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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR A FAMILY OF ANALYTIC FUNCTIONS DEFINED BY A NEW MULTIPLIER DIFFERENTIAL OPERATOR

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Abstract. In this paper, we introduce a new multiplier differential operator $\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)$ in connection with a family of multivalent (or p-valent) functions defined in the unit disk $\mathcal{E}=\{z\in\mathbb{C}:|z|<1\}$. Our investigations include the subordination, superordination and sandwich properties of functions in the new family. Finally, some key corollaries are reported.

1. Introduction and Preliminaries

Let $\mathcal{A}$ represent the family of functions $f(z)$ that are analytic in the unit disk $\mathcal{E}:=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{S}$ be the subfamily of $\mathcal{A}$ of analytic and univalent functions in $\mathcal{E}$, so that f is of the power series representation

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k,\quad z\in\mathcal{E}. \tag{1.1}$$

Also, let $\mathcal{A}(p)$ represent the family of functions of the power series representation

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k,\quad p\in\mathcal{N}:=\{1,2,3,\ldots\}\text{ and } z\in\mathcal{E} \tag{1.2}$$

which are analytic and multivalent (or p-valent) in $\mathcal{E}$. In the sequel, a function $f\in\mathcal{A}(p)$ is said to be a *multivalent convex function of order* α in $\mathcal{E}$ if it satisfies the condition

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad 0\leqq\alpha<p,\ p\in\mathcal{N}\text{ and } z\in\mathcal{E}. \tag{1.3}$$

We denote this family of functions by $\mathcal{C}_p(\alpha)$ so that by equivalence, $\mathcal{C}_1(\alpha)=\mathcal{C}(\alpha)\subset\mathcal{S}$.

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Let $f_1, f_2 \in \mathcal{A}$, then $f_1 \prec f_2$ if there is an analytic function

$$w(z) = \sum_{k=1}^{\infty} w_k z^k, \quad |w(z)| < 1, \ w(0) = 0 \text{ and } z \in \mathcal{E}$$

such that

$$f_1(z) = f_2(w(z)) \iff f_1(0) = f_2(0) \text{ and } f_1(\mathcal{E}) \subset f_2(\mathcal{E}) \text{ for all } z \in \mathcal{E}.$$

Note that "$\prec$" represents subordination. The function f_1 is called subordinate function while the function f_2 is called the superordinate function.

In view of the aforementioned definitions, we now present our novel multiplier differential operator as follows.

Definition 1.1. Let $p \in \mathcal{N}$; $m \in \mathcal{N}_0 := \{0, 1, 2, \ldots\}$; $\alpha, \beta, \lambda \geqq 0$; $0 \leqq \gamma \leqq \lambda$; $0 < \nu \leqq 1$ and $\alpha + \beta > 0$, then for $f \in \mathcal{A}(p)$, we define the *multiplier differential operator* $\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ by

$$\mathcal{D}^{0,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) = f(z) \tag{1.4}$$

$$\begin{aligned}(\alpha+\beta)\mathcal{D}^{1,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) &= [\alpha+\beta+\gamma-(2\nu-1)(\lambda+\mu)]f(z) \\ &- (p-1)\lambda\gamma z^p + \frac{1}{p}[((2\nu-1)(\lambda+\mu)-\gamma)zf'(z) + \lambda\gamma z^2 f''(z)]\end{aligned} \tag{1.5}$$

therefore

$$\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) = \mathcal{D}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu)(\mathcal{D}^{m-1,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z)), \quad m \in \mathcal{N} \tag{1.6}$$

and in general we have that

$$\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{\alpha + [(2\nu-1)(\lambda+\mu) + \gamma(k\lambda-1)](\frac{k}{p}-1) + \beta}{\alpha+\beta}\right]^m a_k z^k \tag{1.7}$$

or for brevity we have

$$\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) = z^p + \sum_{k=p+1}^{\infty} \Omega^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) a_k z^k \tag{1.8}$$

where

$$\Omega^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) = \left[\frac{\alpha + [(2\nu-1)(\lambda+\mu) + \gamma(k\lambda-1)](\frac{k}{p}-1) + \beta}{\alpha+\beta}\right]^m.$$

Remark 1.2. A simple calculation shows that

$$\begin{aligned}\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) &= \left\{1 - \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\alpha+\beta}\right\} \mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z) \\ &- \frac{(p-1)\lambda\gamma}{\alpha+\beta} z^p + \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{p(\alpha+\beta)} z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z))' \\ &+ \frac{\lambda\gamma}{p(\alpha+\beta)} z^2 (\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha, \beta, \gamma, \nu) f(z))''\end{aligned} \tag{1.9}$$

and secondly, the novel multiplier differential operator $\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)$ generalizes many multiplier operators earlier studied by the following authors.

(1) $\mathcal{D}^{m,1}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)=\mathcal{D}^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)$ was studied by Alamoush [2].
(2) $\mathcal{D}^{m,p}_{\lambda,0}(0,1,0,1)f(z)=\mathcal{D}^{m,p}_{\sigma}f(z)$ was studied by Goyal and Goswami [12].
(3) $\mathcal{D}^{m,1}_{1,0}(0,1,0,1)f(z)=\mathcal{D}^{m}f(z)$ was studied by Sălăgean [22].
(4) $\mathcal{D}^{m,1}_{1,0}(1,1,0,1)f(z)=\mathcal{L}^{m}f(z)$ was studied by Uralegaddi and Somanatha [26].
(5) $\mathcal{D}^{m,1}_{1,0}(\alpha,1,0,1)f(z)=\mathcal{L}^{m}_{\alpha}f(z)$ was studied by Cho and Srivastava [8].
(6) $\mathcal{D}^{\rho,1}_{\lambda,0}(0,1,0,1)f(z)=\mathcal{D}^{\rho}_{\lambda}f(z)$ was studied by Acu and Owa [1].
(7) $\mathcal{D}^{m,1}_{\lambda,0}(0,1,0,1)f(z)=\mathcal{D}^{m}_{\lambda}f(z)$ was studied by Al-Oboudi [4].
(8) $\mathcal{D}^{\rho,1}_{\lambda,0}(1,\beta,\gamma,1)f(z)=\mathcal{L}_{1}(\rho,\lambda,\beta)f(z)$ was studied by Cătaş et al. [7].
(9) $\mathcal{D}^{m,1}_{\lambda,0}(\alpha,0,0,1)f(z)=\mathcal{D}^{m}_{\lambda}(\alpha)f(z)$ was studied by Aouf et al. [5].
(10) $\mathcal{D}^{m,1}_{\lambda,0}(0,1,0,\frac{\alpha+\beta}{2})f(z)=\mathcal{D}^{m}_{\alpha,\beta,0,\lambda}f(z)$ was studied by Alamoush and Darus [3].
(11) $\mathcal{D}^{m,p}_{\lambda,0}(0,1,\gamma,1)f(z)=\mathcal{D}^{m,p}_{\lambda,\gamma}f(z)$ was studied by Sambo and Meller [23].
(12) $\mathcal{D}^{m,1}_{\lambda,0}(0,1,\gamma,1)f(z)=\mathcal{D}^{m}_{\lambda,\gamma}f(z)$ was studied by Raducanu and Orhan [21].
(13) $\mathcal{D}^{m,1}_{\lambda,\mu}(\alpha,\beta,0,1)f(z)=\mathcal{D}^{m}_{\lambda}(\alpha,\beta,\mu)f(z)$ was studied by Darus and Faisal [9].
(14) $\mathcal{D}^{m,1}_{\lambda,\mu}(\alpha,0,0,1)f(z)=\mathcal{D}^{m}_{\lambda}(\alpha,\mu)f(z)$ was studied by Darus and Faisal [9].

Motivated by the works of the aforementioned authors and those in [6, 10, 11, 13, 14, 15, 18, 19, 20, 24, 27], we now investigate the subordination, superordination and sandwich properties of the multiplier differential operator. The notion of *differential superordination* was introduced by Miller and Mocanu in [17] which is an extension of the problem of *differential subordination* earlier presented in [16].

Definition 1.3 ([25])**.** Let $Y:\mathbb{C}^3\times\mathcal{E}\to\mathbb{C}$ and let $h(z)$ be univalent in $\mathcal{E}$. If $p(z)$ is analytic in $\mathcal{E}$ and fulfills the second-order differential subordination

$$Y(p(z),zp'(z),z^2p''(z);z)\prec h(z), \tag{1.10}$$

then $p(z)$ is the differential subordination solution of (1.10).

Definition 1.4 ([25])**.** Let $Y_1:\mathbb{C}^3\times\mathcal{E}\to\mathbb{C}$ and let $h(z)$ be univalent in $\mathcal{E}$. If $p(z)$ and

$$Y_1(p(z),zp'(z),z^2p''(z);z)$$

are univalent in $\mathcal{E}$ and $p(z)$ fulfills the second-order differential superordination

$$h(z)\prec Y_1(p(z),zp'(z),z^2p''(z);z), \tag{1.11}$$

then $p(z)$ is the differential superordination solution of (1.11).

2. Relevant Lemmas

We shall need the following lemmas to proof our results.

Lemma 2.1 ([25]). *Let $\Psi(z)$ be convex in $\mathcal{E}$, $\eta_1 \in \mathbb{C}$ and $\eta_2 \in \mathbb{C}^\star$ with*

$$\Re\left(1 + \frac{\Psi''(z)}{\Psi'(z)}\right) > \max\left\{0, -\Re\frac{\eta_1}{\eta_2}\right\}.$$

If $\Phi(z)$ is analytic in $\mathcal{E}$ and

$$\eta_1\Phi(z) + \eta_2 z\Phi'(z) \prec \eta_1\Psi(z) + \eta_2 z\Psi'(z),$$

then $\Phi(z) \prec \Psi(z)$.

Lemma 2.2 ([17]). *Let $\Psi(z)$ be convex in $\mathcal{E}$ and $\Re\,\eta > 0$. If $\Phi(z) \in A[\Psi(0), 1] \cap Q$, $\Phi(z) + \eta z\Phi'(z)$ is univalent in $\mathcal{E}$ and*

$$\Psi(z) + \eta z\Psi'(z) \prec \Phi(z) + \eta z\Phi'(z),$$

then $\Psi(z) \prec \Phi(z)$.

3. Main Results

From now on, we assume that all parameters are as defined in Definition 1.1.

Theorem 3.1. *Let $\Psi(z)$ be convex univalent in $\mathcal{E}$ with $\Psi(0) = 1$, $\omega_1 > 0$, $0 \neq \omega_2 \in \mathbb{C}$ and suppose*

$$\Re\left\{1 + \frac{\Psi''(z)}{\Psi'(z)}\right\} > \max\left\{0, -\Re\frac{\omega_1}{\omega_2}\right\}. \tag{3.1}$$

If $f \in \mathcal{A}(p)$ and satisfies the subordination

$$\left\{1 - \omega_2\frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1)\right\}\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1} +$$
$$\omega_2\left\{\frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)) + p(p-1)z^p + \left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}\right.$$
$$\left.\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}\right\} \prec \Psi(z) + \frac{\omega_2}{\omega_1}z\Psi'(z),$$

then

$$\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1} \prec \Psi(z).$$

Proof. Let $q(z) = \left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}$, then

$$q'(z) = \omega_1\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}\left\{\frac{pz^{p-1}}{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}\right\}$$
$$\frac{pz^{p-1}(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'' - (p(p-1)z^{p-2})(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{(pz^{p-1})^2}$$

$$= \omega_1 \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}} \right\}^{\omega_1} \left\{ \frac{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))''}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} - \frac{(p-1)}{z} \right\}$$

which implies that

$$\frac{zq'(z)}{q(z)} = \omega_1 \left\{ \frac{z^2(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))''}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} - (p-1) \right\}.$$

Using (1.9), then we obtain

$$\frac{zq'(z)}{q(z)} = \omega_1 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)) + \left\{\frac{p[(2\nu-1)(\lambda+\mu)-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} \right.$$
$$\left. + \frac{p(p-1)z^p - \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} - (p-1) \right\}$$

$$= \omega_1 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)) + p(p-1)z^p + \left\{\frac{p[(2\nu-1)(\lambda+\mu)-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} \right.$$
$$\left. - \left\{ \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1) \right\} \right\}$$

so that

$$\frac{\omega_2 zq'(z)}{\omega_1} =$$
$$\omega_2 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)) + p(p-1)z^p + \left\{\frac{p[(2\nu-1)(\lambda+\mu)-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} \right.$$
$$\left. \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}} \right\}^{\omega_1} \right\}$$
$$- \left\{ \left\{ \omega_2 \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1) \right\} \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}} \right\}^{\omega_1} \right\}.$$

Using the hypothesis

$$q(z) + \frac{\omega_2}{\omega_1} zq'(z) \prec \Psi(z) + \frac{\omega_2}{\omega_1} z\Psi'(z)$$

and applying Lemma 2.1 with $\eta_1 = 1$ and $\eta_2 = \frac{\omega_2}{\omega_1}$ complete the proof. $\square$

Corollary 3.2. *Let $\Psi(z)$ be convex univalent in $\mathcal{E}$ with $\Psi(0) = 1$, $\omega_1 > 0$, $0 \neq \omega_2 \in \mathbb{C}$ and suppose*

$$\Re\left\{1 + \frac{\Psi''(z)}{\Psi'(z)}\right\} > \max\left\{0, -\Re\frac{\omega_1}{\omega_2}\right\}. \tag{3.2}$$

If $f \in \mathcal{A}(p)$ and satisfies the subordination

$$\left\{1-\omega_2\frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma}+(p-1)\right\}\left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma}f(z))'}{pz^{p-1}}\right\}^{\omega_1}$$
$$+\,\omega_2\left\{\frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m.p}_{\lambda,\gamma}f(z))+p(p-1)z^p+\left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m.p}_{\lambda,\gamma}f(z)}{z(\mathcal{D}^{m.p}_{\lambda,\gamma}f(z))'}\right.$$
$$\left.\left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma}f(z))'}{pz^{p-1}}\right\}^{\omega_1}\right\}\prec\Psi(z)+\frac{\omega_2}{\omega_1}z\Psi'(z),$$

then

$$\left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma}f(z))'}{pz^{p-1}}\right\}^{\omega_1}\prec\Psi(z).$$

Theorem 3.3. *Let $\Psi(z)$ be convex univalent in $\mathcal{E}$ with $\Psi(0)=1$, $\omega_1>0$, $\Re\,\omega_2>0$. If $f\in\mathcal{A}(p)$,*

$$\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}\in\Theta[q(0),1]\cap Q \tag{3.3}$$

and

$$\left\{1-\omega_2\frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma}+(p-1)\right\}\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}$$
$$+\omega_2\left\{\frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))+p(p-1)z^p+\left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}\right.$$
$$\left.\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}\right\}$$

is univalent in $\mathcal{E}$ and satisfies the superordination

$$\Psi(z)+\frac{\omega_2}{\omega_1}z\Psi'(z)\prec\left\{1-\omega_2\frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma}+(p-1)\right\}\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}$$
$$+\omega_2\left\{\frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))+p(p-1)z^p+\left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\}\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}\right.$$
$$\left.\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}\right\},$$

then

$$\Psi(z)\prec\left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}.$$

Proof. Let $q(z) = \left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1}$, then

$$q'(z) = \omega_1 \left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1} \left\{\frac{pz^{p-1}}{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}\right\}$$
$$\frac{pz^{p-1}(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'' - (p(p-1)z^{p-2})(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{(pz^{p-1})^2}$$
$$= \omega_1 \left\{\frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'}{pz^{p-1}}\right\}^{\omega_1} \left\{\frac{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))''}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} - \frac{(p-1)}{z}\right\}$$

implies that

$$\frac{zq'(z)}{q(z)} = \omega_1 \left\{\frac{z^2(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))''}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)f(z))'} - (p-1)\right\}.$$

Using the hypothesis

$$\Psi(z) + \frac{\omega_2}{\omega_1} z\Psi'(z) \prec q(z) + \frac{\omega_2}{\omega_1} zq'(z)$$

and applying Lemma 2.2 complete the proof. $\square$

Corollary 3.4. *Let $\Psi(z)$ be convex univalent in $\mathcal{E}$ with $\Psi(0) = 1$, $\omega_1 > 0$, $\Re\,\omega_2 > 0$. If $f \in \mathcal{A}(p)$, then*

$$\left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'}{pz^{p-1}}\right\}^{\omega_1} \in \Theta[q(0), 1] \cap Q \tag{3.4}$$

and

$$\left\{1 - \omega_2 \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1)\right\} \left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'}{pz^{p-1}}\right\}^{\omega_1}$$
$$+ \omega_2 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z)) + p(p-1)z^p + \left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\} \mathcal{D}^{m.p}_{\lambda,\gamma} f(z)}{z(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'} \right.$$
$$\left. \left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'}{pz^{p-1}}\right\}^{\omega_1} \right\}$$

is univalent in $\mathcal{E}$ and satisfies the superordination

$$\Psi(z) + \frac{\omega_2}{\omega_1} z\Psi'(z) \prec \left\{1 - \omega_2 \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1)\right\} \left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'}{pz^{p-1}}\right\}^{\omega_1}$$
$$+ \omega_2 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z)) + p(p-1)z^p + \left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\} \mathcal{D}^{m.p}_{\lambda,\gamma} f(z)}{z(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'} \right.$$
$$\left. \left\{\frac{(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'}{pz^{p-1}}\right\}^{\omega_1} \right\},$$

then

$$\Psi(z) \prec \left\{ \frac{(\mathcal{D}^{m.p}_{\lambda,\gamma} f(z))'}{pz^{p-1}} \right\}^{\omega_1}.$$

By combining the two results in Theorems 3.1 and 3.3, we obtain

Theorem 3.5. *Let $\Psi_1(z), \Psi_2(z)$ be convex univalent in $\mathcal{E}$, with $\Psi_1(0) = 1 = \Psi_2(0)$, $\Re\,\omega_2 > 0$,*

$$\Re\left\{1 + \frac{q''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\frac{\omega_1}{\omega_2}\right\}, \tag{3.5}$$

$\omega_1 > 0$ and $0 \neq \omega_2 \in \mathbb{C}$. If $f \in \mathcal{A}(p)$, then

$$\left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'}{pz^{p-1}} \right\}^{\omega_1} \in \Theta[1,1] \cap Q \tag{3.6}$$

and

$$\left\{1 - \omega_2 \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1)\right\} \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'}{pz^{p-1}} \right\}^{\omega_1}$$
$$+\omega_2 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z)) + p(p-1)z^p + \left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\} \mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'} \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'}{pz^{p-1}} \right\}^{\omega_1} \right\}$$

is univalent in $\mathcal{E}$ and satisfies

$$\Psi_1(z) + \frac{\omega_2}{\omega_1} z\Psi_1'(z) \prec \left\{1 - \omega_2 \frac{((2\nu-1)(\lambda+\mu)-\gamma)}{\lambda\gamma} + (p-1)\right\} \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'}{pz^{p-1}} \right\}^{\omega_1}$$
$$+\omega_2 \left\{ \frac{\frac{p(\alpha+\beta)}{\lambda\gamma}(\mathcal{D}^{m+1,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z)) + p(p-1)z^p + \left\{\frac{p[((2\nu-1)(\lambda+\mu))-(\alpha+\beta+\gamma)]}{\lambda\gamma}\right\} \mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z)}{z(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'} \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'}{pz^{p-1}} \right\}^{\omega_1} \right\} \prec \Psi_2(z) + \frac{\omega_2}{\omega_1} z\Psi_2'(z),$$

then

$$\Psi_1(z) \prec \left\{ \frac{(\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu) f(z))'}{pz^{p-1}} \right\}^{\omega_1} \prec \Psi_2(z).$$

Conclusion

In this work, we introduced a novel multiplier differential operator denoted by $\mathcal{D}^{m,p}_{\lambda,\mu}(\alpha,\beta,\gamma,\nu)$. It was shown that the multiplier differential operator generalizes many existing differential operators. Further, our main investigations covered the subordination, superordination and sandwich properties.

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