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RIESZ BASIS AND ITS DUAL IN n-HILBERT SPACE

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ABSTRACT. Concepts of Riesz basis and its dual, very important concept in Hilbert space, have several application in frame theory. These notions have been developed in n-Hilbert space. A relationship between Riesz basis and Bessel sequence in n-Hilbert space are going to be established. Biorthogonal system in n-Hilbert space is discussed. Also, we give a relationship between frame and Riesz basis in a n-Hilbert space. .

1. Introduction and preliminaries

Basis plays an important role for an arbitrary vector space. The main idea of a basis is remain same in finite dimensional or infinite dimensional cases. In both cases, the elements of a basis are linearly independent which allows every element to be uniquely represented as a linear combination of the basis elements. When we are forced to work with infinite series, we shall get some other types of bases also, such as Schauder basis, orthonormal basis etc. A Riesz basis is a slight modification of an orthonormal basis in Hilbert space. A Riesz basis for a Hilbert space H is the image of an orthonormal basis for H under a bijective bounded operator i.e., a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $U : H \to H$ is a bounded bijective operator.

A frame for a separable Hilbert space is a countable family of elements which allows for a decomposition of an arbitrary element into an expansion of the frame elements, but the expansion is not necessarily unique. A frame is more flexible tool than the basis. One can think about a frame as a basis to which one has added more elements. Frame and Riesz basis play important role in theoretical research of wavelet analysis, signal denoising, feature extraction, robust signal processing and so on.

Frame theory in abstract Hilbert spaces has been introduced by Duffin and Schaeffer [6] in 1952. They abstracted the fundamental notion of Gabor frames [7] for studying signal processing. Frames were appeared to generate much interest outside of signal processing after the work of Daubechies et al.[3] in 1986.

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The idea of linear 2-normed space was first introduced by S. Gahler [8]. A generalization of this idea for $n \geq 2$ was developed by H. Gunawan and Mashadi [9]. Diminnie et al.[4, 5] introduced the concept of 2-inner product space. In 1989, A. Misiak [16] developed a generalization of a 2-inner product space for $n \geq 2$ and studied the orthogonality and orthonormality in n-inner product space. Frame in 2-inner product space was introduced by A. A. Arefijamaal, G. Sadeghi [1]. P. Ghosh and T. K. Samanta [11, 12] presented the construction of a frame and atomic system in n-Hilbert space. They also discussed frame in tensor product of n-Hilbert spaces [13]. For few generalizations of frame in n-Hilbert spaces and their tensor products one can go through the papers [14, 15].

In this paper, we discuss the idea of a Riesz basis and the expansion property in a n-Hilbert space. Further we establish a relationship among Riesz basis, Bessel sequence and frame in n-Hilbert space. Finally we give a orthogonality relationship between a Riesz basis and its dual Riesz basis in n-Hilbert space.

Throughout this paper, H will denote separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $l^2(\mathbb{N})$ denote the space of square summable scalar-valued sequences with index set of natural numbers $\mathbb{N}$.

Definition 1.1. [2] Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in H. Then $\{f_k\}_{k=1}^{\infty}$ is said to be a frame for H if there exist positive constants A, B such that

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 \;\; \forall f \in H. \tag{1.1}$$

The constants A and B are called frame bounds. If the sequence $\{f_k\}_{k=1}^{\infty}$ satisfies only the right inequality of (1.1) then it is called a Bessel sequence with bound B.

Theorem 1.2. [2] *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in H. Then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence in H if the series $\sum_{k=1}^{\infty} c_k f_k$ is convergent for all $\{c_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$.*

Definition 1.3. [9] Let X be a linear space of dimension greater than equal to n over the field $\mathbb{K}$ of real or complex numbers. A real valued function $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ is called a n-norm on X if the following conditions hold:

(*i*) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,
(*ii*) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutations of $x_1, x_2, \cdots, x_n$,
(*iii*) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \, \|x_1, x_2, \cdots, x_n\| \quad \forall \alpha \in \mathbb{K}$,
(*iv*) $\|x + y, x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$.

The pair $(X, \|\cdot, \cdots, \cdot\|)$ is then called a linear n-normed space.

Definition 1.4. [16] Let $n \in \mathbb{N}$ and X be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. A function $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle : X^{n+1} \to \mathbb{K}$ is satisfying the following five properties:

(*i*) $\langle x_1, x_1 | x_2, \cdots, x_n \rangle \geq 0$ and $\langle x_1, x_1 | x_2, \cdots, x_n \rangle = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,

(ii) $\langle x, y \,|\, x_2, \cdots, x_n \rangle = \langle x, y \,|\, x_{i_2}, \cdots, x_{i_n} \rangle$ for every permutation $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$,

(iii) $\langle x, y \,|\, x_2, \cdots, x_n \rangle = \overline{\langle y, x \,|\, x_2, \cdots, x_n \rangle}$,

(iv) $\langle \alpha x, y \,|\, x_2, \cdots, x_n \rangle = \alpha \langle x, y \,|\, x_2, \cdots, x_n \rangle$, for all $\alpha \in \mathbb{K}$,

(v) $\langle x + y, z \,|\, x_2, \cdots, x_n \rangle = \langle x, z \,|\, x_2, \cdots, x_n \rangle + \langle y, z \,|\, x_2, \cdots, x_n \rangle$.

is called an n-inner product on X and the pair $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ is called an n-inner product space.

Theorem 1.5. *(Schwarz inequality)*[10] *Let $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ be a n-inner product space and $x, y, x_2, \cdots, x_n \in X$. Then*

$$|\langle x, y \,|\, x_2, \cdots, x_n \rangle| \leq \| x, x_2, \cdots, x_n \| \, \| y, x_2, \cdots, x_n \|,$$

where

$$\| x_1, x_2, \cdots, x_n \| = \sqrt{\langle x_1, x_1 \,|\, x_2, \cdots, x_n \rangle}.$$

Definition 1.6. [9] Let $(X, \| \cdot, \cdots, \cdot \|)$ be a linear n-normed space. A sequence $\{x_k\}$ in X is said to be convergent if there exist an $x \in X$ such that

$$\lim_{k \to \infty} \| x_k - x, e_2, \cdots, e_n \| = 0$$

for every $e_2, \cdots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{k, l \to \infty} \| x_k - x_l, e_2, \cdots, e_n \| = 0$$

for every $e_2, \cdots, e_n \in X$. The space X is said to be complete if every Cauchy sequence in X converges to an element in X. A n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm.

Let $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ be a n-Hilbert space and $a_2, a_3, \cdots, a_n$ be the fixed elements in X. Let L_F denote the linear subspace of X spanned by the non-empty finite set $F = \{a_2, a_3, \cdots, a_n\}$. Then the quotient space X / L_F is a normed linear space with respect to the norm

$$\| x + L_F \|_F = \| x, a_2, \cdots, a_n \| \text{ for every } x \in X.$$

Let M_F be the algebraic complement of L_F, then $X = L_F \oplus M_F$. Define

$$\langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle \text{ on } X.$$

Then $\langle \cdot, \cdot \rangle_F$ is a semi-inner product on X and this semi-inner product induces an inner product on the quotient space X / L_F which is given by

$$\langle x + L_F, y + L_F \rangle_F = \langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle \quad \forall \; x, y \in X.$$

By identifying X / L_F with M_F in an obvious way, we obtain an inner product on M_F. Now for every $x \in M_F$, we define $\| x \|_F = \sqrt{\langle x, x \rangle_F}$ and it can be easily verify that $(M_F, \| \cdot \|_F)$ is a norm space. Let X_F be the completion of the inner product space M_F.

For the remaining part of this paper, $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ is consider to be a n-Hilbert space. I_F will denote the identity operator on X_F and $\mathcal{B}(X_F)$ denote the space of all bounded linear operator on X_F [11].

Definition 1.7. [11] Let $\{ f_k \}_{k=1}^{\infty}$ be a sequence in X. Then $\{ f_k \}_{k=1}^{\infty}$ is said to be a frame associated to $(a_2, \cdots, a_n)$ for X, if for each $f \in X$ there exist constants $A, B > 0$ such that

$$A \, \| f, a_2, \cdots, a_n \|^2 \leq \sum_{k=1}^{\infty} | \langle f, f_k \,|\, a_2, \cdots, a_n \rangle |^2 \leq B \, \| f, a_2, \cdots, a_n \|^2 .$$

The infimum of all such B is called the optimal upper frame bound and supremum of all such A is called the optimal lower frame bound. A sequence $\{ f_k \}_{i=1}^{\infty}$ satisfies only the right inequality of the above inequality is called a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X with bound B.

Theorem 1.8. [11] *A sequence $\{ f_k \}_{k=1}^{\infty} \subseteq X$ is a frame associated to $(a_2, \cdots, a_n)$ with bounds A and B if and only if it is a frame for the Hilbert space X_F with bounds A and B.*

2. Riesz Basis and it's properties in n-Hilbert space

In this section, we shall discuss Riesz basis and its dual in n-Hilbert space and describe some of their properties. The relation among the Bessel sequence, frame and Riesz basis in n-Hilbert space are going to be established.

Theorem 2.1. *A sequence $\{ f_k \}_{k=1}^{\infty} \subseteq X$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ with Bessel bound B if and only if it is a Bessel sequence in the Hilbert space X_F with bound B.*

Proof. Proof of this Theorem directly follows from the Theorem 1.8. □

Definition 2.2. Let $\{ e_i \}_{i=1}^{\infty}$ be a linearly independent set of vectors in X. Then $\{ e_i \}_{i=1}^{\infty}$ is said to be an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X if

(*i*) $\langle e_i, e_j \,|\, a_2, \cdots, a_n \rangle = \delta_{ij} \quad \forall \; i, j \in \mathbb{N}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j . \end{cases}$$

(*ii*) $f = \sum\limits_{i=1}^{\infty} \langle f, e_i \,|\, a_2, \cdots, a_n \rangle \, e_i$, for all $f \in X$.

If $\{ e_i \}_{i=1}^{\infty}$ be an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X, then it is easy to verify that

$$\| f, a_2, \cdots, a_n \|^2 = \sum_{i=1}^{\infty} | \langle f, e_i \,|\, a_2, \cdots, a_n \rangle |^2 ,$$

for all $f \in X$.

Theorem 2.3. *An orthonormal sequence $\{ e_k \}_{k=1}^{\infty}$ associated to $(a_2, \cdots, a_n)$ in X_F is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X.*

Proof. Let $\{d_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$. Then

$$\left\| \sum_{k=1}^{q} d_k e_k - \sum_{k=1}^{p} d_k e_k \right\|_F^2 = \left\| \sum_{k=p+1}^{q} d_k e_k , a_2 , \cdots , a_n \right\|^2$$

$$= \left\langle \sum_{k=p+1}^{q} d_k e_k , \sum_{j=p+1}^{q} d_j e_j \,|\, a_2 , \cdots , a_n \right\rangle$$

$$= \sum_{k=p+1}^{q} d_k \left\langle e_k , \sum_{j=p+1}^{q} d_j e_j \,|\, a_2 , \cdots , a_n \right\rangle$$

$$= \sum_{k=p+1}^{q} d_k \langle e_k , d_{p+1} e_{p+1} + \cdots + d_q e_q \,|\, a_2 , \cdots , a_n \rangle$$

$$= \sum_{k=p+1}^{q} d_k \{ \langle e_k , d_{p+1} e_{p+1} \,|\, a_2 , \cdots , a_n \rangle + \cdots + \langle e_k , d_q e_q \,|\, a_2 , \cdots , a_n \rangle \}$$

$$= \sum_{k=p+1}^{q} d_k \{ \overline{d_{p+1}} \langle e_k , e_{p+1} \,|\, a_2 , \cdots , a_n \rangle + \cdots + \overline{d_q} \langle e_k , e_q \,|\, a_2 , \cdots , a_n \rangle \}$$

$$= \sum_{k=p+1}^{q} d_k \overline{d_k} \; [\text{ since } \langle e_k , e_j \,|\, a_2 , \cdots , a_n \rangle = \delta_{kj} \;\; \forall \; k , j \,]$$

$$= \sum_{k=p+1}^{q} |d_k|^2 .$$

This implies that $\sum\limits_{k=1}^{\infty} d_k e_k$ is convergent in X_F if $\{d_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$. Using the continuity of n-norm function we get $\left\| \sum\limits_{k=1}^{\infty} d_k e_k \right\|_F^2 = \sum\limits_{k=1}^{\infty} |d_k|^2$. By Theorem 1.2, convergence of $\sum\limits_{k=1}^{\infty} d_k e_k$ implies that $\{e_k\}_{k=1}^{\infty}$ is a Bessel sequence in X_F and hence by the Theorem 2.1, $\{e_k\}_{k=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X. This completes the proof. □

Definition 2.4. A Riesz basis associated to $(a_2, \cdots, a_n)$ for X is a family of the form $\{U e_k\}_{k=1}^{\infty}$ where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and $U : X_F \to X_F$ is a bounded linear bijective operator.

Theorem 2.5. *Let $\{f_k\}_{k=1}^{\infty}$ be a Riesz basis associated to $(a_2, \cdots, a_n)$ for X. Then the sequence $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X. Furthermore, there exists a sequence $\{g_k\}_{k=1}^{\infty}$ such that*

$$f = \sum_{k=1}^{\infty} \langle f , g_k \,|\, a_2 , \cdots , a_n \rangle f_k \quad \forall \; f \in X_F .$$

Proof. Since $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis associated to $(a_2, \cdots, a_n)$, we can write $\{f_k\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and $U : X_F \to X_F$ is a bounded linear bijective operator. Now, for each $f \in X_F$, we have

$$\sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2 = \sum_{k=1}^{\infty} |\langle f, U e_k | a_2, \cdots, a_n \rangle|^2$$
$$= \sum_{k=1}^{\infty} |\langle U^* f, e_k | a_2, \cdots, a_n \rangle|^2 = \|U^* f, a_2, \cdots, a_n\|^2$$
$$\leq \|U\|^2 \|f, a_2, \cdots, a_n\|^2.$$

This shows that $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence in X_F and hence by the Theorem 2.1, $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X. Since $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$, we obtain

$$U^{-1} f = \sum_{k=1}^{\infty} \left\langle U^{-1} f, e_k | a_2, \cdots, a_n \right\rangle e_k$$
$$= \sum_{k=1}^{\infty} \left\langle f, \left(U^{-1}\right)^* e_k | a_2, \cdots, a_n \right\rangle e_k.$$

Now,

$$f = U U^{-1} f = U \left(\sum_{k=1}^{\infty} \left\langle f, \left(U^{-1}\right)^* e_k | a_2, \cdots, a_n \right\rangle e_k \right)$$
$$= \sum_{k=1}^{\infty} \left\langle f, \left(U^{-1}\right)^* e_k | a_2, \cdots, a_n \right\rangle U e_k$$
$$= \sum_{k=1}^{\infty} \langle f, g_k | a_2, \cdots, a_n \rangle f_k \;\; \forall f \in X_F.$$

Since $(U^{-1})^*$ is bounded and bijective, so $\{g_k\}_{k=1}^{\infty} = \left\{(U^{-1})^* e_k\right\}_{k=1}^{\infty}$ is also Riesz basis associated to $(a_2, \cdots, a_n)$. This completes the proof. □

Remark 2.6. Let the sequence $\{f_k\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty}$ be a Riesz basis associated to $(a_2, \cdots, a_n)$ for X, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and U be a bounded linear bijective operator on X_F. Then according to the Theorem 2.5, the sequence $\{g_k\}_{k=1}^{\infty} = \left\{(U^{-1})^* e_k\right\}_{k=1}^{\infty}$ is also a Riesz basis associated to $(a_2, \cdots, a_n)$ for X. The sequence $\{g_k\}_{k=1}^{\infty}$ will be called dual Riesz basis associated to $(a_2, \cdots, a_n)$ of $\{f_k\}_{k=1}^{\infty}$.

Definition 2.7. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two sequences in X. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are said to be biorthogonal associated to $(a_2, \cdots, a_n)$ if

$$\langle f_k, g_j | a_2, \cdots, a_n \rangle = \delta_{kj} \;\; \forall k, j \in \mathbb{N}, \quad \text{where } \delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

Theorem 2.8. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be a pair of dual Riesz basis associated to $(a_2, \cdots, a_n)$ for X. Then*

(*i*) $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are biorthogonal associated to* $(a_2, \cdots, a_n)$.

(*ii*) *For every* $f \in X_F$,

$$f = \sum_{k=1}^{\infty} \langle f, g_k | a_2, \cdots, a_n \rangle f_k = \sum_{k=1}^{\infty} \langle f, f_k | a_2, \cdots, a_n \rangle g_k.$$

Proof. (i) Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are pair of dual Riesz basis associated to $(a_2, \cdots, a_n)$, we can write

$$\{f_k\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty} \quad \& \quad \{g_k\}_{k=1}^{\infty} = \left\{ \left(U^{-1}\right)^* e_k \right\}_{k=1}^{\infty},$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and U is a bounded linear bijective operator on X_F. Now,

$$\begin{aligned}
\langle f_k, g_j | a_2, \cdots, a_n \rangle &= \left\langle f_k, \left(U^{-1}\right)^* e_j | a_2, \cdots, a_n \right\rangle \\
&= \left\langle U e_k, \left(U^{-1}\right)^* e_j | a_2, \cdots, a_n \right\rangle \\
&= \left\langle U^{-1} U e_k, e_j | a_2, \cdots, a_n \right\rangle \\
&= \langle e_k, e_j | a_2, \cdots, a_n \rangle = \delta_{kj}.
\end{aligned}$$

Therefore $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are biorthogonal associated to $(a_2, \cdots, a_n)$.

(ii) Now, for each $f \in X_F$,

$$\begin{aligned}
U^* f &= \sum_{k=1}^{\infty} \langle U^* f, e_k | a_2, \cdots, a_n \rangle e_k = \sum_{k=1}^{\infty} \langle f, U e_k | a_2, \cdots, a_n \rangle e_k \\
&= \sum_{k=1}^{\infty} \langle f, f_k | a_2, \cdots, a_n \rangle e_k.
\end{aligned}$$

Therefore, for each $f \in X_F$, we have

$$\begin{aligned}
f &= (U^*)^{-1} U^* f = (U^*)^{-1} \left(\sum_{k=1}^{\infty} \langle f, f_k | a_2, \cdots, a_n \rangle e_k \right) \\
&= \sum_{k=1}^{\infty} \langle f, f_k | a_2, \cdots, a_n \rangle (U^*)^{-1} e_k \\
&= \sum_{k=1}^{\infty} \langle f, f_k | a_2, \cdots, a_n \rangle \left(U^{-1}\right)^* e_k \\
&= \sum_{k=1}^{\infty} \langle f, f_k | a_2, \cdots, a_n \rangle g_k.
\end{aligned}$$

And the first part of (ii) follows from the Theorem 2.5. This completes the proof. $\square$

Theorem 2.9. *Let $\{f_k\}_{k=1}^{\infty}$ be a Riesz basis associated to $(a_2, \cdots, a_n)$ for X. Then for every $f \in X_F$, there exist constants $A, B > 0$ such that*

$$A \|f, a_2, \cdots, a_n\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2 \leq B \|f, a_2, \cdots, a_n\|^2.$$

Proof. Since $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis associated to $(a_2, \cdots, a_n)$ for X, we can write $\{f_k\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and $U : X_F \to X_F$ is a bounded linear bijective operator. By the Theorem 2.5, $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X, so there exists $B > 0$ such that

$$\sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2 \leq B \|f, a_2, \cdots, a_n\|^2 \quad \forall \; f \in X_F,$$

where $B = \|U\|^2$. Now, for each $f \in X_F$, we have

$$\begin{aligned} \|f, a_2, \cdots, a_n\| &= \left\| \left(U^{-1}\right)^* U^* f, a_2, \cdots, a_n \right\| \\ &\leq \left\| \left(U^{-1}\right)^* \right\| \|U^* f, a_2, \cdots, a_n\| \\ &= \left\| U^{-1} \right\| \|U^* f, a_2, \cdots, a_n\|. \end{aligned} \tag{2.1}$$

Since

$$\begin{aligned} \|U^* f, a_2, \cdots, a_n\|^2 &= \sum_{k=1}^{\infty} |\langle U^* f, e_k | a_2, \cdots, a_n \rangle|^2 \\ &= \sum_{k=1}^{\infty} |\langle f, U e_k | a_2, \cdots, a_n \rangle|^2 \\ &= \sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2 \end{aligned}$$

and therefore by (2.1), we have

$$\begin{aligned} \|f, a_2, \cdots, a_n\|^2 &\leq \left\| U^{-1} \right\| \|U^* f, a_2, \cdots, a_n\| \\ &= \left\| U^{-1} \right\|^2 \sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2. \end{aligned}$$

This implies that

$$\frac{1}{\|U^{-1}\|^2} \|f, a_2, \cdots, a_n\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2.$$

Therefore, for each $f \in X_F$, we get

$$A \|f, a_2, \cdots, a_n\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k | a_2, \cdots, a_n \rangle|^2 \leq B \|f, a_2, \cdots, a_n\|^2,$$

where $A = \dfrac{1}{\|U^{-1}\|^2}$ and $B = \|U\|^2$. This completes the proof. □

Theorem 2.10. *A Riesz basis associated to* $(a_2, \cdots, a_n)$ *in* X *is a frame associated to* $(a_2, \cdots, a_n)$ *for* X.

Proof. Let $\{f_k\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty}$ be a Riesz basis associated to $(a_2, \cdots, a_n)$ in X, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and U is a bounded linear bijective operator on X_F, then by Theorem 2.9, for each $f \in X_F$, we have

$$A \, \| f, a_2, \cdots, a_n \|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \, \| f, a_2, \cdots, a_n \|^2,$$

where $A = \dfrac{1}{\| U^{-1} \|^2}$ and $B = \| U \|^2$. Now, according to the Theorem 1.8, it follows that $\{f_k\}_{k=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X. □

Theorem 2.11. *Let* $\{f_k\}_{k=1}^{\infty} \subseteq X$ *be a Riesz basis associated to* $(a_2, \cdots, a_n)$. *Then there exist* $A, B > 0$ *such that for every finite scalar sequence* $\{c_k\}$,

$$A \sum |c_k|^2 \leq \left\| \sum c_k f_k, a_2, \cdots, a_n \right\|^2 \leq B \sum |c_k|^2.$$

Proof. Since $\{f_k\}_{k=1}^{\infty}$ is Riesz basis associated to $(a_2, \cdots, a_n)$, so $\{f_k\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F and U is a bounded linear bijective operator on X_F. Now,

$$\begin{aligned}
\left\| \sum c_k f_k, a_2, \cdots, a_n \right\|^2 &= \left\| \sum c_k U e_k, a_2, \cdots, a_n \right\|^2 \\
&= \left\| \sum U(c_k e_k), a_2, \cdots, a_n \right\|^2 \\
&\leq \| U \|^2 \left\| \sum c_k e_k, a_2, \cdots, a_n \right\|^2.
\end{aligned}$$

Now, we get that

$$\begin{aligned}
&\left\| \sum c_k e_k, a_2, \cdots, a_n \right\|^2 = \sum_j \left| \left\langle \sum_k c_k e_k, e_j \,\middle|\, a_2, \cdots, a_n \right\rangle \right|^2 \\
&= \sum_j \left| \sum_k c_k \langle e_k, e_j \,|\, a_2, \cdots, a_n \rangle \right|^2 \\
&= \sum_k |c_k|^2 \quad [\text{ since for } j = k, \langle e_k, e_j \,|\, a_2, \cdots, a_n \rangle = 1].
\end{aligned}$$

From the above two calculation we get,

$$\left\| \sum_k c_k f_k, a_2, \cdots, a_n \right\|^2 \leq \| U \|^2 \sum_k |c_k|^2. \tag{2.2}$$

Also, we have

$$\sum_k |c_k|^2 = \left\| \sum_k c_k e_k , a_2 , \cdots , a_n \right\|^2$$

$$= \left\| U^{-1} U \left(\sum_k c_k e_k \right), a_2, \cdots, a_n \right\|^2 = \left\| U^{-1} \left(\sum_k c_k U e_k \right), a_2, \cdots, a_n \right\|^2$$

$$= \left\| U^{-1} \left(\sum_k c_k f_k \right), a_2, \cdots, a_n \right\|^2 \leq \| U^{-1} \|^2 \left\| \sum_k c_k f_k, a_2, \cdots, a_n \right\|^2 .$$

Therefore

$$\frac{1}{\| U^{-1} \|^2} \sum_k |c_k|^2 \leq \left\| \sum_k c_k f_k , a_2 , \cdots , a_n \right\|^2 . \tag{2.3}$$

From (2.2) and (2.3), we get

$$\frac{1}{\| U^{-1} \|^2} \sum_k |c_k|^2 \leq \left\| \sum_k c_k f_k , a_2 , \cdots , a_n \right\|^2 \leq \| U \|^2 \sum_k |c_k|^2 .$$

Hence,

$$A \sum_k |c_k|^2 \leq \left\| \sum_k c_k f_k , a_2 , \cdots , a_n \right\|^2 \leq B \sum_k c_k|^2 ,$$

where $A = \dfrac{1}{\| U^{-1} \|^2}$ and $B = \| U \|^2$. This completes the proof. $\square$

Theorem 2.12. *Let $\{ f_k \}_{k=1}^{\infty}$ and $\{ g_k \}_{k=1}^{\infty}$ be two complete Bessel sequences associated to $(a_2, \cdots, a_n)$ in X. If $\{ f_k \}_{k=1}^{\infty}$ and $\{ g_k \}_{k=1}^{\infty}$ are biorthogonal associated to $(a_2, \cdots, a_n)$, then $\{ f_k \}_{k=1}^{\infty}$ is Riesz basis associated to $(a_2, \cdots, a_n)$ for X.*

Proof. Since $\{ f_k \}_{k=1}^{\infty}$ is complete, for every $f \in \operatorname{span} \{ f_k \}_{k=1}^{\infty}$ has a unique representation of the form $f = \sum\limits_{k=1}^{\infty} c_k f_k$, where $\{ c_k \}$ is a finite scalar sequence. Also, $\{ f_k \}_{k=1}^{\infty}$ and $\{ g_k \}_{k=1}^{\infty}$ are biorthogonal associated to $(a_2, \cdots, a_n)$, so by (ii) of the Theorem 2.8, for each $f \in X_F$, we get

$$f = \sum_{k=1}^{\infty} \langle f , g_k | a_2 , \cdots , a_n \rangle f_k .$$

Let $\{ e_k \}_{k=1}^{\infty}$ be an orthonormal basis associated to $(a_2, \cdots, a_n)$ for X_F. Define

$$U : \operatorname{span} \{ f_k \}_{k=1}^{\infty} \to X_F \text{ by } U(f_k) = e_k .$$

Now, for $f \in \operatorname{span} \{ f_k \}_{k=1}^{\infty}$, we have

$$\| U f , a_2 , \cdots , a_n \|^2 = \langle U f , U f | a_2 , \cdots , a_n \rangle$$

$$= \left\langle \sum_{k=1}^{\infty} \langle f, g_k \,|\, a_2, \cdots, a_n \rangle \, e_k, \sum_{j=1}^{\infty} \langle f, g_j \,|\, a_2, \cdots, a_n \rangle \, e_j \,|\, a_2, \cdots, a_n \right\rangle$$

$$= \left\langle \lim_{m \to \infty} \sum_{k=1}^{m} \langle f, g_k \,|\, a_2, \cdots, a_n \rangle \, e_k, \lim_{m \to \infty} \sum_{j=1}^{m} \langle f, g_j \,|\, a_2, \cdots, a_n \rangle \, e_j \,|\, a_2, \cdots, a_n \right\rangle$$

$$= \lim_{m \to \infty} \left\langle \sum_{k=1}^{m} \langle f, g_k \,|\, a_2, \cdots, a_n \rangle \, e_k, \sum_{j=1}^{m} \langle f, g_j \,|\, a_2, \cdots, a_n \rangle \, e_j \,|\, a_2, \cdots, a_n \right\rangle$$

[using the contnuity of n-inner product function]

$$= \lim_{m \to \infty} \sum_{k=1}^{m} \langle f, g_k \,|\, a_2, \cdots, a_n \rangle \left\langle e_k, \sum_{j=1}^{m} \langle f, g_j \,|\, a_2, \cdots, a_n \rangle \, e_j \,|\, a_2, \cdots, a_n \right\rangle$$

$$= \lim_{m \to \infty} \sum_{k=1}^{m} |\langle f, g_k \,|\, a_2, \cdots, a_n \rangle|^2 \; [\text{ since } \langle e_k, e_j \,|\, a_2, \cdots, a_n \rangle = \delta_{kj} \;\; \forall \; k, j \,]$$

$$= \sum_{k=1}^{\infty} |\langle f, g_k \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \, \| f, a_2, \cdots, a_n \|^2$$

[since $\{ g_k \}_{k=1}^{\infty}$ is a Bessel sequence assoicated to $(a_2, \cdots, a_n)$].

By the completeness of $\{ f_k \}_{k=1}^{\infty}$, U has an extension to bounded operator on X_F. By the assumption, we can also extend $V g_k = e_k$ to bounded operator on X_F. Let us now consider the linear combinations of $\{ f_k \}_{k=1}^{\infty}$ and $\{ g_k \}_{k=1}^{\infty}$,

$f = \sum_{k=1}^{\infty} c_k f_k$, and $g = \sum_{k=1}^{\infty} d_k g_k$, where $\{ c_k \}$, $\{ d_k \}$ are finite scalar sequences.

Now

$$\begin{aligned} \langle U f, V g \,|\, a_2, \cdots, a_n \rangle &= \left\langle \sum_{k=1}^{\infty} c_k e_k, \sum_{k=1}^{\infty} d_k e_k \,|\, a_2, \cdots, a_n \right\rangle \\ &= \sum_{k=1}^{\infty} c_k \overline{d_k} = \langle f, g \,|\, a_2, \cdots, a_n \rangle \end{aligned}$$

[since $\{ f_k \}_{k=1}^{\infty}$, $\{ g_k \}_{k=1}^{\infty}$ are biorthogonal sequence assoicated to $(a_2, \cdots, a_n)$].

Thus, by continuity and completeness, we get

$$\langle U f, V g \,|\, a_2, \cdots, a_n \rangle = \langle f, g \,|\, a_2, \cdots, a_n \rangle \;\; \forall \, f, g \in X_F.$$

Therefore, for any $p \in X_F$, we have

$$\begin{aligned} \| p, a_2, \cdots, a_n \|^2 &= \langle p, p \,|\, a_2, \cdots, a_n \rangle = \langle U p, V p \,|\, a_2, \cdots, a_n \rangle \\ &\leq \| U p, a_2, \cdots, a_n \| \, \| V p, a_2, \cdots, a_n \| \\ &\leq \| V \| \, \| p, a_2, \cdots, a_n \| \, \| U p, a_2, \cdots, a_n \| . \end{aligned}$$

This shows that U is injective. Now, for given $q \in X_F$, we can write

$$q = \sum_{k=1}^{\infty} \langle q, e_k | a_2, \cdots, a_n \rangle e_k$$
$$= U \left(\sum_{k=1}^{\infty} \langle q, e_k | a_2, \cdots, a_n \rangle f_k \right) [\text{ since } U f_k = e_k]$$

and this verifies that U is surjective. Also, it is easy to verify that $f_k = U^{-1} e_k$. Hence, $\{ f_k \}_{k=1}^{\infty}$ is Riesz basis associated to $(a_2, \cdots, a_n)$ for X. □

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