv$. The rest of the proof follows from Theorem 3.1. This completes the proof. □

Remark 3.9. Completeness of the space Ξ is relaxed in Theorem 3.1, Theorem 3.4 and Theorem 3.8.

Example 3.10. Let $\Xi = [0, 1]$. We define the function $S\colon \Xi^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{if otherwise}, \end{cases}$$

for all $x, y, z \in \Xi$, then S is an S-metric on Ξ. Define two self-maps $f, g\colon \Xi \to \Xi$ on Ξ by $f(x) = \frac{x}{8}$ and $g(x) = \frac{x}{2}$ for all $x \in \Xi$. Let $x, y, z \in \Xi$ and we assume that $x \geq y \geq z$, then we have

$$\begin{aligned} S(fx, fy, fz) &= S\Big(\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big) = \max\Big\{\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big\} = \frac{x}{8}, \\ S(fx, fx, gz) &= S\Big(\frac{x}{8}, \frac{x}{8}, \frac{z}{2}\Big) = \max\Big\{\frac{x}{8}, \frac{x}{8}, \frac{z}{2}\Big\} = \frac{x}{8}, \\ S(gx, gx, fx) &= S\Big(\frac{x}{2}, \frac{x}{2}, \frac{x}{8}\Big) = \max\Big\{\frac{x}{2}, \frac{x}{2}, \frac{x}{8}\Big\} = \frac{x}{2}, \\ S(gy, gy, fy) &= S\Big(\frac{y}{2}, \frac{y}{2}, \frac{y}{8}\Big) = \max\Big\{\frac{y}{2}, \frac{y}{2}, \frac{y}{8}\Big\} = \frac{y}{2}, \\ S(gx, gx, fz) &= S\Big(\frac{x}{2}, \frac{x}{2}, \frac{z}{8}\Big) = \max\Big\{\frac{x}{2}, \frac{x}{2}, \frac{z}{8}\Big\} = \frac{x}{2}, \\ S(gz, gz, fx) &= S\Big(\frac{z}{2}, \frac{z}{2}, \frac{x}{8}\Big) = \max\Big\{\frac{z}{2}, \frac{z}{2}, \frac{x}{8}\Big\} = \frac{x}{8}. \end{aligned}$$

Now using the inequality of Corollary 3.2, we have

$$\begin{aligned} S(fx, fy, fz) &= \frac{x}{8} \\ &\leq k \max\Big\{\frac{x}{8}, \frac{x}{2}, \frac{y}{2}, \frac{1}{2}[\frac{x}{2} + \frac{x}{8}]\Big\} \\ &= k \max\Big\{\frac{x}{8}, \frac{x}{2}, \frac{y}{2}, \frac{5x}{16}\Big\} \\ &= k\,\frac{x}{2}, \end{aligned}$$

that is,

$$\frac{x}{8} \leq k\,\frac{x}{2},$$

or,

$$k \geq \frac{1}{4},$$

if we take $0 \leq k < 1$, then the assumption for constant in Corollary 3.2 is satisfied. Also, we have $f(\Xi) \subset g(\Xi)$, that is, $[0, 1/8] \subset [0, 1/2]$ and $f(\Xi)$ (or $g(\Xi)$) is a complete subspace of Ξ (see, $[0, 1/8] \subset [0, 1]$ or $[0, 1/2] \subset [0, 1]$). Thus all the assumptions of Corollary 3.2 are verified. Hence by application of Corollary 3.2, f and g have a unique common fixed point, namely $0 \in \Xi$.

4. Conclusion

In this paper, we prove some unique common fixed point theorems in the setting of S-metric spaces by using weakly compatible condition of the mappings and give some corollaries of the main results. Moreover, we give illustrative examples in support of the results. Our results extend, generalize and improve several results from the existing literature.

Acknowledgement. The author would like to thanks the anonymous learned referee and the editor for their careful reading and useful suggestions on the manuscript.

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