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NEIGHBORS DEGREE SUM EXPONENT POLYNOMIAL AND NEIGHBORS DEGREE SUM EXPONENT ENERGY OF GRAPHS

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Abstract. Recently exponent matrix have come out to be working as dynamic tool to study networks by measuring the centrality node. In this work, we define a Neighbors degree sum exponent matrix $NE(G)$ of a graph G whose $(i,j)-th$ entry is $\delta_i^{\delta_j}$ for $i \neq j$, it is zero otherwise, where δ_i is the Neighbors degree sum of a vertex v_i in G. Inspired from the applications of Neighbor degree sum in redefining various degree based topological indices, we introduce characteristic polynomial of $NE(G)$ to be known as Neighbors degree sum exponent polynomial and the sum of absolute value of eigenvalue of $NE(G)$ matrix is called as Neighbors degree sum energy. In this paper, we obtain the Neighbors degree sum exponent polynomial and Neighbors degree sum exponent energy of some graphs.

1. Introduction

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2 \dots v_n\}$ and $E(G) = \{e_1, e_2 \dots e_n\}$ are the *vertex set* and *edge set* of a graph G respectively. If v_1 and v_2 are adjacent vertices of G, then the edge connecting them will be denoted by v_1v_2. The *degree* of a vertex u in G is the number of edges incident to it and is denoted by $d(u)$. If the degree of a each vertex in a graph G is equal to r then it is called as *r-regualr graph*. *Neighborhood* of a vertex u is defined as $N_G(u) = \{v \in V(G) : v \text{ is adjacent to } u \text{ in } G\}$. For graph theoretical notations we follow the book[1].
The *Neighbors degree sum* of a vertex u [4], denoted by $\delta(u)$ and is defined as the sum of degree of adjecent vertices u, that is,

$$\delta(u) = \sum_{v \in N_G(u)} d(v)$$

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Recently, some novel neighbor degree sum polynomial and their energies have been studied in [5]. Degree exponent polynomial and degree exponent energy of graphs have been studied in [6]. In this paper we introduce a new matrix called *Neighbors degree sum exponent matrix* defined as $NE(G) = [ne_{ij}]$, in which

$$ne_{ij} = \begin{cases} \delta_i^{\delta_j}, & if \quad i \neq j \\ 0, & if \quad i = j \end{cases}$$

The characteristic polynomial $P_{NE(G)}(\mu) = det\,(\mu I - NE(G))$ of a matrix $NE(G)$ is called *Neighbors degree sum exponent polynomial* of G, where I is an identity matrix. The roots of the equation $P_{NE(G)}(\mu) = 0$, denoted by $\mu_1, \mu_2, \ldots \mu_n$, are called *Neighbors degree sum exponent eigenvalues* of G and their collection is called *Neighbors degree sum exponent spectrum* of G. Thus the sum of the absolute values of the Neighbors degree sum exponent eigenvalues of G is called *Neighbors degree sum exponent energy* of G and is denoted by $NEE(G)$. Thus

$$NEE(G) = \sum_{i=1}^{n} |\mu_i|. \tag{1.1}$$

It is noted that, the Eq. (1.1) is in full analogy with the *ordinary graph energy*, defined as the sum of absolute values of the eigenvalues of the adjacency matrix of G [2]. For details on the energy of graph, it is suggested to refer the book [3] and the references cited there in.

In this paper we obtain the Neighbors degree sum exponent polynomial of some graphs and thereby obtain Neighbors degree sum exponent energy. Following lemmas are required for the further results.

Let I be the identity matrix and J be the matrix whose all entries are equal to one.

Lemma 1.1. [1] *The eigenvalues of $n \times n$ matrix $aI + bJ$ are a with multiplicity $n-1$ and $a + nb$ with multiplicity one.*

Lemma 1.2. [1] *If a,b,c and d are real numbers then the determinant of order $n_1 + n_2$ of the form*

$$\begin{vmatrix} (\mu + a)I_{n_1} - aJn_1 & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\mu + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be expressed as, $(\mu+a)^{n_1-1}(\mu+b)^{n_2-1}\{[\mu - (n_1-1)a][\mu - (n_2-1)b] - n_1 n_2 cd\}$

Lemma 1.3. [7] *Let G be a graph with n vertices and m edges. Then,*

$$\delta_{\overline{G}}(u) = \delta_G(u) - (n-2)d_G(u) + (n-1)^2 - 2m$$

2. Neighbor degree sum exponent polynomial of some graphs

It is simple to see that if G is an r-regular graph, then its Neighbor degree sum exponent matrix can be expressed as $NE(G) = -r^{2r^2}I + r^{2r^2}J$. Therefore

by Lemma 1.1, for an r-regular graph G of order n,

$$P_{NE(G)}(\mu) = \left(\mu + r^{2r^2}\right)^{n-1} \left(\mu - r^{2r^2}(n-1)\right) \tag{2.1}$$

The complete graph K_n is a $(n-1)$-regular graph and the cycle C_n is a 2-regular graph. Therefore, from Eqn. (2.1),

$$\begin{aligned} P_{NE(K_n)}(\mu) &= \left[\mu + (n-1)^{2n^2-4n+2}\right]^{n-1} \left[\mu - (n-1)^{2n^2-4n+3}\right] \\ P_{NE(C_n)}(\mu) &= [\mu + 256]^{n-1} [\mu - 256(n-1)] \end{aligned}$$

Let $V(G)$ be the vertex set of G. The *complement* of a graph G is a graph $\overline{G}$ with vertex set $V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if these are not adjacent in G. If G is an r-regular graph with n vertices and m edges, neighbor degree sum of every vertex is r^2 then by Lemma 1.3, $\overline{G}$ is an $(n-r-1)$-regular graph on the same number of vertices and edges with neighbor degree sum of every vertex $r^2 - (n-2)r + (n-1)^2 - 2m$. Therefore from Eqn. (2.1), if G is an r-regular graph on n vertices, then the Neighbor degree sum exponent polynomial of its complement graph is

$$\begin{aligned} P_{NE(\overline{G})}(\mu) &= \left[\mu + \left(r^2 - (n-2)r + (n-1)^2 - 2m\right)^{2\left(r^2-(n-2)r+(n-1)^2-2m\right)^2}\right]^{n-1} \\ &\quad \left[\mu - \left(r^2 - (n-2)r + (n-1)^2 - 2m\right)^{2\left(r^2-(n-2)r+(n-1)^2-2m\right)^2} (n-1)\right] \end{aligned}$$

A graph G is said to be *bipartite* if its vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that each edge of G has one end vertex in V_1 and other end vertex in V_2. A *complete bipartite graph* is bipartite graph with partite sets V_1 and V_2 in which every vertex of V_1 is adjacent to all vertices of V_2. The complete bipartite graph is denoted by K_{n_1,n_2}, where $|V_1 = n_1|$ and $|V_2 = n_2|$.

Theorem 2.1. *The Neighbor degree sum exponent polynomial of the complete bipartite graph K_{n_1,n_2} is*

$$(\mu + (n_1n_2)^{n_1n_2})^{n_1+n_2-2} \left\{ \left[(\mu + (n_1-1)(n_1n_2)^{n_1n_2}) (\mu - (n_2-1)(n_1n_2)^{n_1n_2}) - (n_1n_2)^{2n_1n_2+1}\right] \right\}$$

Proof. The Neighbor degree sum exponent matrix of the complete bipartite graph K_{n_1,n_2} is

$$\begin{bmatrix} (n_1n_2)^{(n_1n_2)} J_{n_1} - (n_1n_2)^{(n_1n_2)} I_{n_1} & (n_1n_2)^{(n_1n_2)} J_{n_1\times n_2} \\ (n_1n_2)^{(n_1n_2)} J_{n_2\times n_1} & (n_1n_2)^{(n_1n_2)} J_{n_2} - (n_1n_2)^{(n_1n_2)} I_{n_2} \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{NE(K_{n_1,n_2})}(\mu) &= |\mu I - NE(K_{n_1,n_2})| \\ &= \begin{vmatrix} \left[\mu + (n_1n_2)^{(n_1n_2)}\right] I_{n_1} - (n_1n_2)^{(n_1n_2)} J_{n_1} & -(n_1n_2)^{(n_1n_2)} J_{n_1\times n_2} \\ -(n_1n_2)^{(n_1n_2)} J_{n_2\times n_1} & \left[\mu + (n_1n_2)^{(n_1n_2)}\right] I_{n_2} - (n_1n_2)^{(n_1n_2)} J_{n_2} \end{vmatrix} \end{aligned} \tag{2.2}$$

By Lemma 1.2 to Eqn. (2.2), we get the result.

Corollary 2.2. *Let $S_n = K_{1,n-1}$ be the star graph n vertices. Then the Neighbor degree sum exponent polynomial of S_n is*

$$P_{NE(S_n)}(\mu) = \left[\mu + (n-1)^{n-1}\right]^{n-2} \left[\mu^2 - (n-2)(n-1)^{n-1}\mu - (n-1)^{2n-1}\right]$$

Theorem 2.3. *Let P_n be a path with $n > 4$ vertices. Then the Neighbor degree sum exponent polynomial of P_n is*

$$P_{NE(P_n)}(\mu) = \left(\mu + 4^4\right)^{n-5} \left(\mu + 3^3\right) \left(\mu + 2^2\right) \left[\mu^3 + (1249 - 256n)\mu^2 + (3660 - 2944n)\mu + 23040n - 138240\right]$$

Proof. The characterestic polynomial of Neighbor degree sum exponent matrix of path P_n is

$$\begin{aligned} P_{NE(P_n)}(\mu) &= |\mu I - NE(P_n)| \\ &= \begin{vmatrix} \mu & -2^3 & -2^4 & -2^4 & \cdots & -2^4 & -2^3 & -2^2 \\ -3^2 & \mu & -3^4 & -3^4 & \cdots & -3^4 & -3^3 & -3^2 \\ -4^2 & -4^3 & \mu & -4^4 & \cdots & -4^4 & -4^3 & -4^2 \\ -4^2 & -4^3 & -4^4 & \mu & \cdots & -4^4 & -4^3 & -4^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -4^2 & -4^3 & -4^4 & -4^4 & \cdots & \mu & -4^3 & -4^2 \\ -3^2 & -3^3 & -3^4 & -3^4 & \cdots & -3^4 & \mu & -3^2 \\ -2^2 & -2^3 & -2^4 & -2^4 & \cdots & -2^4 & -2^3 & \mu \end{vmatrix} \end{aligned} \tag{2.3}$$

Substract third column from the columns $4, 5 \ldots n-2$ and substract first column from the last column and also second column from $n-1$ columns of Eqn. (2.3) to obtain Eqn. (2.4)

$$= \begin{vmatrix} \mu & -2^3 & -2^4 & 0 & \cdots & 0 & 0 & -2^2 - \mu \\ -3^2 & \mu & -3^4 & 0 & \cdots & 0 & -3^3 - \mu & 0 \\ -4^2 & -4^3 & \mu & -4^4 - \mu & \cdots & -4^4 - \mu & 0 & 0 \\ -4^2 & -4^3 & -4^4 & \mu + 4^4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -4^2 & -4^3 & -4^4 & 0 & \cdots & \mu + 4^4 & 0 & 0 \\ -3^2 & -3^3 & -3^4 & 0 & \cdots & 0 & \mu + 3^3 & 0 \\ -2^2 & -2^3 & -2^4 & 0 & \cdots & 0 & 0 & \mu + 2^2 \end{vmatrix} \tag{2.4}$$

Adding last row to the first row and adding second row to the $(n-1)^{th}$ row and then adding rows $4, 5 \ldots n-2$ to the row of Eqn. (2.4) to obtain Eqn. (2.5).

$$= \begin{vmatrix} \mu - 2^2 & 2(-2^3) & 2(-2^4) & 0 & \cdots & 0 & 0 & 0 \\ 2(-3^2) & \mu - 3^3 & 2(-3^4) & 0 & \cdots & 0 & 0 & 0 \\ (n-4)(-4^2) & (n-4)(-4^3) & \mu - 4^4(n-5) & 0 & \cdots & 0 & 0 & 0 \\ -4^2 & -4^3 & -4^4 & \mu + 4^4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -4^2 & -4^3 & -4^4 & 0 & \cdots & \mu + 4^4 & 0 & 0 \\ -3^2 & -3^3 & -3^4 & 0 & \cdots & 0 & \mu + 3^3 & 0 \\ -2^2 & -2^3 & -2^4 & 0 & \cdots & 0 & 0 & \mu + 2^2 \end{vmatrix} \tag{2.5}$$

Eqn. (2.5) can be reduced to

$$\begin{aligned} P_{NE(P_n)}(\mu) &= \left(\mu+4^4\right)^{n-5}\left(\mu+3^3\right)\left(\mu+2^2\right)\begin{vmatrix} \mu-2^2 & 2(-2^3) & 2(-2^4) \\ 2(-3^2) & \mu-3^3 & 2(-3^4) \\ (n-4)(-4^2) & (n-4)(-4^3) & \mu-4^4(n-5) \end{vmatrix} \\ &= \left(\mu+4^4\right)^{n-5}\left(\mu+3^3\right)\left(\mu+2^2\right) \\ &\quad \left[\mu^3+(1249-256n)\mu^2+(3660-2944n)\mu+23040n-138240\right] \end{aligned}$$

The wheel W_n is a graph obtained from the cycle C_n-1 by adding a new vertex u and joining it to all vertices of C_n-1.

Theorem 2.4. *Let W_n be a wheel graph with n vertices. Then the Neighbor degree sum exponent polynomial is*

$$P_{NE(W_n)}(\mu)=\left[\mu+(n+5)^{(n+5)}\right]^{n-2}\left[\mu^2-\mu(n-2)(n+5)^{(n+5)}-(3n-3)^{(n+5)}(n-1)(n+5)^{3(n-1)}\right]$$

Proof. The characterestic polynomial of Neighbor degree sum exponent matrix of path W_n is

$$\begin{aligned} P_{NE(W_n)}(\mu) &= |\mu I - NE(W_n)| \\ &= \begin{vmatrix} \mu & -[3(n-1)]^{(n+5)} & -[3(n-1)]^{(n+5)} & \cdots & -[3(n-1)]^{(n+5)} \\ -(n+5)^{3(n-1)} & \mu & -(n+5)^{(n+5)} & \cdots & -(n+5)^{(n+5)} \\ -(n+5)^{3(n-1)} & -(n+5)^{(n+5)} & \mu & \cdots & -(n+5)^{(n+5)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n+5)^{3(n-1)} & -(n+5)^{(n+5)} & -(n+5)^{(n+5)} & \cdots & \mu \end{vmatrix} \end{aligned} \tag{2.6}$$

Substract second column from the columns $3,4\ldots n$ of Eqn. (2.6) to obtain Eqn. (2.7)

$$= \begin{vmatrix} \mu & -[3(n-1)]^{(n+5)} & 0 & \cdots & 0 \\ -(n+5)^{3(n-1)} & \mu & -(n+5)^{(n+5)}-\mu & \cdots & -(n+5)^{(n+5)}-\mu \\ -(n+5)^{3(n-1)} & -(n+5)^{(n+5)} & \mu+(n+5)^{(n+5)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n+5)^{3(n-1)} & -(n+5)^{(n+5)} & 0 & \cdots & \mu+(n+5)^{(n+5)} \end{vmatrix} \tag{2.7}$$

Adding rows $3,4\ldots n$ to the second row in Eqn. (2.7) to obtain Eqn. (2.8)

$$= \begin{vmatrix} \mu & -[3(n-1)]^{(n+5)} & 0 & \cdots & 0 \\ -(n-1)(n+5)^{3(n-1)} & \mu-(n-2)(n+5)^{(n+5)} & 0 & \cdots & 0 \\ -(n+5)^{3(n-1)} & -(n+5)^{(n+5)} & \mu+(n+5)^{(n+5)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n+5)^{3(n-1)} & -(n+5)^{(n+5)} & 0 & \cdots & \mu+(n+5)^{(n+5)} \end{vmatrix} \tag{2.8}$$

On simplifying Eqn. (2.8), we get,

$$\begin{aligned} P_{NE(W_n)}(\mu) &= \left[\mu+(n+5)^{(n+5)}\right]^{n-2}\begin{vmatrix} \mu & -[3(n-1)]^{(n+5)} \\ -(n-1)(n+5)^{3(n-1)} & \mu-(n-2)(n+5)^{(n+5)} \end{vmatrix} \\ &= \left[\mu+(n+5)^{(n+5)}\right]^{n-2}\left[\mu^2-\mu(n-2)(n+5)^{(n+5)}-(3n-3)^{(n+5)}(n-1)(n+5)^{3(n-1)}\right] \end{aligned}$$

3. Neighbor degree sum exponent Energy

The results of this section are straightforward by using the results of Section 2 via Eq. (1.1).
By Eqn. (2.1), the Neighbor degree sum exponent eigenvalues of an r-regular graph G are $-r^{2r^2}(n-1$ times) and $r^{2r^2}(n-1)$. Therefore for an $r-$ regular graph G on $n-$ vertices, $NEE(G) = 2r^{2r^2}(n-1)$. Hence, $NEE(K_n) = 2(n-1)^{2n^2-4n+3}$ and $NEE(C_n) = 512(n-1)$.

If G is an r-regular graph on n vertices then

$$NEE(\overline{G}) = 2(n-1)\left[r^2-(n-2)r+(n-1)^2-2m\right]^{2\left[r^2-(n-2)r+(n-1)^2-2m\right]^2}$$

By Theorem 2.1,

$$NEE(K_{n_1,n_2}) = \sqrt{[(n_1n_2)^{n_1n_2}(n_1-n_2)]^2+4n_1n_2^{2n_1n_2}(2n_1n_2-n_1-n_2+1)}+(n_1+n_2-2)(n_1n_2^{n_1n_2})$$

For a wheel W_n,

$$NEE(W_n) = (n-2)(n+5)^{n+5}+\sqrt{[(n+5)^{n+5}(n-2)]^2+4(n-1)(3n-3)^{n+5}(n+5)^{3n-3}}$$

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