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CONSTRUCTION OF 3-GDDS WITH BLOCK SIZE 4 AND 2 GROUPS USING 1 AND 2-FACTORIZATIONS OF COMPLETE GRAPHS

WILBROAD BEZIRE[1], FREDRICK NYAMWALA[2]

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Abstract. Recently we presented construction of 3-GDD$(n, 2, 4; \lambda_1, \lambda_2)$ using 3-designs and Steiner quadruple systems and established that necessary conditions for existence of a 3-GDD$(n, 2, 4; 9, 1)$ are sufficient for $n \equiv 3 \pmod{12}$ and $n = 15$, 27. In this paper, we present the construction of three group divisible designs of block size four and two groups using cycling a one and two factorization of complete graphs. We show that necessary conditions for existence of a 3-GDD$(n, 2, 4; \lambda_1, \lambda_2)$ are sufficient for $\lambda_1 = 0$ and $\lambda_2 = 1, 2$.

1. ntroduction

One and two factorizations of complete graphs are combinatorial tools used in the construction of new families of designs. The construction of a 3-GDD$(n, 2, 4; 9, 1)$ was done in [1] where the authors established that the necessary conditions for existence of a 3-GDD$(n, 2, 4; 9, 1)$ are sufficient for $n \equiv 3 \pmod{12}$ and $n \neq 3$. It was also proved that for $n \equiv 10 \pmod{12}$ necessary conditions for existence of a 3-GDD$(n, 2, 4; 6s, 8t)$ are sufficient. The constructions were done using 3-designs and Steiner quadruple systems. In this paper we use cycling a one and two-factorization of complete graphs to construct new designs.

1.1. Group divisible designs.

Definition 1.1. [4] A design is a pair $(X, \mathcal{A})$ where X is a set of elements called points, and $\mathcal{A}$ is a collection of nonempty subsets of X called blocks.

Two identical blocks (repeated blocks) in a design are allowed and hence we refer to $\mathcal{A}$ as a multiset rather than a set of blocks. A design is simple if it does not contain repeated blocks.

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[1] Corresponding author.
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Example 1.2. let $V = \{1, 2, 3, 4, 5, 6, 7\}$ be a set of 7 elements and $\mathcal{B} = \{\{4, 5, 6, 7\}, \{2, 3, 6, 7\}, \{2, 3, 4, 5\}, \{1, 3, 5, 7\}, \{1, 3, 4, 6\}, \{1, 2, 5, 6\}, \{1, 2, 4, 7\}\}$ be a collection of four-subsets of V.

We note here that there are no repeated blocks. Hence the design is a simple design [8].

Definition 1.3. [4, 8] A group divisible design, $GDD(n, m, k; \lambda_1, \lambda_2)$, is a collection of k-element subsets, called blocks, of an nm-set X, where

(i) the elements of X are partitioned into m subsets (called groups) of size n each;
(ii) pairs of distinct elements within the same group are called first associates of each other and appear together in λ_1 blocks and
(iii) any two elements not in the same group are called second associates and appear together in λ_2 blocks.

Example 1.4. A GDD$(3, 2, 4; 3, 1)$ with two groups $\{1, 2, 3\}$ and $\{4, 5, 6\}$ is; $\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5, 6, 1\}, \{4, 5, 6, 2\}, \{4, 5, 6, 3\}\}$.

In the above example we note the following:

(i) Every element occurs in exactly 4 blocks. In any GDD where the groups are of the same size, every element occurs a fixed number of times. This is called the replication number and is denoted by r.
(ii) There are only two groups of the same size that is 3.
(iii) Each block intersects a group in exactly three points or in exactly one point.

In (iii) above, we note that GDD$(3, 2, 4; 3, 1)$ is an odd group divisible design.

Definition 1.5. [6] Group Divisible Association: When v elements are divided into m groups of size n each, then any two elements from the same group are called the first associates and any two elements from different groups are called the second associates. Such an association scheme is called Group Divisible Association Scheme.

Each element has $(n-1)$ first associates and $m(n-1)$ second associates. According to their parametric relationships, GDDs are classified into three classes:

(i) Singular Group Divisible Designs (SGD) if $r - \lambda_1 = 0$.
(ii) Semi-regular Group Divisible Designs (SRGD) if $r-\lambda_1 > 0$ and $rk-v\lambda_2 = 0$.
(iii) Regular Group Divisible Designs (RGD) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

Remark 1.6. For singular GDDs, the necessary condition $r = \lambda_1$, forces that if an element is in the block, all first associates must be in the block and hence group size divides the block size, and therefore k must be a composite number. Thus SGDs do not exist for $k = 5$ and $k = 7$ since these are prime numbers. In general SGDs do not exist when k is a prime number.

1.2. **Balanced Incomplete Block Designs (BIBDs).**

Definition 1.7. [8] Let v, k and λ be positive integers such that $v > k \geq 2$. A (v, k, λ) - balanced incomplete block design ((v, k, λ)-BIBD) is a design $(X, \mathcal{A})$ such that the following properties are satisfied.

(i) $|X| = v$.
(ii) each block contains exactly k points.
(iii) every pair of distinct points is contained in exactly λ blocks.

Property (iii) in the definition above is the "balance" property. A $BIBD$ is called incomplete because $v > k$, hence all its blocks are incomplete blocks.

Example 1.8. A $(7, 3, 1)$-BIBD, $X = \{1, 2, 3, 4, 5, 6, 7\}$.
$\mathcal{A} = \{123, 145, 167, 246, 257, 347, 356\}$. This is represented by the following funel plane.

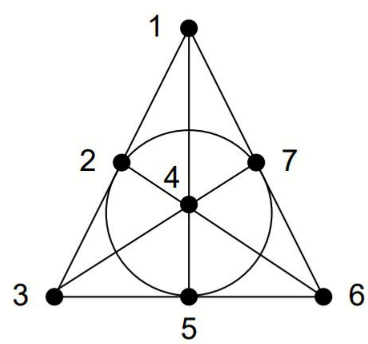

FIGURE 1. $(7, 3, 1)$-BIBD

Example 1.9. A $(9, 3, 1)$-BIBD with $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ we have $\mathcal{A} = \{123, 456, 789, 147, 258, 369, 159, 267, 348, 168, 249, 357\}$.

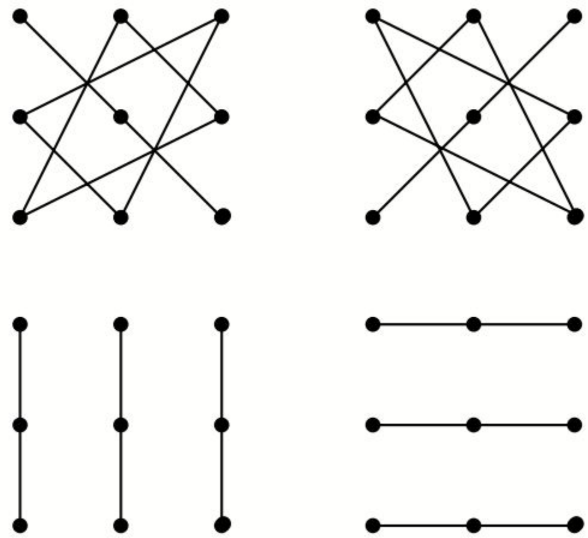

FIGURE 2. $(9, 3, 1)$-BIBD

The following are the basic properties of BIBDs.

Theorem 1.10. [8] *In a (v, k, λ)-BIBD every point occurs in exactly* $r = \dfrac{\lambda(v-1)}{k-1}$ *blocks.*

Theorem 1.11. [8] *A (v, k, λ)-BIBD has exactly* $b = \dfrac{vr}{k} = \dfrac{v.\lambda(v-1)}{k(k-1)} = \dfrac{\lambda(v^2-v)}{(k^2-k)}$ *blocks.*

To record the five parameters of a BIBD, we use the notation (v, b, r, k, λ). We note that b and r must be positive integers. Because of this, we are able to establish that BIBDs with particular parameters don't exist. We can also denote a BIBD as (v, k, λ). This is because other parameters like b and r can be obtained using v, k and λ.

Corollary 1.12. [8] *If a (v, k, λ)-BIBD exists, then $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $\lambda(v-1) \equiv 0 \pmod{k-1}$.*

From the corollary above, a $(8, 3, 1)$-BIBD does not exist. Similarly a $(19, 4, 1)$-BIBD does not exist. The set $\mathcal{M}$ of all subsets of a v-set $\mathcal{V}$ gives a BIBD$(v, b = \binom{v}{k}, r = \binom{v-1}{k-1}, k, \lambda = \binom{v-2}{k-2})$.

Example 1.13. Set $\mathcal{M}$ of $\binom{5}{3}$ blocks is a BIBD$(5, 10, 4, 2, 1)$ on a set $\mathcal{V} = \{p, q, r, s, t\}$. The corresponding value of $\mathcal{V} = \{\{p, q\}, \{p, r\}, \{p, s\}, \{p, t\}, \{q, r\}, \{q, s\}, \{q, t\}, \{r, s\}, \{r, t\}, \{s, t\}\}$

We note that when $v = k$, a $BIBD(v, k, \lambda)$ becomes a $BIBD(v, v, \lambda)$. $\mathcal{B}$ is treated as a multi-set that is a set of subsets of V. Here repeats count but the order of blocks in the list doesn't matter. Such example is illustrated below.

Example 1.14. Let $\mathcal{B} = \{\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$. This forms a $BIBD(5, 5, 4, 4, 3)$ on $V = \{a, b, c, d, e\}$. In the above example, we observe that no block is repeated and therefore the design is a simple design.

Example 1.15. Let $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Forming blocks of size three from V we obtain the following blocks.

$$\begin{gathered}\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\}\\ \{2,5,6\},\{3,6,9\},\{1,5,9\},\{2,6,7\}\\ \{3,4,8\},\{1,6,8\},\{2,4,9\},\{3,5,7\}\end{gathered}$$

The blocks above form a $BIBD(9, 12, 4, 3, 1)$.

By repeating each block in example 1.15, we obtain a $BIBD(9, 24, 4, 3, 2)$. From this, we see that when a design with a lower index exists, a multiple of such index gives a design. For any design, there exists a corresponding incidence matrix of which we define below.

Definition 1.16. The incidence matrix of a $BIBD(v, b, r, k, \lambda)$, $(\{x_1, x_2, \cdots, x_v\}, \{B_1, B_2, \cdots, B_b\})$, is a matrix $A = a_{ij}$ where $a_{ij} = \begin{cases} 1, x_i \in \mathcal{B}_j, \\ 0, otherwise \end{cases}$.

Example 1.17. For a $BIBD(9, 12, 4, 3, 1)$, the corresponding incidence matrix is given by

$$M = \begin{bmatrix} 1&0&0&1&0&0&1&0&0&1&0&0\\ 1&0&0&0&1&0&0&1&0&0&1&0\\ 1&0&0&0&0&1&0&0&1&0&0&1\\ 0&1&0&1&0&0&0&0&1&0&1&0\\ 0&1&0&0&1&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&1&1&0&0&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0&1&1&0&0\\ 0&0&1&0&0&1&1&0&0&0&1&0 \end{bmatrix}$$

Incidence matrix of a BIBD$(9, 12, 4, 3, 1)$

From the above matrix, we observe the following: Each row has four 1's. This the replication number r. Each column has three 1's. This is the block size k. The inner product of any pair of distinct rows is 1. This is the design index λ.

Definition 1.18. A symmetric balanced incomplete block design($SBIBD$) is a $BIBD$ in which $b = v$.

Example 1.19. Let $v = \{1, 2, 3, 4, 5, 6, 7\}$. Then, the blocks of size 6 are:
$\{2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 6, 7\}, \{1, 2, 3, 5, 6, 7\}, \{1, 2, 4, 5, 6, 7\}, \{1, 3, 4, 5, 6, 7\}$. This forms a $BIBD(7, 6, 5)$ and is symmetric.

1.3. **Resolvable BIBD.**

Definition 1.20. [8] A block design is α resolvable if a collection of blocks B can be partitioned into say, t, classes such that each element of v is replicated exactly

λ-times in each class. Thus for each an α resolvable design, $r = \alpha t$. For $\alpha = 1$, classes are called parallel classes and the design is called a resolvable design.

The necessary conditions for existence of RBIBD (v, k, λ) are:

(i) $\lambda(v-1) \equiv 0 \pmod{k-1}$ and

(ii) $v \equiv 0 \pmod{k}$.

For $k = 5$, there are three basic cases; $\lambda = 1, 2$ and 4, since necessary conditions on v depend only on whether $\lambda \equiv 1 \pmod 2$, $\lambda \equiv 2 \pmod 4$, or $\lambda \equiv 0 \pmod 4$. These conditions are summarized in [8] as follows:

Theorem 1.21. *Necessary conditions are sufficient for existence of a RBIBD* $(v, 5, \lambda)$ *are* $\lambda(v-1) \equiv 0 \pmod 4$ *and* $v \equiv 0 \pmod 5$. *For* $\lambda = 1, 2$ *and* 4, *these conditions are sufficient except* $(v, \lambda) \in \{(10, 4), (15, 2)\}$ *and possibly in the following cases:*

- $\lambda = 1$ *and* $v \in \{45, 225, 345, 465, 645\}$.
- $\lambda = 2$ *and* $v \in \{45, 115, 135, 195, 215, 225, 235, 295, 315, 335, 345, 395\}$.
- $\lambda = 4$ *and* $v \in \{15, 70, 90, 135, 160, 190, 195\}$.

1.4. **Latin Squares.** Latin squares are combinatorial structures that are used to construct new designs.

Definition 1.22. [8, 4] A Latin square of order n is an $n \times n$ array with entries from n distinct symbols, say $\{1, 2, ...n\}$, such that each element occurs exactly once in each row and each column.

Example 1.23. The following is an example of a Latin square of order 4.

$$L_4 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 4 & 1 & 2 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 2 & 3 & 4 & 1 \\ \hline \end{array}$$

Example 1.24. A latin square of order 7 can be obtained from cyclicing a set of seven elements. Consider a set $\{a, b, c, d, e, f, g\}$. The corresponding latin square is given by:

$$L_7 = \begin{array}{|c|c|c|c|c|c|c|} \hline \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} \\ \hline \text{g} & \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} \\ \hline \text{f} & \text{g} & \text{a} & \text{b} & \text{c} & \text{d} & \text{e} \\ \hline \text{e} & \text{f} & \text{g} & \text{a} & \text{b} & \text{c} & \text{d} \\ \hline \text{d} & \text{e} & \text{f} & \text{g} & \text{a} & \text{b} & \text{c} \\ \hline \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{a} & \text{b} \\ \hline \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{a} \\ \hline \end{array}$$

We note that the addition table of $\mathbb{Z}_n$ also provides a latin square of order n. Therefore there exists a latin square of order n for each non negative integer n.

1.5. **One and Two Factorizations of complete graphs.**

Definition 1.25. A complete graph K_n is a graph on n vertices where each distinct pair of vertices is connected by an edge.

The figure below shows a a complete graph of order 7 that is K_7.

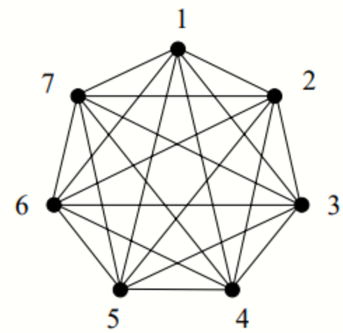

FIGURE 3. K_7

Example 1.26. Let G be a graph on a set of vertices (nodes) $V = \{a, b, c, d\}$ with set of edges $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$. This is a complete graph K_4.

Definition 1.27. A factor of a graph G, is a spanning a sub-graph of G. A k-factor of a graph is a k-regular sub-graph and a k-factorization partitions the edge set of the graph into distinct k-factors.

Definition 1.28. A 1-factor of a graph G is a set of pairwise disjoint edges which partition the vertex set of G.

Definition 1.29. A 1-factorization of a graph G is a partition of the edge set of G into 1-factors.

Example 1.30. Let G be a graph with vertex set $X = \{1, 2, 3, 4\}$. 1-factors are $F_1 = \{\{1, 2\}, \{3, 4\}\}$, $F_2 = \{\{1, 4\}, \{2, 3\}\}$, $F_3 = \{\{1, 3\}, \{2, 4\}\}$. The corresponding 1-factorization of K_4 is $F = \{F_1, F_2, F_3\}$.

It is well known that a complete graph K_v on even number of vertices v where v is a positive integer, has a 1-factorization. In fact, for all $v \equiv 2, 4 \pmod 6$ with $v \geq 4$, there exists a 1-factorization $[F_1, ..., F_v]$ of K_v. One-factorizations of K_v for even v are often written as pairs $(X; F)$ where X is a vertex set of K_v and $F = \{F_1, F_2, \cdot\cdot\cdot, F_v\}$ is a set of factors. We can also find a one-factorization of K_n by using idempotent quasi groups of order $n-1$. We note here that n is even and therefore the order $n-1$ is odd. Suppose that $(X; \circ)$ is an idempotent quasi group of order $n-1$, where $X = \{1, 2, \cdot\cdot\cdot, n-1\}$. Let $F_i = \{\{i, n\} \cup \{x \circ y\} : x \circ y = y \circ x = i\}$, then $F = \{F_1, F_2, \cdot\cdot\cdot, F_{n-1}\}$ is a one factorization of K_n on a set $S = \{1, 2, \cdot\cdot\cdot, n\}$. The vertices of Kn are labelled with distinct integers $1, 2, \cdot\cdot\cdot, n$ where the edge (e, f) is the line joining vertices e and f.

Definition 1.31. A 2-factor is a set of edges from a complete graph in which each vertex appears twice.

Definition 1.32. A 2-factorization of a complete graph K_n is a set of 2-factors that partitions the edges of a complete graph.

Example 1.33. Let G be a graph with vertex set $V = \{1,2,3,4,5\}$. The 2-factors are:
$F_1 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$, $F_2 = \{\{1,3\},\{3,5\},\{5,2\},\{2,4\},\{4,1\}\}$.
A 2-factorization of K_5 is $\{F_1, F_2\}$.

1.6. **Steiner triple systems(STS).**

Definition 1.34. A triple system is a balanced incomplete block design $BIBD(v,b,r,k,\lambda)$ with $k = 3$. That is $BIBD(v,3,\lambda)$

Definition 1.35. A Steiner triple system(STS) is an ordered pair (S,T), where S is a finite set of symbols and T is a set of 3-subset of S called triples, such that every pair of distinct elements of S occurs together in exactly one triple.(see figure below).

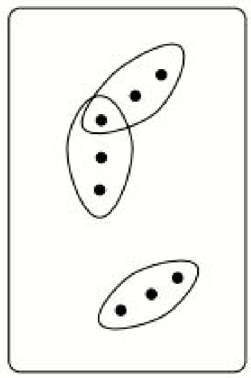

FIGURE 4. Steiner triple system (V,B)

The order of the Steiner triple system (S,T) is the size of the set S, and is denoted bt $|S|$.

Example 1.36. For $S = \{1,2,3,4,5,6,7\}$,
$T = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\}$

(S,T) of order $|S|$ is equivalent to a complete graph $K_{|S|}$, that is the equivalence between the a Steiner triple system and decomposition of K_n into triangles.

Theorem 1.37. *A Steiner triple system of order v exists if and only if $v \equiv 1,3 \pmod 6$.*

Proof. If (S,T) is a triple system of order v, any triangle $\{a,b,c\}$ contains 2-element subjects $\{a,b\},\{b,c\},\{a,c\}$ and S contains a total of $\binom{v}{2} = \dfrac{v(v-1)}{2}$ 2-element subsets. Since every pair of distinct elements of S occurs together in exactly one triple of T, we have $3|T| = \binom{v}{2}, \Rightarrow |T| = \dfrac{\binom{v}{2}}{3}$, giving us $|T| = \dfrac{v(v-1)}{6}$. Since $(v-1)$ is even, then v must $v \equiv 1,3,5, \pmod 6$. For $v = 5$, $|T| = \dfrac{5(4)}{6} \neq$ integer, hence $v \equiv 1,3 \pmod 6$. ∎

1.7. **The Bose construction (for the Steiner triple system of order $v \equiv 3 \pmod 6$).** Let $v = 6n+3$ and let $(Q;\circ)$ be an idempotent commutative quasi-group of order $2n+1$, where $Q = \{1, 2, \cdots, 2n+1\}$. Let $S = Q \times \{1, 2, 3\}$ and define T to contain the following two types of triples.
Type 1: for $1 \le i \le 2n+1$, $\{(i,1), (i,2), (i,3)\} \in T$.
Type 2 : for $1 \le i < j \le 2n+1$, $\{(i,1), (j,1), (i \circ j, 2)\}, \{(i,2), (j,2), (i \circ j, 3)\}, \{(i,3), (j,3), (i \circ j, 1)\}$.

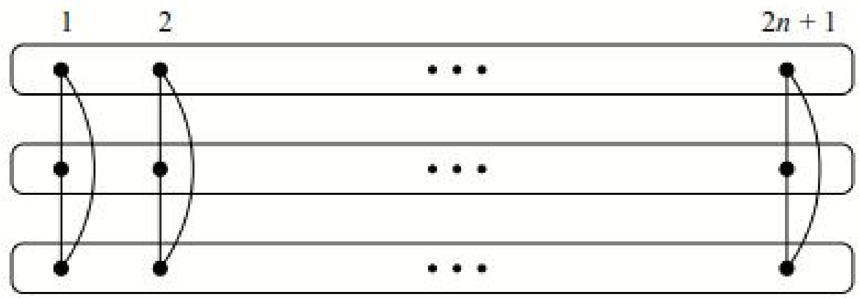

FIGURE 5. Type 1 triples

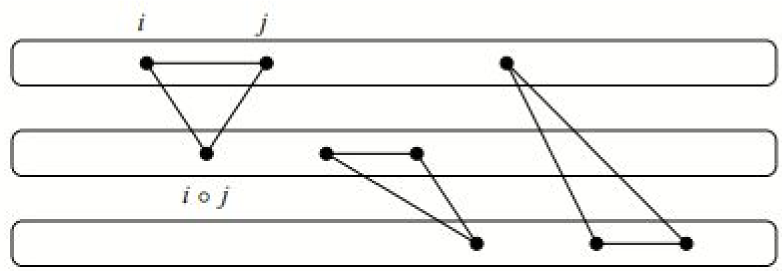

FIGURE 6. Type 2 tiples

We now prove that (S, T) is a Steiner triple system.

Proof. We begin by counting the number of triples in T. The number of type 1 triples is clearly $2n+1$. Defining type 2 triples, we have $\binom{2n+1}{2} = \dfrac{(2n+1)(2n)}{2}$ choices for i and j each giving rise to 3 triples of type 2. There are $|T| = 2n+1+\frac{3(2n+1)(2n)}{2}$, $|T| = (2n+1)(3n+1)$, but $v = \dfrac{v-3}{6}$ and we obtain $|T| = \dfrac{v(v-1)}{6}$. □

1.8. **Steiner Quadruple Systems (SQS).**

Definition 1.38. [4] A Steiner quadruple system is an ordered pair (V, B) where V is a finite set of v symbols and B a collection of 4 element subsets of V called blocks (quadruples) with the property that every three element subset of V is a subset of exactly one quadruple B. $|V|$ is called the order of the Steiner quadruple system. (See Figure below).

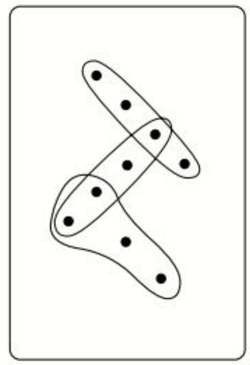

FIGURE 7. Steiner quadruple system (V, B)

We can easily find the number of quadruples in any Steiner quadruple system (V, B) of order v. Each quadruple $\{a, b, c, d\}$ in B contains the four 3-element sub-sets $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ and $\{b, c, d\}$, and altogether there are $\binom{v}{3}$ 3-element subsets of V. Since the blocks in B partition there are $\binom{v}{3}$ 3-element subsets, we have the following result.

Theorem 1.39. *The number of quadruples in an SQS(v), (V, B) is* $\dfrac{\binom{v}{3}}{3} = \dfrac{v(v-1)(v-2)}{24}$.

A well known example of a SQS is a SQS(2^n). For all $w \equiv 2, 4 \pmod 6$, there exists an SQS(w). Further more, an SQS(w) can be constructed which contains an SQS(8) except if $w \in \{1, 2, 4, 10, 14\}$.

1.9. **A classical construction of SQS(2^n).** Let V be a set of binary words of length n. Define addition coordinate-wise modulo 2. For example; Let $X = (a_1, a_2, \cdots, a_n)$, $Y = (b_1, b_2, \cdots, b_n)$.
We have $X+Y = ((a_1+b_1) \pmod 2, (a_2+b_2) \pmod 2, \cdots, (a_n+a_n) \pmod 2)$. Then (V, B) is a SQS where the set of blocks $B = \{x, y, z, w\}$, for which x, y, z and w are 4 distinct binary words of length n such that $x+y+z+w = 0 = (0, 0, 0, \cdots, 0)$. We need to prove that given any triple of distinct words say x, y, z, it is at most one block and second it is in exactly one block. We note that x, y and z are distinct. Let $w = x+y+z$. Then, $x+y+z+w = 0$. So $\{x, y, z, w\}$ may be a block containing the triple and if $x+y+z+w = 0$, then $x+y+z = w$ so the triple will be in at most one block. Note: If x, y, z and w are not all distinct, then no block contains x, y and z and if they are distinct, exactly one block will contain them. Now we must show that in fact x, y, z are distinct. To show that x, y, z are distinct, let $x = w$ then $x+w = 0$. So $x+y+z+w = 0$ becomes $y+z = 0$ and $y = z$ and we have a contradiction. So for every triple is in exactly one quadruple and we have a SQS on 2^n elements

Example 1.40. We can find the Boolean Steiner quadruple system (V, B) of order 8 as follows. Since $8 = 2^3$, V consists of all binary words of length 3; so $V = \{000, 001, 010, 011, 100, 101, 110, 111\}$. We find the $|B| = \frac{8\times7\times6}{24} = 14$ quadruples by making sure that if a, b, c and d are all different, and if $\{a + b + c + d = 0\}$, then $\{a, b, c, d\}$ is a block. For example, if $a = 100, b = 101, c = 110$ and $d = 111$, then they are all different, and $a + b + c + d = 100 + 101 + 110 + 111 = 000$; so $\{100, 101, 110, 111\} \in B$. Thus we obtain the following blocks.

{000, 001, 010, 011}	{100, 101, 110, 111}
{000, 001, 100, 101}	{010, 011, 110, 111}
{000, 001, 110, 111}	{010, 011, 100, 101}
{000, 010, 100, 110}	{001, 011, 101, 111}
{000, 010, 101, 111}	{001, 011, 100, 110}
{000, 011, 100, 111}	{001, 010, 101, 110}
{000, 011, 101, 110}	{001, 010, 100, 111}

2. Construction of a 3-GDD$(n, 2, 4; \lambda_1, \lambda_2)$ using one and two factorizations of complete graphs

In this section we give construction of 3-GDD$(n, 2, 4; \lambda_1, \lambda_2)$ using 1 and 2- factorizations of complete graphs. In particular, we consider only cases of $\lambda_1 = 0$ and $\lambda_2 = 1, 2$ which is our main result.

2.1. **Construction of 3-GDD$(n, 2, 4; 0, 1)$.** Let G_1 and G_2 be two groups of cardinality n, that is $G_1 = \{x_1, x_2, \cdots, x_n\}$ and $G_2 = \{y_1, y_2, \cdots, y_n\}$. Then by Kn on G_i we mean the vertices of a complete graph are labeled with the elements of G_i for $i = 1, 2$. Let n be even that is $n = 2t$ for any positive integer t. Then a complete graph Kn on G_1 and G_2 has a 1-factorization. Let E and F be 1-factorization on G_1 and G_2 respectively that is $E = \{E_1, E_2, \cdots, E_{n-1}\}$ on G_1 and $F = \{F_1, F_2, \cdots, F_{n-1}\}$ on G_2. Let $t = 1, 2, \cdots, n-1$, then E and F become $E = \{E_1, E_2, \cdots, E_t\}$ on G_1 and $F = \{F_1, F_2, \cdots, F_t\}$. We form blocks of size four from $\{E_i \cup F_i\} \cup \{E_i \cup F_{i+1}\} \cup \cdots \cup \{E_i \cup F_{i+t}\}$ on G_2. We have a 3-GDD$(n, 2, 4, 0, 1)$ as follows: No blocks contain three elements from the same group and hence $\lambda_1 = 0$. For every pair say $\{x, y\}$ of elements from the same group is in exactly 1- factor as an edge e. Suppose that e is in 1-factor E. The blocks which contain a pair $\{x, y\}$ that is the edge e are precisely $e \cup f_i$, for $i = 1, 2, \cdots, t$. Hence a triple of of elements in $\{x, y, z\}$ where z is an element in G_2 occurs in exactly one block. By symmetry, a triple containing two elements from G_2 and the third element from G_1 also occurs in one block. Hence $\lambda_2 = 1$.

Example 2.1. Let $X = \{1, 2, 3, 4, a, b, c, d\}$. Let $G_1 = \{1, 2, 3, 4\}$ and $G_2 = \{a, b, c, d\}$ be two groups of cardinality 4. The one factorization of a complete graph K_4 on G_1 and G_2 gives the following one factors. $E_1 = \{\{1, 2\}, \{3, 4\}\}$, $E_2 = \{\{1, 4\}, \{2, 3\}\}$, $E_3 = \{\{1, 2\}, \{2, 4\}\}$, $F_1 = \{\{a, b\}, \{c, d\}\}$, $F_2 = \{\{a, d\}, \{b, c\}\}$, $F_3 = \{\{a, c\}, \{b, d\}\}$. Forming blocks of $E_i \cup F_j$ of size four we obtain the following blocks:

$\{1,2,a,b\},\{1,2,c,d\},\{3,4,a,b\},\{3,4,c,d\},\{1,3,a,d\},\{1,3,b,c\}\{2,4,a,b\},\{2,4,b,c\},$ $\{1,4,a,c\},\{1,4,b,d\},\{2,3,a,c\},\{2,3,b,d\}$. Hence we have a 3-GDD$(n,2,4;0,1)$.

Lemma 2.2. *Necessary conditions for existence of a* 3-*GDD*$(n,2,4;0,1)$ *are sufficient.*

In example 2.1, we have constructed, a 3-GDD$(n,2,4;0,1)$

2.2. **Construction of 3-GDD($n,2,4;0,2$).** To demonstrate the construction of a 3-GDD$(n,2,4;0,2)$, we use example 2.1 for a 2-factorization of K_4 which gives a 3-GDD$(4,2,4;0,2)$.

Example 2.3. Let $G_1=\{1,2,3,4\}$ and $G_2=\{a,b,c,d\}$. For a 2-factorization of K_4 we get the following 2-factors. There is only one factor $F_1=\{1,2,3,4\}$ on G_1 and one factor $F_2=\{a,b,c,d\}$ on G_2. If we form groups of size four by cycling two factors of K_4 we obtain the blocks of size four. The blocks are shown in columns below.

1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
2	2	2	2	3	3	3	3	4	4	4	4	1	1	1	1
a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d
b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a

we notice that no triple can be obtained from each quadruple, that is no subset of three can be obtained from each group. Hence $\lambda_1=0$. Forming a triple where one element is from G_1 and two elements from G_2 we obtain two triples and hence $\lambda_2=2$ giving us a 3-GDD$(4,2,4;0,2)$. Hence for even n we obtain a 3-GDD$(n,2,4;0,2)$.

Example 2.4. Consider two groups G_1 and G_2 of group size 5 as shown. Let $G_1=\{1,2,3,4,5\}$ and $G_2=\{a,b,c,d,e\}$. Since $n=5$, we have two 2-factorizations of K_5 of each group. Let F_1, F_2 be two factorozations of G_1 and E_1, E_2 be two-factorizationsof G_2. Then, $F_1=\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$, $F_2=\{1,3\},\{3,35\},\{5,2\},\{2,4\},\{4,1\}\}$, $E_1=\{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,a\}\}$ and $E_2=\{\{a,c\},\{c,e\},\{e,b\},\{b,d\},\{d,a\}\}$. Forming blocks of size four we obtain 50 blocks as follows:

1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
2	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5	5	5	5
a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e
b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a
5	5	5	5	5	1	1	1	1	1	3	3	3	3	3	5	5	5	5	5
1	1	1	1	1	3	3	3	3	3	5	5	5	5	5	2	2	2	2	2
a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e
b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a
2	2	2	2	2	4	4	4	4	4										
4	4	4	4	4	1	1	1	1	1										
a	b	c	d	e	a	b	c	d	e										
b	c	d	e	a	b	c	d	e	a										

We again notice that no triple can be obtained from each quadruple, that is no subset of three can be obtained from each group. Hence $\lambda_1 = 0$. Forming a triple where one element is from G_1 and two elements from G_2 we obtain two triples and hence $\lambda_2 = 2$ which gives a 3-GDD$(5, 2, 4, 0, 2)$. Hence for odd n, a 3-GDD$(n, 2, 4, 0, 2)$ exists.

We generalize the above result as follows:

Theorem 2.5. *Necessary conditions for existence of a* 3-*GDD*$(n, 2, 4; \lambda_1, \lambda_2)$ *are sufficient for indices* $\lambda_1 = 0$ *and* $\lambda_2 = 1, 2$.

In examples 2.3 and 2.4, we have constructed a 3-GDD$(n, 2, 4; 0, 2)$ by cycling a two-factorization of complete graphs K_4 and K_5 respectively.

References

1. Bezire W, Owino I, Nyamala F, Construction of 3-GDDs using 3-designs. *Annals of Mathematics and Computer science* (22)
2. Hanani H, On some tactical configurations, *Canad. J. Math.*,**15(4)**(1963)702-722.
3. Lapchinda W, Punnem N and Pabhapote, GDD with associate classes and with three groups of sizes $1, n, n$ and λ_1, λ_2, *Thialand-Japan joint conference on combinatorial geometry and graph* 101 - 109.
4. Sarvate D. G., Bezire W., Group divisible designs with block size 4, *Bulletin of Institute of Combinatorics and its Applications*, (82) 2018, 56-71.
5. Sarvate D. G., Cowden W., A simple construction of 3-GDDs with block size 4 using SQS(v), *Bulletin of Institute of Combinatorics and its Applications*, (84) 2018, 40-44.
6. Sarvate D. G., Mishra N., Namyalo K., Group divisible designs with block size five from Clatworthy's table, *Communications in Statistics- Theory and Methods* 2016.
7. Sarvate D. G., Zhang L., Group divisible designs of three groups and block size five with configurations(1,2,2), *Australian Journal of Combinatorics*, 66(2) 2016, 333-343.
8. Stinson D., Combinatorial Designs: Construction and Analysis, *Springer*, 2004.
9. Shen H., Shen J., Existence of resolvable group divisible designs with block size four, *Discrete Mathematics*, (254) 2002, 513-525.
10. Zhu M., Ge G., Mixed group divisible designs with three groups and block size four, *Discrete Mathematics*, (310) 2010, 2323-2326

[1] Department of Science, Kampala International University, Uganda.
Email address: wilbroad@kiu.ac.ug

[2] Department of Mathematics, Physics and Computing, Moi University, Kenya.
Email address: nyamalafredrick27@gmail.com