

A STUDY OF COMPLETE LATTICES OF COVERING SPACES

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ABSTRACT. Let $C(X)$ denote the set of all covering spaces $(\tilde{X}, \tilde{x}, p)$ of (X, x) where (X, x) are path connected, locally path connected and semilocally simply connected pointed topological spaces. In [1], it is shown that $(C(X), \geq)$ is a lattice with respect to the partial order relation $' \geq'$. Also $(C(X), \geq)$ is a modular, bounded and complete lattice when $\pi(X, x)$ is abelian. In this paper, we will study some properties of the complete lattice $(C(X), \geq)$.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper we assume that all the spaces (X, x) are path connected, locally path connected and semilocally simply connected pointed topological spaces and maps are base point preserving continuous maps.

A covering space of a space (X, x) is a triple $(\tilde{X}, \tilde{x}, p)$ consisting of a pointed space $(\tilde{X}, \tilde{x})$ and a continuous surjective map $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ such that each point $x \in X$ has a path connected open neighborhood U such that each path component of $p^{-1}(U)$ is mapped homeomorphically onto U by p .

Let $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2)$ be two covering spaces of (X, x) . A homomorphism of $(\tilde{X}_1, \tilde{x}_1, p_1)$ into $(\tilde{X}_2, \tilde{x}_2, p_2)$ is a base point preserving continuous map $f : (\tilde{X}_1, \tilde{x}_1) \rightarrow (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 f = p_1$. If in particular, f is a homeomorphism, then the coverings $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2)$ are said to be isomorphic. Let $C(X)$ denote the set of all covering spaces $(\tilde{X}, \tilde{x}, p)$ of (X, x) . Then for each $(\tilde{X}, \tilde{x}, p) \in C(X)$, the map $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ induces a monomorphism $p_* : \pi((\tilde{X}, \tilde{x})) \rightarrow \pi(X, x)$ in the corresponding fundamental groups. The image group $H = p_* \pi((\tilde{X}, \tilde{x}))$ depends on the choice of the base point $\tilde{x} \in p^{-1}(x)$.

Let $(S, \leq)$ be a partially ordered set and a, b are any two elements of S . The least upper bound of $\{a, b\}$ in $(S, \leq)$, if it exists, is denoted by $a \vee b$. Similarly the greatest lower bound of $\{a, b\}$ in $(S, \leq)$, if it exists, is denoted by $a \wedge b$.

A poset $(S, \leq)$ is called an upper semi lattice if $a \vee b$ exists in S for all $a, b \in S$. Similarly a poset $(S, \leq)$ is called a lower semi lattice if $a \wedge b$ exists in S for all $a, b \in S$.

A poset $(L, \leq)$ is called a lattice if $a \vee b$ and $a \wedge b$ exists in L for all $a, b \in L$.

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Let $(L, \leq)$ be a lattice and L' be a nonempty subset of L such that $a \vee b$ and $a \wedge b$ exists in L' for all $a, b \in L'$, then $(L', \leq)$ is called a sublattice of $(L, \leq)$.

A lattice $(L, \leq)$ is called a modular lattice if for all $a, b, c \in L$, $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$ and is called a distributive lattice if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

A lattice $(L, \leq)$ is called a complete lattice if every subsets of L have both a lub and a glb.

2. USEFUL RESULTS TO PROVE OUR THEOREMS

Lemma 2.1. ([2]) *Two covering spaces $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2)$ of (X, x) such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$ are isomorphic if and only if the subgroups $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$ and $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ belong to the same conjugacy class in $\pi(X, x)$.*

Lemma 2.2. ([2]) *Given a subgroup H of $\pi(X, x)$, there exists a covering space $(\tilde{X}, \tilde{x}, p)$ of (X, x) such that $p_*\pi(\tilde{X}, \tilde{x}) = H$.*

Let $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2) \in C(X)$. Define a binary relation ρ on $C(X)$ by $(\tilde{X}_1, \tilde{x}_1, p_1)\rho(\tilde{X}_2, \tilde{x}_2, p_2) \Leftrightarrow p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$. Then ρ is an equivalence relation.

Let $C(X)/\rho$ denote the set of all ρ equivalence classes $(\tilde{X}, \tilde{x}, p)\rho$ of the coverings of (X, x) . Define $' \geq'$ on $C(X)/\rho$ by $(\tilde{X}_1, \tilde{x}_1, p_1)\rho \geq (\tilde{X}_2, \tilde{x}_2, p_2)\rho \Leftrightarrow p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) \subseteq p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$.

Proposition 2.3. ([1]) *$' \geq'$ is a partial order relation on $C(X)/\rho$.*

Theorem 2.4. ([1]) *The partially ordered set $(C(X)/\rho, \geq)$ is a semilattice.*

Theorem 2.5. ([1]) *Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a lattice.*

Theorem 2.6. ([1]) *Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a bounded lattice.*

Theorem 2.7. ([1]) *Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a complete lattice.*

Definition 2.8. For the poset $(C(X), \geq)$, a map $f : (C(X), \geq) \rightarrow (C(X), \geq)$ is called a closure operator on $(C(X), \geq)$ if for all $(\tilde{X}_1, \tilde{x}_1, p_1), (\tilde{X}_2, \tilde{x}_2, p_2) \in C(X)$

- (clo1) $f((\tilde{X}_1, \tilde{x}_1, p_1)) \geq (\tilde{X}_1, \tilde{x}_1, p_1)$,
- (clo2) $(\tilde{X}_2, \tilde{x}_2, p_2) \geq (\tilde{X}_1, \tilde{x}_1, p_1) \implies f((\tilde{X}_2, \tilde{x}_2, p_2)) \geq f((\tilde{X}_1, \tilde{x}_1, p_1))$,
- (clo3) $f(f((\tilde{X}_1, \tilde{x}_1, p_1))) = f((\tilde{X}_1, \tilde{x}_1, p_1))$.

Remark 2.9. If $P(X)$ be a nonempty subset of the complete lattice $(C(X), \geq)$, with the inherited order, then for a subset $S(X)$ of $P(X)$, we have

$$\bigvee_{C(X)} S(X) \in P(X) \implies \bigvee_{P(X)} S(X) \text{ exists and equals to } \bigvee_{C(X)} S(X);$$

$$\bigwedge_{C(X)} S(X) \in P(X) \implies \bigwedge_{P(X)} S(X) \text{ exists and equals to } \bigwedge_{C(X)} S(X).$$

Definition 2.10. For the poset $(C(X), \geq)$, let $g : (C(X), \geq) \rightarrow (C(X), \geq)$ be a map. We say that $(\tilde{X}, \tilde{x}, p) \in C(X)$ is a fixed point of g if $g((\tilde{X}, \tilde{x}, p)) = (\tilde{X}, \tilde{x}, p)$ and a prefix point of g if $(\tilde{X}, \tilde{x}, p) \geq g((\tilde{X}, \tilde{x}, p))$.

Let g_{fix} denotes the set of all fixed points of g and g_{pre} denotes the set of all prefix points of g . Let the least element of g_{fix} , when it exists, is denoted by $\beta(g)$ and similarly the greatest element of g_{fix} , when it exists, is denoted by $\gamma(g)$.

3. SOME PROPERTIES OF THE COMPLETE LATTICE $(C(X), \geq)$

With respect to the partial order relation ' $\geq$ ', $(C(X), \geq)$ is a lattice (**Theorem 2.5**). Also, $(C(X), \geq)$ is a bounded and complete lattice when the fundamental group $\pi(X, x)$ is abelian (**Theorem 2.6 and Theorem 2.7**). Now, in this section, we will study some properties of the complete lattice $(C(X), \geq)$.

Theorem 3.1. *For the poset $(C(X), \geq)$, assume that $\bigwedge S(X)$ exists in $C(X)$ for every nonempty subset $S(X)$ of $C(X)$. Then $\bigvee S(X)$ exists in $C(X)$ for every subset $S(X)$ of $C(X)$ which has an upper bound in $C(X)$; indeed $\bigvee S(X) = \bigwedge S^u(X)$, where $S^u(X) := \{(\tilde{X}', \tilde{x}', p') \in C(X) : (\forall (\tilde{X}, \tilde{x}, p) \in S(X)) (\tilde{X}', \tilde{x}', p') \geq (\tilde{X}, \tilde{x}, p)\}$*

Proof. Let $S(X)$ be a nonempty subset of $C(X)$ and assume that $S(X)$ has an upper bound in $C(X)$; thus $S^u(X) \neq \emptyset$. Now, by assumption $(\tilde{X}'', \tilde{x}'', p'') := \bigwedge S^u(X)$ exists in $C(X)$. Hence, $(\tilde{X}'', \tilde{x}'', p'')$ is the least upper bound of $S(X)$ (by definition of $S^u(X)$). Hence $\bigvee S(X) = (\tilde{X}'', \tilde{x}'', p'')$. $\square$

Theorem 3.2. *For the complete lattice $(C(X), \geq)$, the following are equivalent:*

- (a) $C(X)$ is a complete lattice;
- (b) $\bigwedge S(X)$ exists in $C(X)$ for every subset $S(X)$ of $C(X)$;
- (c) $C(X)$ has a top element $\top$, and $\bigwedge S(X)$ exists in $C(X)$ for every nonempty subset $S(X)$ of $C(X)$.

Proof. (a) $\implies$ (b) is trivial.

(b) $\implies$ (c) since the 'glb' of the empty subset $\emptyset$ of $C(X)$ exists only if $C(X)$ has a top element. (Also lub $\emptyset$ exists only if $C(X)$ has a bottom element).

(c) $\implies$ (a) follows from **Theorem 3.1**. $\square$

Theorem 3.3. *For the complete lattice $(C(X), \geq)$, let $f : (C(X), \geq) \rightarrow (C(X), \geq)$ be a monotone map. Then $Q(X) := \{(\tilde{X}, \tilde{x}, p) \in C(X) : (\tilde{X}, \tilde{x}, p) \geq f((\tilde{X}, \tilde{x}, p))\}$ is a complete lattice.*

Proof. To prove that $Q(X)$ is a complete lattice, by **Theorem 3.2**, it suffices to show that arbitrary 'glbs' exist in $Q(X)$. By **Remark 2.9**, this happens if for every subset $S(X)$ of $C(X)$, we have $(\tilde{X}_1, \tilde{x}_1, p_1) := \bigvee_{C(X)} S(X) \in Q(X)$ (and then $\bigvee_{Q(X)} S(X) = (\tilde{X}_1, \tilde{x}_1, p_1)$). But, $\forall (\tilde{X}, \tilde{x}, p) \in S(X)$, $(\tilde{X}, \tilde{x}, p) \geq (\tilde{X}_1, \tilde{x}_1, p_1) \implies f((\tilde{X}, \tilde{x}, p)) \geq f((\tilde{X}_1, \tilde{x}_1, p_1))$ (by definition of $Q(X)$ and as f is monotone) $\implies (\tilde{X}_1, \tilde{x}_1, p_1) \geq f((\tilde{X}_1, \tilde{x}_1, p_1))$ (by definition of $\bigwedge$). Therefore, $(\tilde{X}_1, \tilde{x}_1, p_1) \in Q(X)$. Hence $Q(X)$ is a complete lattice. $\square$

Theorem 3.4. *For the complete lattice $(C(X), \geq)$, let $f : (C(X), \geq) \rightarrow (C(X), \geq)$ be a closure operator on $C(X)$.*

Then $f(C(X)) = C(X)_f := \{(\tilde{X}, \tilde{x}, p) \in C(X) : f((\tilde{X}, \tilde{x}, p)) = (\tilde{X}, \tilde{x}, p)\}$ is a complete lattice, in which $\bigwedge_{C(X)_f} S(X) = \bigwedge_{C(X)} S(X)$ and $\bigvee_{C(X)_f} S(X) = f(\bigvee_{C(X)} S(X))$ and $T_{f(C(X))} = f(T_{C(X)})$.

Proof. $(\tilde{X}_1, \tilde{x}_1, p_1) \in C(X)_f \implies (\tilde{X}_1, \tilde{x}_1, p_1) = f((\tilde{X}_1, \tilde{x}_1, p_1)) \in f(C(X))$, while if $(\tilde{X}_1, \tilde{x}_1, p_1) = f((\tilde{X}, \tilde{x}, p))$ for some $(\tilde{X}, \tilde{x}, p) \in C(X)$, then $f((\tilde{X}_1, \tilde{x}_1, p_1)) = f(f((\tilde{X}, \tilde{x}, p))) = f((\tilde{X}, \tilde{x}, p)) = (\tilde{X}_1, \tilde{x}_1, p_1)$ (by **(clo3)** of **Definition 2.8**). So, $(\tilde{X}_1, \tilde{x}_1, p_1) \in C(X)_f$. Thus $f(C(X)) = C(X)_f$. By **(clo3)** of **Definition 2.8** this set is just the set $Q(X)$ of the **Theorem 3.3**. Hence $C(X)_f$ is a complete lattice (by **Theorem 3.3**), whose 'glbs' coincide with those in $C(X)$, that is, $\bigwedge_{C(X)_f} S(X) = \bigwedge_{C(X)} S(X)$.

Now, let $(\tilde{X}_2, \tilde{x}_2, p_2) := f(\bigvee_{C(X)} S(X))$, where $S(X)$ is the subset of $C(X)_f$. Certainly, $(\tilde{X}_2, \tilde{x}_2, p_2) \in f(C(X)) = C(X)_f$. Also, $(\tilde{X}, \tilde{x}, p) \in S(X) \implies \bigvee_{C(X)} S(X) \geq (\tilde{X}, \tilde{x}, p)$ (by definition of $\bigvee_{C(X)}$) $\implies (\tilde{X}_2, \tilde{x}_2, p_2) = f(\bigvee_{C(X)} S(X)) \geq f((\tilde{X}, \tilde{x}, p)) = (\tilde{X}, \tilde{x}, p)$ (by **(clo2)** of **Definition 2.8**). So, $(\tilde{X}_2, \tilde{x}_2, p_2)$ is an upper bound for $S(X)$ in $C(X)_f$. Also, for any upper bound $(\tilde{X}_3, \tilde{x}_3, p_3)$ for $S(X)$ with $(\tilde{X}_3, \tilde{x}_3, p_3) \in C(X)_f$, $\forall (\tilde{X}, \tilde{x}, p) \in S(X)$, $(\tilde{X}_3, \tilde{x}_3, p_3) \geq (\tilde{X}, \tilde{x}, p) \implies (\tilde{X}_3, \tilde{x}_3, p_3) \geq \bigvee_{C(X)} S(X)$ (by definition of $\bigvee_{C(X)}$) $\implies (\tilde{X}_3, \tilde{x}_3, p_3) = f((\tilde{X}_3, \tilde{x}_3, p_3)) \geq f(\bigvee_{C(X)} S(X)) = (\tilde{X}_2, \tilde{x}_2, p_2)$

(by **(col2)** of **Definition 2.8**). So, $(\tilde{X}_2, \tilde{x}_2, p_2)$ is the least upper bound (lub). Finally, $T_{f(C(X))} = f(T_{C(X)})$ (by **(col1)** of **Definition 2.8**). $\square$

Theorem 3.5. *For the poset $(C(X), \geq)$, let $g : (C(X), \geq) \rightarrow (C(X), \geq)$ be a monotone map. Assume that g possesses a least prefix point $\alpha'(g)$. Then the least fixed point $\alpha(g)$ exists and satisfies $(\tilde{X}, \tilde{x}, p) \geq g((\tilde{X}, \tilde{x}, p)) \implies (\tilde{X}, \tilde{x}, p) \geq \alpha(g)$. If $(C(X), \geq)$ is a complete lattice, then $\alpha(g) = \alpha'(g)$.*

Proof. Let $\alpha'(g)$ exists. We have to show that $\alpha'(g) \in g_{fix}$. Since g_{fix} is a subset of g_{pre} , we must have that $\alpha(g) = \alpha'(g)$.

As $\alpha'(g) \in g_{pre}$, by **Definition 2.10**, we have $\alpha'(g) \geq g(\alpha'(g))$. As g is monotone, then $g(\alpha'(g)) \geq g(g(\alpha'(g)))$, so that $g(\alpha'(g)) \in g_{pre}$. Since $\alpha'(g)$ is the least element of g_{pre} , we then have $g(\alpha'(g)) \geq \alpha'(g)$. Hence, $\alpha'(g)$ is indeed a fixed point of g .

Now, if $C(X)$ is a complete lattice, then there is an obvious for $\alpha'(g)$, namely $\bigwedge g_{pre}$. Monotonicity of g implies $\alpha'(g) = \bigwedge g_{pre} \in g_{pre}$, that is, $\alpha'(g) = \alpha(g)$. $\square$

Theorem 3.6. *For the complete lattice $(C(X), \geq)$, let $g : (C(X), \geq) \rightarrow (C(X), \geq)$ be a monotone map. Then $\bigwedge \{(\tilde{X}, \tilde{x}, p) \in C(X) : (\tilde{X}, \tilde{x}, p) \geq g((\tilde{X}, \tilde{x}, p))\}$ is the least fixed point of g . Dually, g has a greatest fixed point, given by $\bigvee \{(\tilde{X}, \tilde{x}, p) \in C(X) : g((\tilde{X}, \tilde{x}, p)) \geq (\tilde{X}, \tilde{x}, p)\}$.*

Proof. This follows from **Theorem 3.5** and its dual. $\square$

Theorem 3.7. *g_{pre} and g_{fix} are complete lattices when $(C(X), \geq)$ is a complete lattice.*

Proof. g_{pre} is a complete lattice follows from **Theorem 3.3**.

Let $S(X)$ be a subset of $C(X)$. Define $P(X) := \{(\tilde{X}', \tilde{x}', p') \in C(X) : (\tilde{X}, \tilde{x}, p) \geq g((\tilde{X}', \tilde{x}', p')) \geq (\tilde{X}', \tilde{x}', p') \forall (\tilde{X}, \tilde{x}, p) \in S(X)\}$ and let $\delta := \bigvee_{C(X)} P(X)$. But by Definitions of $P(X)$ and g_{fix} , it follows that $\delta = \bigwedge_{g_{fix}} S(X)$. Hence by **Theorem 3.2**, g_{fix} is a complete lattice. $\square$

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