

SLIDING WINDOW METHOD FOR MEASURABLE FUNCTIONS ON NEUTROSOPHIC NORMED SPACES

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ABSTRACT. We aim to introduce the nonnegative real-valued functions class and the research of Mursaleen and Mohiuddine. We introduce the sliding window idea approaches within the neutrosophic normed spaces range $[0, \infty)$ for real-valued measurable functions in order to accomplish aim. Additionally, using summability techniques, we derive a number of theorems and samples.

1. INTRODUCTION

The initial work on theorem among fuzzy sets were written from Zadeh (1965), and after that this discipline is familiar with ongoing research along with major advancements, serving as the foundation for numerous studies when it comes to the scientific and technological research, including human race trends the findings of Barros et al. (2000). Additionally, fuzzy real numbers get exposed by Felbin (1992) along within his fuzzy functional analysis as the fuzzy norm. Park (2004) first coined the terms metric space of Intuitionistic Fuzzy Metric Spaces (IFMS), followed by Saadati along with Park improved that earlier term IFMS to intuitionistic fuzzy norm spaces. The convergence of lacunary statistical led Fast (1951) along with Schoenberg (1978) proposed, was also laid out by Mursaleen and Mohiuddine (2009).

Neutrosophic set was a generic concept that first emerged in 1998, according to Smarandache (2005). The extent of uncertainty in the neutrosophic set allows it to be more general compared to both fuzzy and intuitionistic fuzzy sets. Baset et al (2020) looked into the assumption towards neutrosophic theory were examined by in a variety of sectors, including taking decisions, financial performance, and the assessment of production industries. These fields included an approach to handle the risk in the supply chain sustainably and equalizing shortages of supplies in building endeavors. Kirisci et al. (2020) suggested an alternation to IFMS in 2019 that yields theorems of fixed point to provide complete Neutrosophic Metric Space [NMS]. In NMS, fixed point outcomes were demonstrated in

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In this research, we explore sliding window techniques to determine in Neutrosophic Normed Spaces for measurable function, there is summability. The fact that this study expands by Mursaleen and Mohiuddine earlier research falls under a real-valued domain of nonnegative Lebesgue function in between $[0, \infty]$ and proves several fascinating theorems.

2. PRELIMINARIES

Definition 2.1. The 7-tuple $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ is referred to as a Neutrosophic Normed Space [NNS], if $*$ acts as $\mathbf{t}$ -norm which is continuous, $\diamond$ and $\odot$ act as $\mathbf{t}$ -co norms which are continuous along with Ξ is a vector space, $\dot{\zeta}, \ddot{\eta}, \ddot{i}$ are fuzzy sets on $\Xi \times (0, \infty)$.

- (cn1) $\dot{\zeta}(\tilde{\zeta}, \varrho) + \ddot{\eta}(\tilde{\zeta}, \varrho) + \ddot{i}(\tilde{\zeta}, \varrho) \leq 3$;
- (cn2) $0 \leq \dot{\zeta}(\tilde{\zeta}, \varrho) \leq 1; 0 \leq \ddot{\eta}(\tilde{\zeta}, \varrho) \leq 1$ and $0 \leq \ddot{i}(\tilde{\zeta}, \varrho) \leq 1$;
- (cn3) $\dot{\zeta}(\tilde{\zeta}, \varrho) > 0$;
- (cn4) $\dot{\zeta}(\tilde{\zeta}, \varrho) = 1 \Leftrightarrow \tilde{\zeta} = 0$ for all $\varrho = 0$;
- (cn5) $\dot{\zeta}(\gamma\tilde{\zeta}, \varrho) = \dot{\zeta}\left(\tilde{\zeta}, \frac{\varrho}{|\gamma|}\right)$, for all $\gamma \in \mathbb{R}$ and $\gamma \neq 0$;
- (cn6) $\dot{\zeta}(\tilde{\zeta} + z, \varrho + \bar{\alpha}) \geq \dot{\zeta}(\tilde{\zeta}, \varrho) * \dot{\zeta}(z, \bar{\alpha})$ for all $\tilde{\zeta}, z \in \Xi$ and $\varrho, \bar{\alpha} > 0$;
- (cn7) $\dot{\zeta}(\tilde{\zeta}, \cdot) : (0, \infty) \rightarrow [0, 1]$ and it is continuous,
- (cn8) $\lim_{\varrho \rightarrow \infty} \dot{\zeta}(\tilde{\zeta}, \varrho) = 1$ and $\lim_{\varrho \rightarrow 0} \dot{\zeta}(\tilde{\zeta}, \varrho) = 0$;
- (cn9) $\ddot{\eta}(\tilde{\zeta}, \varrho) < 1$;
- (cn10) $\ddot{\eta}(\tilde{\zeta}, \varrho) = 0 \Leftrightarrow \tilde{\zeta} = 0$ for all $\varrho > 0$;
- (cn11) $\ddot{\eta}(\gamma\tilde{\zeta}, \varrho) = \ddot{\eta}\left(\tilde{\zeta}, \frac{\varrho}{|\gamma|}\right)$, for all $\gamma \in \mathbb{R}$ and $\gamma \neq 0$;
- (cn12) $\ddot{\eta}(\tilde{\zeta} + z, \varrho + \bar{\alpha}) \leq \ddot{\eta}(\tilde{\zeta}, \varrho) \diamond \ddot{\eta}(z, \bar{\alpha})$ for all $\tilde{\zeta}, z \in \Xi$ and $\varrho, \bar{\alpha} > 0$;
- (cn13) $\ddot{\eta}(\tilde{\zeta}, \cdot) : (0, \infty) \rightarrow [0, 1]$ and it is continuous;
- (cn14) $\lim_{\varrho \rightarrow \infty} \ddot{\eta}(\tilde{\zeta}, \varrho) = 0$ and $\lim_{\varrho \rightarrow 0} \ddot{\eta}(\tilde{\zeta}, \varrho) = 1$;
- (cn15) $\ddot{i}(\tilde{\zeta}, \varrho) < 1$;
- (cn16) $\ddot{i}(\tilde{\zeta}, \varrho) = 0 \Leftrightarrow \tilde{\zeta} = 0$ for all $\varrho > 0$;
- (cn17) $\ddot{i}(\gamma\tilde{\zeta}, \varrho) = \ddot{i}\left(\tilde{\zeta}, \frac{\varrho}{|\gamma|}\right)$, for all $\gamma \in \mathbb{R}$ and $\gamma \neq 0$;
- (cn18) $\ddot{i}(\tilde{\zeta} + z, \varrho + \bar{\alpha}) \leq \ddot{i}(\tilde{\zeta}, \varrho) \odot \ddot{i}(z, \bar{\alpha})$, for all $\tilde{\zeta}, z \in \Xi$ and $\varrho, \bar{\alpha} > 0$;
- (cn19) $\ddot{i}(\tilde{\zeta}, \cdot) : (0, \infty) \rightarrow [0, 1]$ and it is continuous;
- (cn20) $\lim_{\varrho \rightarrow \infty} \ddot{i}(\tilde{\zeta}, \varrho) = 0$ and $\lim_{\varrho \rightarrow 0} \ddot{i}(\tilde{\zeta}, \varrho) = 1$;

Whenever this occurs, $(\dot{\zeta}, \ddot{\eta}, \ddot{i})$ is labeled as a NNS. Let $(\Xi, ||\cdot||)$ be a normed space as an illustration and let $\tilde{\varphi} * \psi = \tilde{\varphi}\psi$, $\tilde{\varphi} \diamond \psi = \min\{\tilde{\varphi} + \psi, 1\}$ and $\tilde{\varphi} \odot \psi = \min\{\tilde{\varphi} + \psi, 1\}$ for all $\tilde{\varphi}, \psi \in [0, 1]$. For all $\tilde{\zeta} \in \Xi$ and every $\varrho, \bar{\alpha} > 0$, consider

$$(\dot{\zeta})(\tilde{\zeta}, \varrho) := \frac{\varrho}{\varrho + ||\tilde{\zeta}||}, \ddot{\eta}(\tilde{\zeta}, \varrho) := \frac{||\tilde{\zeta}||}{\varrho + ||\tilde{\zeta}||} \text{ and } \ddot{i}(\tilde{\zeta}, \varrho) := \frac{||\tilde{\zeta}||}{\varrho}.$$

Then, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ which is a NNS.

3. $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})}$ CONVERGENCE IN NNS

This paragraph explores the idea of $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})}$ -Statistical Convergent (SC) functions in NNS is revealed. In the following article, $\mathfrak{R}$ indicate Lebesgue measure appears as well via a half-line $[0, \infty)$, especially $\mathfrak{m}$. Here, each of us method to employ the pair of Sliding Window (SW) $(\dot{\beta}, \dot{\delta})$ the fact Connor and Savas (2015) provided.

- (1) The real-valued measurable functions $\dot{\beta}$ and $\dot{\delta}$ both are nondecreasing, nonnegative functions specified on $\mathfrak{R}$,
- (2) $\dot{\beta}(\bar{\alpha}) < \dot{\delta}(\bar{\alpha})$ for each $\bar{\alpha} \in \mathbb{R}^+$, and $\dot{\delta}(\bar{\alpha}) \rightarrow \infty$ as $\bar{\alpha} \rightarrow \infty$.
- (3) $\liminf_{\bar{\alpha}} (\dot{\delta}(\bar{\alpha}) - \dot{\beta}(\bar{\alpha})) > 0$ and
- (4) $(0, \dot{\delta}(\bar{\alpha})) = \cup \{ \dot{\beta}(\ddot{\omega}), \dot{\delta}(\ddot{\omega}) : \ddot{\omega} \leq \bar{\alpha} \}$ for each $\bar{\alpha} > 0$.

We also consider $\mathfrak{J}_{\bar{\alpha}} = (\dot{\beta}(\bar{\alpha}), \dot{\delta}(\bar{\alpha}))$ and $\dot{\delta}(\bar{\alpha}) - \dot{\beta}(\bar{\alpha}) = \mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})$ where the Lebesgue measurable of the set $\mathfrak{E}$ is indicated by $\mathfrak{m}(\mathfrak{E})$.

Definition 3.1. Let $(\Xi, \dot{\zeta}, \dot{\eta}, \dot{i}, *, \diamond, \odot)$ be a NNS. The measurable function $\ell : \mathfrak{R} \rightarrow \mathbb{R}$ is therefore stated to be convergent to $\bar{\lambda} \in \Xi$ relate to the Neutrosophic Norm (NN) $(\dot{\zeta}, \dot{\eta}, \dot{i})$ given that every $\vartheta > 0$ and $\varrho > 0$, the fact that $\check{\xi}_0 \in \mathbb{N}$ such that $(\dot{\zeta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \vartheta$, $(\dot{\eta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > \vartheta$ and $(\dot{i})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > \vartheta$ for all $\check{\xi} \geq \check{\xi}_0$. Consequently, we represent $(\dot{\zeta}, \dot{\eta}, \dot{i}) - \lim \ell(\check{\xi}) = \bar{\lambda}$ or $\ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \dot{\eta}, \dot{i})} \bar{\lambda}$ as $\check{\xi} \rightarrow \infty$.

Definition 3.2. Let $(\Xi, \dot{\zeta}, \dot{\eta}, \dot{i}, *, \diamond, \odot)$ be a NNS. The measurable function $\ell : \mathfrak{R} \rightarrow \mathbb{R}$ is therefore stated to be SC to $\bar{\lambda} \in \Xi$ relate to the NN given that for each $\vartheta > 0$ and $\varrho > 0$.

$$\lim_{\mathfrak{w} \rightarrow \infty} \frac{1}{\mathfrak{w}} \mathfrak{m} \left(\begin{array}{l} \check{\xi} \leq \mathfrak{w} : (\dot{\zeta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \dot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \dot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right) = 0$$

or something equivalent

$$\lim_{\mathfrak{w} \rightarrow \infty} \frac{1}{\mathfrak{w}} \mathfrak{m} \left(\begin{array}{l} \check{\xi} \leq \mathfrak{w} : (\dot{\zeta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \vartheta \text{ or} \\ \dot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta \text{ and } \dot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta \end{array} \right) = 1.$$

Consequently, we represent $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$ or $\ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \dot{\eta}, \dot{i})} \bar{\lambda}(\mathfrak{S})$, where $\bar{\lambda}$ is consider being $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})} - \lim \ell(\check{\xi})$ additionally, we give classifications for every $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})}$ -SC functions relating towards NN $(\dot{\zeta}, \dot{\eta}, \dot{i})$ along $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})}$.

Definition 3.3. Let $(\dot{\beta}, \dot{\delta})$ represent the pair of SW, $(\Xi, \dot{\zeta}, \dot{\eta}, \dot{i}, *, \diamond, \odot)$ be a NNS. The measurable function $\ell : \mathfrak{R} \rightarrow \mathbb{R}$ is therefore stated to be $S_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \dot{\eta}, \dot{i})}$ -SC to the number $\bar{\lambda} \in \Xi$ relate to the NN $(\dot{\zeta}, \dot{\eta}, \dot{i})$ if for each $\vartheta > 0$ and $\varrho > 0$,

$$\lim_{\rho \rightarrow \infty} \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \mathfrak{m} \left(\begin{array}{l} \check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : (\dot{\zeta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \dot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \dot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right) = 0$$

or equivalently

$$\lim_{\rho \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\begin{array}{l} \check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : (\dot{\zeta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta \end{array} \right) = 1.$$

We represent $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$ or $\ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \bar{\lambda}(\mathfrak{S}_{(\dot{\beta}, \dot{\delta})})$, where $\bar{\lambda}$ is consider being $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi})$ additionally, we give classification for every lacunary $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -SC functions in relate to the NN $(\dot{\zeta}, \ddot{\eta}, \ddot{i})$ by $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$.

Definition 3.4. Let $(\dot{\beta}, \dot{\delta})$ be a pair of SW, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ be a NNS, and the measurable function $\ell : \mathfrak{K} \rightarrow \mathbb{R}$. Then, ℓ has $\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -summable towards $\bar{\lambda} \in \Xi$ relate to the NN $(\dot{\zeta}, \ddot{\eta}, \ddot{i})$ if for each $\vartheta > 0$ and $\varrho > 0$

$$\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\begin{array}{l} (\dot{\zeta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right) d\check{\xi} = 0.$$

We represent $\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$ or $\ell(\check{\xi}) \rightarrow \bar{\lambda} \left(\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \right)$, and additionally, a list regarding such functions will be provided by $\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$.

We point out that Definition (3.1) include a general of more idea than that which Mursaleen along with Mohiuddin (2009) emphasized. Regarding about lacunary $\bar{\theta} = (\rho_i)$, the sequence (η_j) is lacunary statistical summable to $\bar{o}$ relate to the NN $(\dot{\zeta}, \ddot{\eta}, \ddot{i})$, if every $\vartheta > 0$ and $\varrho > 0$.

$$\lim_i \frac{1}{\rho_i - \rho_{i-1}} \mathbf{m} \left(\begin{array}{l} \rho_i < j < \rho_{i-1} : \dot{\zeta}(\eta_j - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\eta_j - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \ddot{i}(\eta_j - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right) = 0.$$

It is clear from this that the observed statistics are based on a discontinuous interval $(\rho_{i-1}, \rho]$. However, under our concept of statistical $(\dot{\beta}, \dot{\delta})$ convergence does not need the interval $(\dot{\beta}(\bar{\alpha}), \dot{\delta}(\bar{\alpha})]$ to be disjoint. If we take into $\dot{\beta}(\bar{\alpha}) = 0$ and $\dot{\delta}(\bar{\alpha}) = \bar{\alpha}$ and in NNS, We express that, $\mathfrak{S}(\dot{\beta}, \dot{\delta})$ and $\mathfrak{N}(\dot{\beta}, \dot{\delta})$ are, respectively. In NNS, there is significant Cesaro summability with strong and SC for measurable functions that have not yet been studied.

Lemma 3.5. Let $(\dot{\beta}, \dot{\delta})$ be a pair of SW, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ be a NNS. Then, for each $\vartheta > 0$ and $\varrho > 0$, the phrases listed below are equivalent:

- (1) $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$.
- (2) $\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \right) = 0;$
- $\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : (\ddot{\eta})(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \right) = 0;$
- $\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \right) = 0.$

- (3) $\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}), \varrho \right) > 1 - \vartheta, \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta \text{ and } \ddot{\imath}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta = 1.$
- (4) $\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \vartheta) = \lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta) = \lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \ddot{\imath}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta) = 1.$
- (5) $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) = 1, \mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) = 0, \text{ and } \mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \ddot{\imath}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) = 0$

Theorem 3.6. *Let $(\check{\beta}, \check{\delta})$ be a pair of SW, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{\imath}, *, \diamond, \odot)$ be a NNS. If a function is $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})}$ -SC relate to the NN $(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})$, then unique specifies $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})}$ -limit.*

Proof. Assume that $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \ell(\check{\xi}) = \bar{\lambda}_1, \mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \ell(\check{\xi}) = \bar{\lambda}_2$ and $\bar{\lambda}_1 \neq \bar{\lambda}_2$. Given $\vartheta > 0$ and let us pick $\hat{\imath} > 0$ so the fact $(1 - \hat{\imath}) * (1 - \hat{\imath}) > 1 - \vartheta, \hat{\imath} \diamond \hat{\imath} < \vartheta$ and $\hat{\imath} \odot \hat{\imath} < \vartheta$. Then, for any $\varrho > 0$, Establishing the sets in a subsequent manner:

$$\begin{aligned} \mathfrak{K}_{\dot{\zeta}, 1}(\hat{\imath}, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \dot{\zeta} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \leq 1 - \hat{\imath} \right\}, \\ \mathfrak{K}_{\dot{\zeta}, 2}(\hat{\imath}, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \dot{\zeta} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) \leq 1 - \hat{\imath} \right\}, \\ \mathfrak{K}_{\ddot{\eta}, 1}(\hat{\imath}, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \ddot{\eta} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \geq \hat{\imath} \right\}, \\ \mathfrak{K}_{\ddot{\eta}, 2}(\hat{\imath}, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \ddot{\eta} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) \geq \hat{\imath} \right\} \text{ and} \\ \mathfrak{K}_{\ddot{\imath}, 1}(\hat{\imath}, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \ddot{\imath} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \geq \hat{\imath} \right\}, \\ \mathfrak{K}_{\ddot{\imath}, 2}(\hat{\imath}, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \ddot{\imath} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) \geq \hat{\imath} \right\}. \end{aligned}$$

Since $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \ell(\check{\xi}) = \bar{\lambda}_1$, for all $\varrho > 0$ we have to use Lemma (3.5).

$$\begin{aligned} &\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathbb{N} : \dot{\zeta} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \leq 1 - \vartheta \right) \\ &= \lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathbb{N} : \ddot{\eta} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \geq \vartheta \right) \\ &= \lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathbb{N} : \ddot{\imath} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \geq \vartheta \right) = 0. \end{aligned}$$

Also, using $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\imath})} - \lim \ell(\check{\xi}) = \bar{\lambda}_2$, for all $\varrho > 0$ we are granted

$$\begin{aligned} &\lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathbb{N} : \dot{\zeta} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) \leq 1 - \vartheta \right) \\ &= \lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathbb{N} : \ddot{\eta} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) \geq \vartheta \right) \end{aligned}$$

$$= \lim_{\bar{\alpha} \rightarrow \infty} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathbb{N} : \ddot{\iota} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\vartheta}{2} \right) \geq \vartheta \right) = 0.$$

We now let

$$\mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{\iota}}(\vartheta, \varrho) = (\mathfrak{K}_{\dot{\zeta}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\dot{\zeta}, 2}(\vartheta, \varrho)) \cap (\mathfrak{K}_{\ddot{\eta}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\ddot{\eta}, 2}(\vartheta, \varrho)) \cap (\mathfrak{K}_{\ddot{\iota}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\ddot{\iota}, 2}(\vartheta, \varrho)).$$

Then, we show that $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{\iota}}(\vartheta, \varrho) \right) = 0$.

It, implies $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathbb{N} \setminus \mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{\iota}}(\vartheta, \varrho) \right) = 1$.

If $\check{\xi} \in \mathbb{N} \setminus \mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{\iota}}(\vartheta, \varrho)$, after we get three terms;

$\check{\xi} \in \mathbb{N} \setminus (\mathfrak{K}_{\dot{\zeta}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\dot{\zeta}, 2}(\vartheta, \varrho))$, $\check{\xi} \in \mathbb{N} \setminus (\mathfrak{K}_{\ddot{\eta}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\ddot{\eta}, 2}(\vartheta, \varrho))$ and

$\check{\xi} \in \mathbb{N} \setminus (\mathfrak{K}_{\ddot{\iota}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\ddot{\iota}, 2}(\vartheta, \varrho))$.

We consider on the first part $\check{\xi} \in \mathbb{N} \setminus (\mathfrak{K}_{\dot{\zeta}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\dot{\zeta}, 2}(\vartheta, \varrho))$. Then, we obtain

$$\dot{\zeta}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) \geq \dot{\zeta} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) * \dot{\zeta} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) > (1 - \hat{\iota}) * (1 - \hat{\iota}).$$

Since $(1 - \hat{\iota}) * (1 - \hat{\iota}) > 1 - \vartheta$, $\dot{\zeta}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) > 1 - \vartheta$.

Since $\vartheta > 0$, we have $\dot{\zeta}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) = 1$ for all $\varrho > 0$, this yields $\bar{\lambda}_1 = \bar{\lambda}_2$.

On the second part, if $\check{\xi} \in \mathbb{N} \setminus (\mathfrak{K}_{\ddot{\eta}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\ddot{\eta}, 2}(\vartheta, \varrho))$, then

$$\ddot{\eta}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) \geq \ddot{\eta} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \diamond \ddot{\eta} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) > \hat{\iota} \diamond \hat{\iota}.$$

Using $\hat{\iota} \diamond \hat{\iota} < \vartheta$, we get $\ddot{\eta}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) < \vartheta$.

Therefore, $\ddot{\eta}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) = 0$ for all $\varrho > 0$, this implies $\bar{\lambda}_1 = \bar{\lambda}_2$.

On the other part, if $\check{\xi} \in \mathbb{N} \setminus (\mathfrak{K}_{\ddot{\iota}, 1}(\vartheta, \varrho) \cup \mathfrak{K}_{\ddot{\iota}, 2}(\vartheta, \varrho))$, then

$$\ddot{\iota}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) \geq \ddot{\iota} \left(\ell(\check{\xi}) - \bar{\lambda}_1, \frac{\varrho}{2} \right) \odot \ddot{\iota} \left(\ell(\check{\xi}) - \bar{\lambda}_2, \frac{\varrho}{2} \right) > \hat{\iota} \odot \hat{\iota}.$$

By using $\hat{\iota} \odot \hat{\iota} < \vartheta$, we obtain $\ddot{\iota}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) < \vartheta$.

Therefore, $\ddot{\iota}(\bar{\lambda}_1 - \bar{\lambda}_2, \varrho) = 0$ for all $\varrho > 0$, this implies $\bar{\lambda}_1 = \bar{\lambda}_2$.

Consequently, there are three distinct possibilities that a unique $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\iota})}$ -limits. $\square$

Theorem 3.7. Let $(\dot{\beta}, \dot{\delta})$ be a pair of SC, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{\iota}, *, \diamond, \odot)$ be a NNS. If $(\dot{\zeta}, \ddot{\eta}, \ddot{\iota}) - \lim \ell(\check{\xi}) = \bar{\lambda}$, then $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{\iota})} - \lim \ell(\check{\xi}) = \bar{\lambda}$. Even though, converse is not true.

Proof. Let $(\dot{\zeta}, \ddot{\eta}, \ddot{\iota}) - \lim \ell(\check{\xi}) = \bar{\lambda}$. Afterward, for all $\vartheta > 0$ and $\varrho > 0$, it has $\check{\xi}_0 \in \mathbb{N}$ thus, $\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \vartheta$, $\ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta$ and $\ddot{\iota}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \vartheta$ forever $\check{\xi} \geq \check{\xi}_0$. So, the set

$$\left\{ \check{\xi} \in \mathbb{N} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or } \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \ddot{\iota}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \right\}$$

has a set with finite number of entries. Since the consistence zero exists for each finite subset of $\mathbb{N}$, we can

$$\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\begin{array}{c} \check{\xi} \in \mathbb{N} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta, \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \ddot{\iota}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta, \end{array} \right)$$

that is $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\check{\zeta}, \check{\eta}, \check{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$. We set up the following illustration for conserve: $\square$

Example 3.8. Let $(\mathbb{R}, |\cdot|)$ use the usual norm to indicate the region of all real numbers, and provide $\tilde{\varphi} * \psi = \tilde{\varphi}\psi$, $\tilde{\varphi} \diamond \psi = \min\{\tilde{\varphi} + \psi, 1\}$ and $\tilde{\varphi} \odot \psi = \min\{\tilde{\varphi} + \psi, 1\}$ for all $\tilde{\varphi}, \psi \in [0, 1]$. For all $\ell : [0, \infty) \rightarrow \mathbb{R}$ and every $\varrho > 0$, consider

$$(\check{\zeta})(\ell, \varrho) := \frac{\varrho}{(\varrho + |\ell|)}, \check{\eta}(\ell, \varrho) := \frac{|\ell|}{(\varrho + |\ell|)} \text{ and } \check{i}(\ell, \varrho) := \frac{|\ell|}{\varrho}$$

Then, $(\mathbb{R}, \check{\zeta}, \check{\eta}, \check{i}, *, \diamond, \odot)$ be a NNS. Taking $\ell(\check{\xi})$ is a function that is specified by

$$\ell(\check{\xi}) = \begin{cases} \check{\xi}; & \text{for } \delta(\bar{\alpha}) - [\log \mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})] + 1 \leq \check{\xi} \leq \delta(\bar{\alpha}), \bar{\alpha} \in \mathbb{N} \\ 0; & \text{otherwise} \end{cases}$$

For each $\vartheta > 0$ and $\varrho > 0$, we have

$$\mathfrak{K}_{\bar{\alpha}}(\vartheta, \varrho) = \left\{ \check{\xi} \in \mathbb{N} : (\check{\zeta})(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or } \check{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and } \check{i}(\ell(\check{\xi}), \varrho) \geq \vartheta \right\}.$$

Then,

$$\begin{aligned} \mathfrak{K}_{\bar{\alpha}}(\vartheta, \varrho) &= \left\{ \check{\xi} \in \mathbb{N} : \frac{\varrho}{\varrho + |\ell(\check{\xi})|} \leq 1 - \vartheta \text{ or } \frac{|\ell(\check{\xi})|}{\varrho + |\ell(\check{\xi})|} \geq \vartheta \text{ and } \frac{|\ell(\check{\xi})|}{\varrho} \geq \vartheta \right\} \\ &= \left\{ \check{\xi} \in \mathbb{N} : |\ell(\check{\xi})| \geq \frac{\vartheta \varrho}{1 - \vartheta} > 0 \right\} = \left\{ \check{\xi} \in \mathbb{N} : \ell(\check{\xi}) = \check{\xi} \right\} \\ &= \left\{ \check{\xi} \in \mathbb{N} : \delta(\bar{\alpha}) - [\log \mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})] + 1 \leq \check{\xi} \leq \delta(\bar{\alpha}), \bar{\alpha} \in \mathbb{N} \right\}. \end{aligned}$$

Then, we obtain the form

$$\begin{aligned} \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \mathfrak{m}(\mathfrak{K}_{\bar{\alpha}})(\vartheta, \varrho) &\leq \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \mathfrak{m}(\check{\xi} \in \mathbb{N} : \delta(\bar{\alpha}) - [\log \mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})] + 1 \leq \check{\xi} \leq \delta(\bar{\alpha}), \bar{\alpha} \in \mathbb{N}) \\ &\leq \frac{\log \mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \end{aligned}$$

which implies that $\lim_{\bar{\alpha}} \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \mathfrak{m}(\mathfrak{K}_{\bar{\alpha}})(\vartheta, \varrho) = 0$.

Therefore, $\lim_{\bar{\alpha}} \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \mathfrak{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \mathfrak{K}_{\bar{\alpha}}(\vartheta, \varrho)) = \frac{\log \mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} = 0$ as $\bar{\alpha} \rightarrow \infty$ implies that $\ell(\check{\xi}) \xrightarrow{(\check{\zeta}, \check{\eta}, \check{i})} 0 \left(\mathfrak{S}_{(\check{\zeta}, \check{\eta}, \check{i})} \right)$.

However, $\ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} 0$, since

$$\begin{aligned}\dot{\zeta}(\ell(\check{\xi}), \varrho) &= \frac{\varrho}{\varrho + |\ell(\check{\xi})|} \\ &= \begin{cases} \frac{\varrho}{\varrho + \check{\xi}}, & \text{for } \delta(\bar{\alpha}) - [\log \mathbf{m}(\mathfrak{J}_{\bar{\alpha}})] + 1 \leq \check{\xi} \leq \delta(\bar{\alpha}), (\bar{\alpha} \in \mathbb{N}) \\ 1, & \text{otherwise;} \end{cases} \leq 1, \\ \ddot{\eta}(\ell(\check{\xi}), \varrho) &= \frac{|\ell(\check{\xi})|}{\varrho + |\ell(\check{\xi})|} \\ &= \begin{cases} \frac{\check{\xi}}{\varrho + \check{\xi}}, & \text{for } \delta(\bar{\alpha}) - [\log \mathbf{m}(\mathfrak{J}_{\bar{\alpha}})] + 1 \leq \check{\xi} \leq \delta(\bar{\alpha}), (\bar{\alpha} \in \mathbb{N}) \\ 0, & \text{otherwise;} \end{cases} \geq 0 \text{ and} \\ \ddot{i}(\ell(\check{\xi}), \varrho) &= \frac{|\ell(\check{\xi})|}{\varrho} \\ &= \begin{cases} \frac{\check{\xi}}{\varrho}, & \text{for } \delta(\bar{\alpha}) - [\log \mathbf{m}(\mathfrak{J}_{\bar{\alpha}})] + 1 \leq \check{\xi} \leq \delta(\bar{\alpha}), (\bar{\alpha} \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases} \geq 0.\end{aligned}$$

Theorem 3.9. *Let $(\dot{\beta}, \dot{\delta})$ be a pair of SC, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ be a NNS, then $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$ if and only if a subset exist $\mathfrak{K} = \{j_1 < j_2 < \dots\} \subseteq N$ such that*

$$\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{K}) = 1 \text{ and } (\dot{\zeta}, \ddot{\eta}, \ddot{i}) - \lim_{j \rightarrow \infty} \ell(\check{\xi}_j) = \bar{\lambda}.$$

Proof. Necessity. Presume that $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$

$$\begin{aligned}\mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho) &= \left\{ \begin{array}{l} \check{\xi} \in \mathbb{N} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \frac{1}{\hat{i}}, \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \frac{1}{\hat{i}} \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \frac{1}{\hat{i}} \end{array} \right\} \text{ and} \\ \mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho) &= \left\{ \begin{array}{l} \check{\xi} \in \mathbb{N} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \frac{1}{\hat{i}}, \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \frac{1}{\hat{i}} \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \frac{1}{\hat{i}} \end{array} \right\},\end{aligned}$$

Then $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho)) = 0$.

Since $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$. Also, $\mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho) \supset \mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i} + 1, \varrho)$

and $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho)) = 1$ for each $\varrho > 0$ and $\bar{\alpha} = 1, 2, \dots$.

Here by indicating, we have $\check{\xi} \in \mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho), \ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \bar{\lambda}$.

Assumable for certain $\check{\xi} \in \mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{i}, \varrho), \ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \bar{\lambda}$.

Therefore, there is $\emptyset > 0$ and ξ_0 a positive integer in terms of that

$$\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \emptyset \text{ or } \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \emptyset \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \emptyset \text{ for all } \check{\xi} \geq \xi_0.$$

Let $\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \emptyset$, or $\ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \emptyset$ and $\ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \emptyset$ for all $\check{\xi} < \xi_0$.

Then,

$$\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m} \left(\begin{array}{c} \check{\xi} \in \mathbb{N} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \emptyset, \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \emptyset \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \emptyset \end{array} \right) = 0.$$

Since $\emptyset > \frac{1}{5}$, we have $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{M}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\hat{\ell}, \varrho)) = 0$.

It contradicts (3.6). Hence, $\ell(\check{\xi}) \xrightarrow{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \bar{\lambda}$.

Sufficiency. Assume that a subset exist $\mathfrak{K} = \{j_1 < j_2 < \dots\} \subseteq N$ so that $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{k}) = 1$ and $(\dot{\zeta}, \ddot{\eta}, \ddot{i}) - \lim_{j \rightarrow \infty} \ell(\check{\xi}_j) = \bar{\lambda}$, and there exists $\mathfrak{N} \in \mathbb{N}$ such that for every $\emptyset > 0$ and $\varrho > 0$

$$\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) > 1 - \emptyset, \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \emptyset \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) < \emptyset.$$

Now,

$$\mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\emptyset, \varrho) := \left\{ \begin{array}{c} \check{\xi} \in \mathbb{N} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \emptyset \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \emptyset \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \emptyset \end{array} \right\} \subseteq \mathbb{N} - \{j_{N+1}, j_{N+2}, \dots\}.$$

Thus, $\lim_{\bar{\alpha}} \frac{1}{\mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \mathbf{m}(\check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \check{\xi} \in \mathfrak{K}_{\dot{\zeta}, \ddot{\eta}, \ddot{i}}(\emptyset, \varrho)) \leq 0$.

Therefore, $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \bar{\lambda}$. □

Theorem 3.10. *If ℓ is $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -SC and $\mathfrak{S}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -SC, then $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = \mathfrak{S}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi})$.*

Proof. Assume that without limiting uniformity, the measurable function ℓ becomes $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = 1$ and $\mathfrak{S}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell(\check{\xi}) = 0$ and consider the contradiction. Initially, we pick Q so that $\gamma > Q$ implies that for $\varrho > 0$

$$\mathbf{m} \left(\begin{array}{c} \check{\xi} \in \mathfrak{J}_{\gamma} : \dot{\zeta}(\ell(\check{\xi}) - 1, \varrho) \leq 1 - \frac{3}{4} \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - 1, \varrho) \geq \frac{3}{4} \text{ and } \ddot{i}(\ell(\check{\xi}) - 1, \varrho) \geq \frac{3}{4} \end{array} \right) < \frac{1}{6} \mathbf{m}(\mathfrak{J}_{\gamma})$$

and thus,

$$\mathbf{m} \left(\begin{array}{c} \check{\xi} \in \mathfrak{J}_{\gamma} : \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \frac{1}{4} \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \end{array} \right) \geq \mathbf{m}(\mathfrak{J}_{\gamma}) - \frac{1}{6} \mathbf{m}(\mathfrak{J}_{\gamma}) = \frac{5}{6} \mathbf{m}(\mathfrak{J}_{\gamma}).$$

Since $\delta(\bar{\alpha}) \rightarrow \infty$, then we could pick $\mathfrak{S} > Q$ in which $\bar{\alpha} > \mathfrak{S}$ gives $\frac{\delta(Q)}{\delta(\bar{\alpha})} < \frac{1}{12}$.

Let $0 < \vartheta < \frac{1}{12}$ to be determined, and then choose an secure space

$\{\mathfrak{J}_{\gamma_j} : Q < \gamma_j \leq \bar{\alpha} : j = 1, 2, \dots, n\}$, such that $\int_{j=i}^n \mathbf{m}(\mathfrak{J}_{\gamma_j}) \geq \frac{\delta(\bar{\alpha})}{3} - \delta(Q) - \vartheta$.

Now,

$$\begin{aligned}
& \mathbf{m} \left(\check{\xi} \leq \delta(\bar{\alpha}) : \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \frac{1}{4} \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \right) \\
& \geq \int_{\mathbf{j}}^n \mathbf{m} \left(\check{\xi} \in \mathfrak{J}_{\gamma_j} : \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \frac{1}{4} \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \right) \\
& \geq \frac{5}{6} \int_{\mathbf{j}}^n \mathbf{m}(\mathfrak{J}_{\gamma_j}) > \frac{5}{6} \left(\frac{\delta(\bar{\alpha})}{3} - \delta(Q) - \vartheta \right)
\end{aligned}$$

and since $\vartheta < \frac{1}{12}$

$$\frac{1}{\delta(\bar{\alpha})} \mathbf{m} \left(\check{\xi} \leq \delta(\bar{\alpha}) : \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \frac{1}{4} \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \frac{1}{4} \right) > \frac{1}{9}.$$

Thus, contrary to the ℓ hypothesis and ℓ be a limit of statistical does not equal to zero. $\square$

Theorem 3.11. Let $(\dot{\beta}, \dot{\delta})$ be a pair of SW, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ be a NNS, the measurable function ℓ and the real number $\bar{\lambda}$.

- (i) If ℓ is $\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -summable to $\bar{\lambda} \in \Xi$, then ℓ is $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -convergence to $\bar{\lambda} \in \Xi$.
- (ii) If ℓ is bounded, and $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -convergent to $\bar{\lambda} \in \Xi$, then ℓ is $\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -summable to $\bar{\lambda} \in \Xi$.

Proof. (i) First, we determine $\mathfrak{N}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell = \bar{\lambda}$ implies $\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} - \lim \ell = \bar{\lambda}$ in general. If any, then $\vartheta > 0$ and $\varrho > 0$, we have

$$\int \left(\begin{array}{c} \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \text{ and} \\ \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \end{array} \right) \geq \varepsilon \mathbf{m} \left(\begin{array}{c} \check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right)$$

and so

$$\frac{1}{\varepsilon \mathbf{m}(\mathfrak{J}_{\bar{\alpha}})} \int \left(\begin{array}{c} \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and} \\ \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right) d\check{\xi} \geq \varepsilon \mathbf{m} \left(\begin{array}{c} \check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and} \\ \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \end{array} \right)$$

- (ii) Presume that $\ell(\check{\xi}) \rightarrow \mathfrak{L} \left(\mathfrak{S}_{(\dot{\beta}, \dot{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \right)$ and $\ell(\check{\xi})$ is enclosed by $\tilde{\mathfrak{B}} > 0$ in such a way $\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \tilde{\mathfrak{B}}$ or $\ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq \tilde{\mathfrak{B}}$ and $\ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq \tilde{\mathfrak{B}}$ for every $\varrho > 0$.

Every $\vartheta > 0$, we get the following

$$\begin{aligned} & \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \int \left(\begin{array}{l} \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \end{array} \right) d\check{\xi} \geq \vartheta. \\ &= \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \int \times (\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \text{ or } \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho)) d\check{\xi} \\ &+ \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \int \times (\dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \text{ or } \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho)) d\check{\xi} \\ &\leq \frac{\tilde{\mathfrak{B}}}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \mathfrak{m} \left(\begin{array}{l} \check{\xi} \in \mathfrak{J}_{\bar{\alpha}} : \dot{\zeta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \leq 1 - \frac{\vartheta}{2} \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \frac{\vartheta}{2} \text{ and } \ddot{i}(\ell(\check{\xi}) - \bar{\lambda}, \varrho) \geq \frac{\vartheta}{2} \end{array} \right) + \frac{\vartheta}{2}. \end{aligned}$$

Consequently, $\mathfrak{S}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -convergent implies $\mathfrak{N}_{(\check{\beta}, \check{\delta})}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})}$ -summable. $\square$

In the following, we show an equivalent result:

Theorem 3.12. *Let $((\check{\beta}_1), (\check{\delta}_1))$ and $((\check{\beta}_2), (\check{\delta}_2))$ be two pairs of SW, $(\Xi, \dot{\zeta}, \ddot{\eta}, \ddot{i}, *, \diamond, \odot)$ be a NNS, and the measurable function ℓ is a statistically bounded. Assuming $\tilde{\mathfrak{B}} > 0$ as the value $|\check{\beta}_1(\check{\xi}) - \check{\beta}_2(\check{\xi})| + |\check{\delta}_1(\check{\xi}) - \check{\delta}_2(\check{\xi})| \leq \tilde{\mathfrak{B}}$ for all $\check{\xi}$ and $(\check{\delta}_1(\check{\xi}) - \check{\delta}_2(\check{\xi})) = \infty$, then $\ell \rightarrow \bar{\lambda} \left(\mathfrak{N}_{(\check{\beta}_1), (\check{\delta}_1)}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \right)$ if and only if $\ell \rightarrow \bar{\lambda} \left(\mathfrak{N}_{(\check{\beta}_2), (\check{\delta}_2)}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \right)$.*

Proof. From Theorem (3.11), it is essential to prove that $\ell \rightarrow \bar{\lambda} \left(\mathfrak{N}_{(\check{\beta}_1), (\check{\delta}_1)}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \right)$ if and only if $\ell \rightarrow \bar{\lambda} \left(\mathfrak{N}_{(\check{\beta}_2), (\check{\delta}_2)}^{(\dot{\zeta}, \ddot{\eta}, \ddot{i})} \right)$. Let us consider $\bar{\lambda} = 0$ and $\varrho > 0$, We have established that

$$\mathfrak{A} = \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \int_{\check{\beta}_2(\bar{\alpha})}^{\check{\delta}_2(\bar{\alpha})} \left(\begin{array}{l} \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and} \\ \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta \end{array} \right) d\check{\xi} - \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \int_{\check{\beta}_1(\bar{\alpha})}^{\check{\delta}_1(\bar{\alpha})} \left(\begin{array}{l} \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and} \\ \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta \end{array} \right) d\check{\xi}$$

and $\mathfrak{A} \rightarrow 0$ as $\bar{\alpha} \rightarrow \infty$. Note, that when $\bar{J}_{\bar{\alpha}} = (\check{\beta}_2(\bar{\alpha}), \check{\delta}_2(\bar{\alpha})]$ and $\mathfrak{m}(\bar{J}_{\bar{\alpha}}) = \check{\delta}_2(\bar{\alpha}) - \check{\beta}_2(\bar{\alpha})$

$$\begin{aligned} \mathfrak{A} &= \frac{1}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \left(\begin{array}{l} \int_{\check{\beta}_2(\bar{\alpha})}^{\check{\beta}_1(\bar{\alpha})} (\dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta) d\check{\xi} \\ - \int_{\check{\delta}_2(\bar{\alpha})}^{\check{\delta}_1(\bar{\alpha})} (\dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta) d\check{\xi} \end{array} \right) \\ &+ \left(\frac{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}}) - \mathfrak{m}(\bar{J}_{\bar{\alpha}})}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \right) \frac{1}{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}})} \int_{\check{\beta}_1(\bar{\alpha})}^{\check{\delta}_1(\bar{\alpha})} \left(\begin{array}{l} \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and} \\ \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta \end{array} \right) d\check{\xi} \end{aligned}$$

Now, we presume that $|\ell(\check{\xi})| \leq \mathfrak{M}$ for all $\check{\xi} \geq 0$. Then,

$$\begin{aligned} & \frac{1}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \left(\begin{array}{l} \int_{\check{\beta}_2(\bar{\alpha})}^{\check{\beta}_1(\bar{\alpha})} (\dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta) d\check{\xi} \\ - \int_{\check{\delta}_2(\bar{\alpha})}^{\check{\delta}_1(\bar{\alpha})} (\dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or } \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and } \ddot{i}(\ell(\check{\xi}), \varrho) \geq \vartheta) d\check{\xi} \end{array} \right) \\ &\leq \frac{\mathfrak{M}}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \left(|\check{\beta}_1(\check{\xi}) - \check{\beta}_2(\check{\xi})| + |\check{\delta}_1(\check{\xi}) - \check{\delta}_2(\check{\xi})| \leq \frac{\tilde{\mathfrak{B}} \mathfrak{M}}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \right) \end{aligned}$$

$$\text{And } \left(\frac{\mathfrak{m}(\mathfrak{J}_{\bar{\alpha}}) - \mathfrak{m}(\bar{J}_{\bar{\alpha}})}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \right) \frac{1}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})} \int_{\delta_1(\bar{\alpha})}^{\beta_1(\bar{\alpha})} \left(\begin{array}{l} \dot{\zeta}(\ell(\check{\xi}), \varrho) \leq 1 - \vartheta \text{ or} \\ \ddot{\eta}(\ell(\check{\xi}), \varrho) \geq \vartheta \text{ and} \\ \ddot{\iota}(\ell(\check{\xi}), \varrho) \geq \vartheta \end{array} \right) d\check{\xi} \leq \frac{\check{\mathfrak{M}}}{\mathfrak{m}(\bar{J}_{\bar{\alpha}})}$$

and thus, since $\lim_{\bar{\alpha}} \mathfrak{m}(\bar{J}_{\bar{\alpha}}) = \infty$, both terms tends to 0 as $\bar{\alpha} \rightarrow \infty$. Thus, it follows

$$\ell \rightarrow 0 \left(\mathfrak{N}_{(\beta_1), (\delta_1)}^{(\zeta, \ddot{\eta}, \ddot{\iota})} \right) \text{ if and only if } \ell \rightarrow 0 \left(\mathfrak{N}_{(\beta_2), (\delta_2)}^{(\zeta, \ddot{\eta}, \ddot{\iota})} \right). \quad \square$$

4. CONCLUSION

Many Researchers searched into certain NNS summability techniques. Considering the sequence within IFNS and we have NNS, Mursaleen and Mohiuddine have newly discovered convergent of Lacunary Statistically. Fortunately, by considering on more basic concepts that includes measurable functions.

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