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$\mathcal{A}$-POLYNOMIAL FOR SOME FAMILY OF CAVEMAN GRAPHS

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Abstract. The caveman graph is a graph formed by modifying a set of isolated k-cliques (or caves) by removing one edge from each clique and using it to connect to a neighboring clique along a central cycle such that all n cliques form a single unbroken loop. Caveman graph containing n-copies of k-cliques is denoted as, (n,k)-caveman graph. For a graph G, the $\mathcal{A}$-matrix is defined as, $\mathcal{A}(G) = [a_{ij}]$ in which $a_{ij} = 1$ if vertices v_i and v_j are adjacent; and 0 otherwise. The polynomial associated with $\mathcal{A}$-matrix is the $\mathcal{A}$-polynomial. In this work, we study $\mathcal{A}$-polynomial for some family of caveman graphs.

1. Introduction and preliminaries

Throughout the study we consider simple, finite and undirected graphs. A graph G consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ consisting of distinct, unordered pair of vertices. If $V(G)$ has cardinality n then G is a graph of order n. Whenever two vertices share a common edge, they are said to be adjacent (or neighbors) and such an edge is said to incident on those vertices. The number of edges incident to a vertex gives degree of that vertex. If all vertices of G have the same degree say r, then G is called an r-regular graph. For a graph G of order n, the adjacency matrix or $\mathcal{A}$-matrix is defined as, $\mathcal{A} = \mathcal{A}(G) = [a_{ij}]_{n\times n}$ in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent; and 0 otherwise. The polynomial associated to $\mathcal{A}$ defined by, $\phi_{\mathcal{A}}\big(G;x\big) = \det\big(xI_n - \mathcal{A}(G)\big)$ is called the adjacency polynomial or $\mathcal{A}$-polynomial. Roots of the equation $\phi_{\mathcal{A}}\big(G;x\big) = 0$ are called $\mathcal{A}$-eigenvalues, their collection is the $\mathcal{A}$-spectrum. A clique of a graph G is a complete subgraph of G; clique of size k is called a k-clique. For undefined graph theoretical terminologies and notations, we follow the book [1].

Definition 1.1. The caveman graph [3] is a graph formed by modifying a set of isolated k-cliques (or caves) by removing one edge from each clique and using it to connect to a neighboring clique along a central cycle such that all n cliques form a single unbroken loop. Caveman graph containing n-copies of k-cliques is denoted as, (n,k)-caveman graph.

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Definition 1.2. For a labeled graph G of order n with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the generalized composition (or joined union) [2] of n arbitrary graphs $G_1, G_2, \ldots, G_n$ is the graph obtained from the union of graphs $G_1, G_2, \ldots, G_n$ by joining every vertex of G_i to each vertex of G_j whenever v_i and v_j are adjacent in G, which is denoted by $G[G_1, G_2, \ldots, G_n]$.

Definition 1.3. A partition $\Pi : V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$ of the vertex set $V(G)$ of a graph G is equitable [2] if the number of neighbors in V_j for a vertex v in V_i is a constant q_{ij}, independent of v for all i, j with $1 \leq i, j \leq k$.

The partition of $V(G)$ into singletons is always equitable. In generalized composition of graphs, if a graph is regular then whole vertex set of that regular graph will be taken as a partite set in an equitable partition.

For an equitable partition, $\Pi : V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$ with k partite sets, we associate a $k \times k$ matrix $Q = [q_{ij}]$, called a quotient matrix.

Theorem 1.4. [2] *If $\Pi : V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$ is an equitable partition of the vertex set $V(G)$ of a graph G, then $\phi(Q : x)$ divides $\phi\big(\mathcal{A}(G) : x\big)$ i.e., polnomial due to the quotient matrix Q will be a factor of that due to $\mathcal{A}$-matrix.*

Proposition 1.5. [1] *For any positive integer n, the adjacency A-polynomial of K_n is*

$$\phi_{\mathcal{A}}(K_n : x) = \Bigg(x - (n-1)\Bigg)(x+1)^{n-1}.$$

2. $\mathcal{A}$-POLYNOMIAL FOR SOME CLASS OF CAVEMAN GRAPHS

In this section, we study $\mathcal{A}$-polynomial for some family of caveman graphs.

Theorem 2.1. *$\mathcal{A}$-polynomial of the class of $(2, k)$-caveman graphs is,*

$$\begin{aligned}\phi_{\mathcal{A}}(G; x) = (x+1)^{2k-8}\Bigg(&x^8 - 2(k-4)x^7 + (k^2 - 14k + 28)x^6 + 2(3k^2 - 19k + 24)x^5 \\ &+(11k^2 - 42k + 26)x^4 + 2(2k^2 + k - 16)x^3 - 5(k-2)(k-4)x^2 \\ &-2k(k-3)x + (k-3)^2\Bigg).\end{aligned}$$

Proof. Since $(2, k)$-caveman graphs contain two copies of k-cliques, it contains $2k$ vertices. Making these $2k$ vertices into eight partite sets: $V_1, V_2, \ldots, V_8$. With a suitable labelling of the vertices, the quotient matrix corresponding to the equitable partition containing these eight partite sets is:

$$Q = \begin{pmatrix} 0 & 1 & 0 & 1 & k-3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & k-3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & k-3 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & k-3 & 0 \\ 1 & 1 & 0 & 0 & k-4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & k-4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & k-3 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \phi(Q:x) = \Big(& x^8 - 2(k-4)x^7 + (k^2-14k+28)x^6 + 2(3k^2-19k+24)x^5 \\ & +(11k^2-42k+26)x^4 + 2(2k^2+k-16)x^3 - 5(k-2)(k-4)x^2 \\ & -2k(k-3)x + (k-3)^2 \Big). \end{aligned}$$

As $(2,k)$-caveman graphs can be viewed in terms of the structure of joined union, the part $(x+1)^{2k-8}$ of the polynomial $\phi_{\mathcal{A}}(G;x)$ is due to the partitions V_5 and V_7 each of which contain a copy of complete graph of order $k-3$ (complete graphs being regular graphs, form partite sets V_5 ad V_7); will give eigenvalue -1 occuring $k-4$ times for each partition V_5 and V_7, together form eigenvalue -1 occuring $2k-8$ times.
Result follows from $\phi(Q;x)$ and the eigenvalues obtained from V_5 and V_7. $\square$

Example 2.2. Consider $(2,6)$-caveman graph as in Figure 1 with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. The $\mathcal{A}$-polynomial of $(2,6)$-caveman graph is studied according to Theorem 2.1 by taking the partite sets: $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5, v_6, v_7\}$, $V_6 = \{v_8\}$, $V_7 = \{v_9, v_{10}, v_{11}\}$ and $V_8 = \{v_{12}\}$.

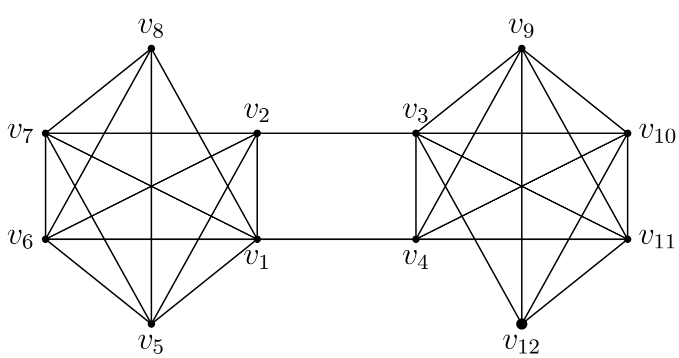

FIGURE 1. $(2,6)$-caveman graph

Theorem 2.3. *$\mathcal{A}$-polynomial of the class of $(3,k)$-caveman graphs is*

$$\phi_{\mathcal{A}}(G;x) = (x+1)^{3k-12}\Big(x^{12} - 3(k-4)x^{11} + 3(k^2-11k+22)x^{10}$$
$$-(k^3-30k^2+159k-208)x^9 - 3(3k^3-41k^2+139k-128)x^8$$
$$-3(10k^3-84k^2+193k-144)x^7 - (42k^3-222k^2+243k+83)x^6$$
$$-6(2k^3+8k^2-67k+81)x^5 + 6(k-1)(4k^2-31k+52)x^4 + 2(7k^3-30k^2+3k+62)x^3$$
$$-3(k-3)(2k^2-15k+19)x^2 - 3k(k-3)^2x + (k-3)^3\Big).$$

Proof. Since $(3,k)$-caveman graphs contain three copies of k-cliques, it contains $3k$ vertices. Making these $3k$ vertices into twelve partite sets: V_1, V_2, $\ldots$, V_{12}. With a suitable labelling of the vertices, the quotient matrix corresponding to the equitable partition containing these twelve partite sets is:

$$Q = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & k-3 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & k-3 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & k-3 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & k-3 & 1\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & k-4 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & k-4 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & k-4 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0
\end{pmatrix}.$$

Thus,

$$\phi(Q:x) = \Big(x^{12} - 3(k-4)x^{11} + 3(k^2-11k+22)x^{10} - (k^3-30k^2+159k-208)x^9$$
$$-3(3k^3-41k^2+139k-128)x^8 - 3(10k^3-84k^2+193k-144)x^7$$
$$-(42k^3-222k^2+243k+83)x^6 - 6(2k^3+8k^2-67k+81)x^5$$
$$+6(k-1)(4k^2-31k+52)x^4 + 2(7k^3-30k^2+3k+62)x^3$$
$$-3(k-3)(2k^2-15k+19)x^2 - 3k(k-3)^2x + (k-3)^3\Big).$$

As $(3,k)$-caveman graphs can be viewed in terms of the structure of joined union, the part $(x+1)^{3k-12}$ of the polynomial $\phi_{\mathcal{A}}(G;x)$ is due to the partitions V_7, V_9 and V_{11} each of which contain a copy of complete graph of order $k-3$ (complete graphs being regular graphs, form partite sets V_7, V_9 and V_{11}); will give eigenvalue -1 occuring $k-4$ times for each partition V_7, V_9 and V_{11}, together form eigenvalue -1 occuring $3k-12$ times.
Result follows from $\phi(Q;x)$ and the eigenvalues obtained from V_7, V_9 and V_{11}. $\square$

Example 2.4. Consider $(3,5)$-caveman graph as in Figure 2 with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$. The $\mathcal{A}$-polynomial of $(3,5)$-caveman graph is studied according to Theorem 2.3 by taking the partite sets: $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5\}$, $V_6 = \{v_6\}$, $V_7 = \{v_7, v_8\}$, $V_8 = \{v_9\}$, $V_9 = \{v_{10}, v_{11}\}$, $V_{10} = \{v_{12}\}$, $V_{11} = \{v_{13}, v_{14}\}$ and $V_{12} = \{v_{15}\}$.

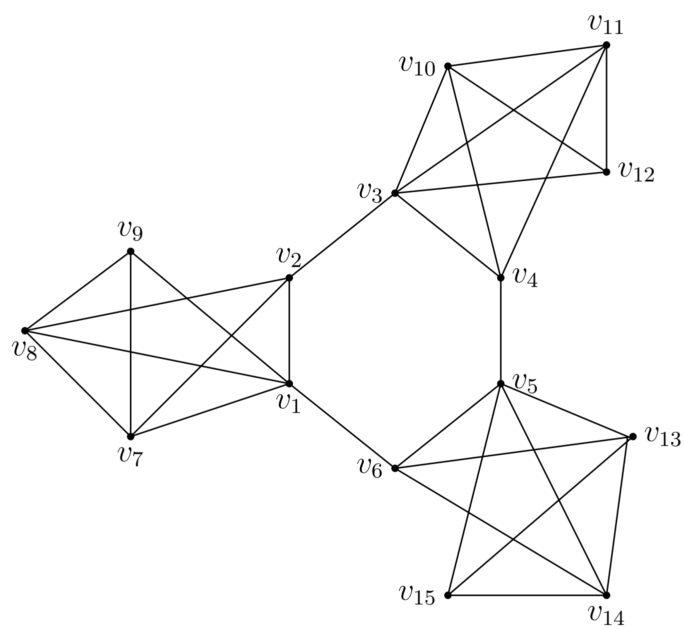

Figure 2. $(3,5)$-caveman graph

Theorem 2.5. *$\mathcal{A}$-polynomial of the class of $(4,k)$-caveman graphs is,*

$$\begin{aligned}\phi_{\mathcal{A}}(G;x) = (x+1)^{4k-16}\Big(& x^{16} - 4(k-4)x^{15} + 6(k^2-10k+20)x^{14} - 4(k^3 - 21k^2 \\ & +103k - 136)x^{13} + (k^4 - 52k^3 + 522k^2 - 1668k + 1608)x^{12} + 4(3k^4 \\ & -72k^3 + 462k^2 - 1063k + 770)x^{11} + 2(29k^4 - 432k^3 + 1957k^2 - 3290k \\ & +1674)x^{10} + 4(35k^4 - 357k^3 + 1123k^2 - 1155k + 128)x^9 + (159k^4 \\ & -996k^3 + 1026k^2 + 2796k - 4018)x^8 + 4(6k^4 + 128k^3 - 1004k^2 + 2109k \\ & -1272)x^7 - 2(58k^4 - 640k^3 + 2131k^2 - 2374k + 432)x^6 - 4(18k^4 - 93k^3 \\ & -35k^2 + 689k - 744)x^5 + (31k^4 - 476k^3 + 2174k^2 - 3708k + 1944)x^4 \\ & +28(k-2)(k-3)(k^2-3k-3)x^3 - 2(k-3)^2(3k^2-30k+38)x^2 - 4k(k-3)^3x + (k-3)^4\Big).\end{aligned}$$

Proof. Since $(4,k)$-caveman graphs contain four copies of k-cliques, it contains $4k$ vertices. Making these $4k$ vertices into sixteen partite sets: V_1, V_2, $\ldots$, V_{16}. With a suitable labelling of the vertices, the quotient matrix corresponding to the equitable partition containing these sixteen partite sets is:

$$Q = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & k-3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-4 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & k-4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k-3 & 0
\end{pmatrix}.$$

Thus,

$$\begin{aligned}
\phi(Q:x) = \Big(& x^{16} - 4(k-4)x^{15} + 6(k^2-10k+20)x^{14} - 4(k^3-21k^2+103k-136)x^{13} \\
& +(k^4-52k^3+522k^2-1668k+1608)x^{12} + 4(3k^4-72k^3+462k^2-1063k+770)x^{11} \\
& +2(29k^4-432k^3+1957k^2-3290k+1674)x^{10} + 4(35k^4-357k^3+1123k^2-1155k+128)x^9 \\
& +(159k^4-996k^3+1026k^2+2796k-4018)x^8 + 4(6k^4+128k^3-1004k^2+2109k-1272)x^7 \\
& -2(58k^4-640k^3+2131k^2-2374k+432)x^6 - 4(18k^4-93k^3-35k^2+689k-744)x^5 \\
& +(31k^4-476k^3+2174k^2-3708k+1944)x^4 + 28(k-2)(k-3)(k^2-3k-3)x^3 \\
& -2(k-3)^2(3k^2-30k+38)x^2 - 4k(k-3)^3x + (k-3)^4\Big).
\end{aligned}$$

As $(4,k)$-caveman graphs can be viewed in terms of the structure of joined union, the part $(x+1)^{4k-16}$ of the polynomial $\phi_{\mathcal{A}}(G;x)$ is due to the partitions V_9, V_{11}, V_{13} and V_{15} each of which contain a copy of complete graph of order $k-3$ (complete graphs being regular graphs, form partite sets V_9, V_{11}, V_{13} and V_{15}); will give eigenvalue -1 occuring $k-4$ times for each partition V_9, V_{11}, V_{13} and V_{15}, together form eigenvalue -1 occuring $4k-16$ times.
Result follows from $\phi(Q;x)$ and the eigenvalues obtained from V_9, V_{11}, V_{13} and V_{15}. □

Example 2.6. Consider $(4,5)$-caveman graph as in Figure 3 with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}\}$. The $\mathcal{A}$-polynomial of $(4,5)$-caveman graph is studied according to Theorem 2.5 by taking the partite sets: $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5\}$,

$V_6 = \{v_6\}$, $V_7 = \{v_7\}$, $V_8 = \{v_8\}$, $V_9 = \{v_9, v_{10}\}$, $V_{10} = \{v_{11}\}$, $V_{11} = \{v_{12}, v_{13}\}$, $V_{12} = \{v_{14}\}$, $V_{13} = \{v_{15}, v_{16}\}$, $V_{14} = \{v_{17}\}$, $V_{15} = \{v_{18}, v_{19}\}$ and $V_{16} = \{v_{20}\}$.

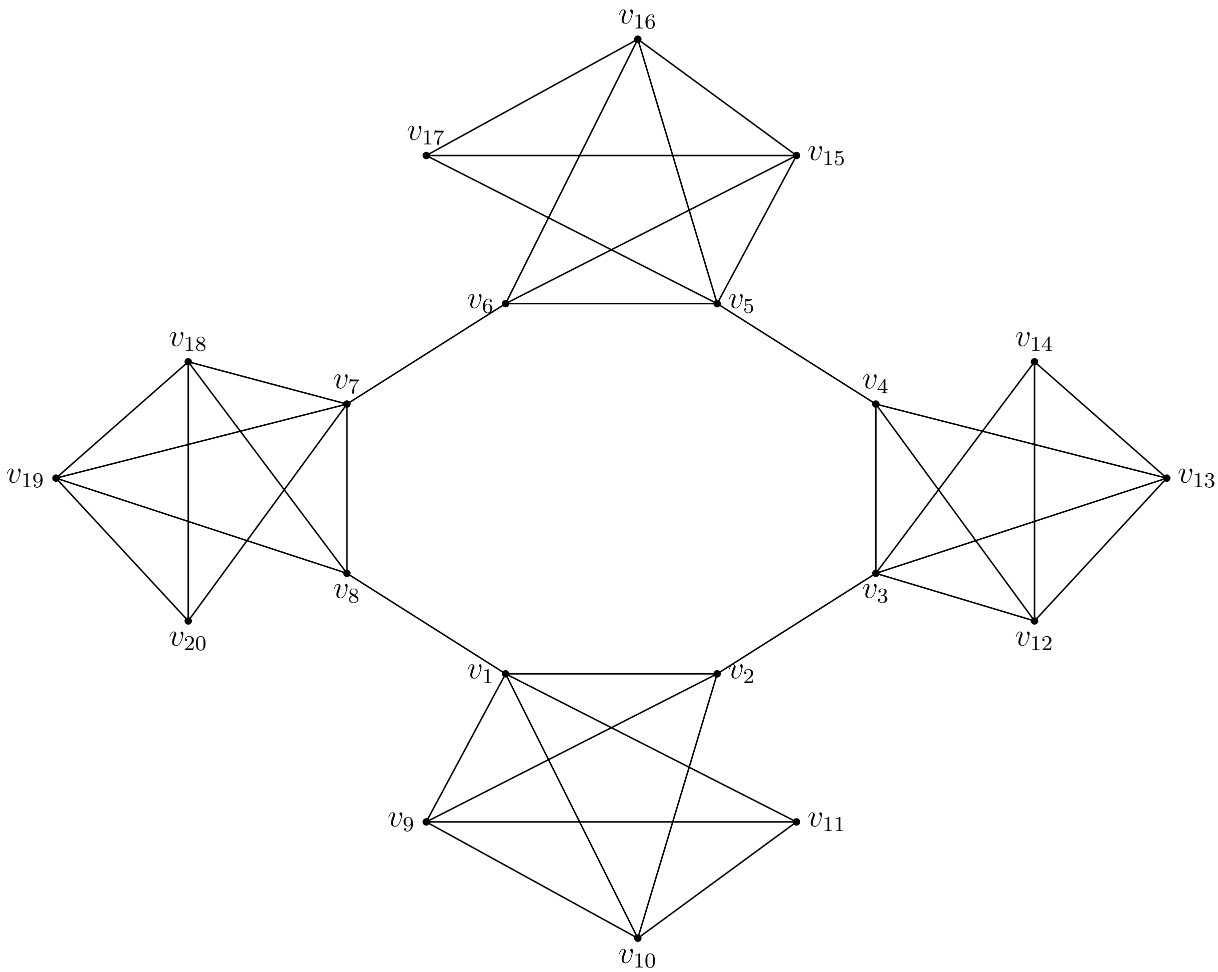

Figure 3. $(4, 5)$-caveman graph

References

[1] Cvetković, D., Rawlinson, P., Simić, S.: An Introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge, 2010.
[2] Schwenk, A. J.: *Computing the characteristic polynomial of a graph*, Lecture Notes in Math., **406** (1974), 153–172.
[3] Watts, D. J.: *Networks, Dynamics, and the Small-World Phenomenon*, Am. J. Sociol., **105** (2) (1999), 493–527.

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