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A NOTE ON MAXIMAL NON-MAXIMAL PRIME SUBMODULE

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Abstract. The rings considered in this article are commutative with identity $1 \neq 0$ and the modules considered in this article are modules over commutative rings and are unitary. We say that a proper submodule N of an R-module M is a maximal non-prime submodule if N is not a prime submodule of M but any proper submodule K of M with $N \subseteq K$ and $N \neq K$ is a prime submodule. That is, among all the proper submodules of M, N is maximal with respect to the property of being not a prime submodule. The concept of maximal non-maximal submodule, maximal non-primary submodule and maximal non-irreducible submodule of a module can be similarly defined. The aim of this article is to characterize submodules N of an R-module M such that N is a maximal non-prime (respectively, a maximal non-maximal, a maximal non-primary, a maximal non-irreducible) submodule of M.

1. Introduction

The rings considered in this article are commutative with identity $1 \neq 0$ and the modules considered in this article are modules over commutative rings and are unitary.

In [9], S. Visweswaran and A. Parmar introduced and studied the concept of a maximal non-prime ideal. Let R be a non-zero commutative ring with identity. A proper ideal I of R is said to be a *maximal non-prime ideal* of R if the following conditions hold: (i) I is not a prime ideal of R and (ii) If A is any proper ideal of R such that $A \supset I$, then A is a prime ideal of R. Similarly, we can define the concept of a *maximal non-maximal ideal*, a *maximal non-primary ideal* or a *maximal non-irreducible ideal* of a ring R.

Let R be a ring. Let I be a proper radical ideal of R. It is proved in [9, Proposition 3.2] that the following statements are equivalent : (i) I is a maximal non-primary ideal of R. (ii) $I = \mathfrak{m}_1 \cap \mathfrak{m}_2$ for some distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ of R. (iii) I is a maximal non-maximal ideal of R. (iv) I is a maximal non-prime ideal of R. It is also proved in Proposition 5.1 of [9] that for a proper radical

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ideal I of R, I is a maximal non-irreducible ideal of R if and only if $I = \mathfrak{m}_1 \cap \mathfrak{m}_2$ for some distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ of R.

Let M be a module over a ring R. Let N_1 and N_2 be submodules of M. Recall from [1, Page 19] that $(N_1 :_R N_2) = \{r \in R : rN_2 \subseteq N_1\}$ is an ideal of R. In particular, $((0) :_R M) = \{r \in R : rM = (0)\}$ is called the *annihilator* of M and is denoted by $Ann_R(M)$ or by $Ann_R M$. M is said to be *faithful* if $Ann_R M = (0)$. Recall from [2, Page 765] that a proper submodule P of M is said to be a *prime submodule* if $Ann_R(K/P) = Ann_R(M/P)$ for every submodule K with $P \subset K \subseteq M$. The above definition of a prime submodule P of M is equivalent to the conditions that $P \neq M$ and if $rm \in P$ for some $r \in R$ and $m \in M$, then either $m \in P$ or $r \in (P :_R M)$. In [7], R. L. McCasland and M. E. Moore proved that if N is prime submodule of M, then $(N :_R M)$ is prime ideal of R, say $\mathfrak{p}$. In this case, N is said to be a $\mathfrak{p}$-prime submodule of M. Recall from [7] that a proper submodule N of M is said to be a *maximal* submodule of M if K is any submodule of M with $N \subseteq K$, then either $K = N$ or $K = M$. Recall from [7] that the *M-radical* of a proper submodule N of M, denoted by $rad(N)$, is the intersection of all prime submodule of M containing N. If there is no prime submodule of M containing N, then $rad(N)$ is defined to be M (that is, $rad(N) = M$). A proper submodule N of M is called a *radical submodule* if $rad(N) = N$ [7].

Recall from [1, Page 58] that a proper submodule Q of M is said to be a *primary submodule* of M if every zero-divisor in M/Q is nilpotent. In other words, whenever $rm \in Q$ for some $r \in R$ and $m \in M$ implies that either $m \in Q$ or $r \in \sqrt{(Q :_R M)}$. If Q is primary submodule of M, then $(Q :_R M)$ is primary ideal of R. Hence, $\sqrt{(Q :_R M)}$ is a prime ideal of R, say $\mathfrak{p}$. In this case, Q is said to be a $\mathfrak{p}$-primary submodule of M [1, Page 58]. Recall from [6, Page 40] that a submodule N of M is said to be an *irreducible submodule* of M if $N \neq N_1 \cap N_2$ for any submodules N_1, N_2 of M with $N \subset N_i$ for each $i \in \{1, 2\}$. In other words, if $N = N_1 \cap N_2$ for some submodules N_1, N_2 of M, then either $N_1 = N$ or $N_2 = N$. M is called a *multiplication module* if for each submodule N of M, there exists an ideal I of R such that $N = IM$ [2].

Let M be a module over a ring R. A proper submodule N of M is called a *maximal non-prime submodule* of M if the following conditions hold:

(i) N is not a prime submodule of M.

(ii) If P is any proper submodule of M such that $P \supset N$, then P is a prime submodule of M.

Similarly, we can define the concept of a *maximal non-maximal submodule*, a *maximal non-primary submodule* or a *maximal non-irreducible submodule* of M.

In Section 2 of this article, we prove some theorems over a multiplication module. Let R be a commutative ring with identity and M be a faithful finitely generated multiplication R-module. Let N be a proper radical submodule of M. It is proved in Theorem 2.8 that the following statements are equivalent: (i) N is a maximal non-primary submodule of M. (ii) $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M. (iii) N is a maximal non-maximal submodule of M. (iv) N is a maximal non-prime submodule of M. It is also proved in

Theorem 2.11 that the following statements are equivalent: (i) N is a maximal non-irreducible submodule of M. (ii) $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M.

2. Over the Multiplication module

The aim of this section is to characterize proper radical submodules N of a faithful finitely generated multiplication module M over a ring R such that N is a maximal non-prime submodule of M.

We often use [2, Theorem 3.1] of in our following discussion. We mention [2, Theorem 3.1] here as Proposition 2.1 for convenient reference.

Proposition 2.1. [2, Theorem 3.1] *Let R be a commutative ring with identity and M a faithful multiplication R-module. Then the following statements are equivalent:*

(i) *M is finitely generated.*
(ii) *If A and B are ideals of R such that $AM \subseteq BM$, then $A \subseteq B$.*
(iii) *For each submodule N of M, there exists a unique ideal I of R such that $N = IM$.*
(iv) *$M \neq AM$ for any proper ideal A of R.*
(v) *$M \neq \mathfrak{p}M$ for any maximal ideal $\mathfrak{p}$ of R.*

Remark 2.2. Let R be a commutative ring with identity and M a faithful multiplication R-module. It follows from $(i) \Leftrightarrow (iii)$ of Proposition 2.1 that M is finitely generated if and only if for each submodule N of M, there exists a unique ideal I of R such that $N = IM$. In fact, $I = (N :_R M)$.

Proposition 2.3. *Let M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is a maximal non-prime submodule of M if and only if $(N :_R M)$ is a maximal non-prime ideal of R.*

Proof. Suppose that N is a maximal non-prime submodule of M. Now, $N = (N :_R M)M$. Since N is not a prime submodule of M, it follows from $(iii) \Rightarrow (i)$ in [2, Corollary 2.11] that $(N :_R M) \notin Spec(R)$. Let J be any proper ideal of R with $(N :_R M) \subset J$. Let $P = JM$. From $(N :_R M) \subset J$, it follows from $(i) \Rightarrow (iii)$ of Proposition 2.1 that $N = (N :_R M)M \subset JM = P$. Since N is a maximal non-prime submodule of M, we obtain that $P = JM$ is a prime submodule of M. Hence, it follows from $(i) \Rightarrow (iii)$ in [2, Corollary 2.11] that $J \in Spec(R)$. This proves that $(N :_R M)$ is a maximal non-prime ideal of R.

Conversely, assume that $(N :_R M)$ is a maximal non-prime ideal of R. As $(N :_R M) \notin Spec(R)$, it follows from $(i) \Rightarrow (iii)$ of [2, Corollary 2.11] that $N = (N :_R M)M$ is not a prime submodule of M. Let P be any proper submodule of M with $N \subset P$. As $N = (N :_R M)M$ and $P = (P :_R M)M$, it follows from $N \subset P$ that $(N :_R M) \subset (P :_R M)$. Since $(P :_R M) \neq R$, we get that $(P :_R M) \in Spec(R)$ and so, we obtain from $(iii) \Rightarrow (i)$ of [2, Corollary 2.11] that $P = (P :_R M)M$ is a prime submodule of M. This proves that N is a maximal non-prime submodule of M. $\square$

Proposition 2.4. *Let M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is a maximal non-maximal submodule of M if and only if $(N :_R M)$ is a maximal non-maximal ideal of R.*

Proof. Suppose that N is a maximal non-maximal submodule of M. Now, $N = (N :_R M)M$. Since N is not a maximal submodule of M, it follows from [2, Theorem 2.5(ii)] that $(N :_R M) \notin Max(R)$. Let J be any proper ideal of R with $(N :_R M) \subset J$. Let $P = JM$. From $(N :_R M) \subset J$, it follows from $(i) \Rightarrow (iii)$ of Proposition 2.1 that $N = (N :_R M)M \subset JM = P$. Since N is a maximal non-maximal submodule of M, we obtain that $P = JM$ is a maximal submodule of M. Hence, it follows from [2, Theorem 2.5(ii)] that $J \in Max(R)$. This proves that $(N :_R M)$ is a maximal non-maximal ideal of R.

Conversely, assume that $(N :_R M)$ is a maximal non-maximal ideal of R. As $(N :_R M) \notin Max(R)$, it follows from [2, Theorem 2.5(ii)] that $N = (N :_R M)M$ is not a maximal submodule of M. Let P be any proper submodule of M with $N \subset P$. As $N = (N :_R M)M$ and $P = (P :_R M)M$, it follows from $N \subset P$ that $(N :_R M) \subset (P :_R M)$. Since $(P :_R M) \neq R$, we get that $(P :_R M) \in Max(R)$ and so, we obtain from [2, Theorem 2.5(ii)] that $P = (P :_R M)M$ is a maximal submodule of M. This proves that N is a maximal non-maximal submodule of M. $\square$

Let R be a ring and let M be a faithful finitely generated multiplication R-module. We prove in Proposition 2.6 that a proper submodule N of M is a maximal non-primary submodule if and only if $(N :_R M)$ is a maximal non-primary ideal of R. We use Lemma 2.5 in the proof of Proposition 2.6.

Lemma 2.5. *Let M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is a primary submodule of M if and only if $(N :_R M)$ is a primary ideal of R.*

Proof. Suppose that N is a primary submodule of M. Since $N \neq M$, it follows that $(N :_R M) \neq R$. Let $ab \in (N :_R M)$ for some $a, b \in R$ and $a \notin (N :_R M)$. Therefore, $am \notin N$ for some $m \in M$. As $ab \in (N :_R M)$, we get $abm \in N$. Hence, $b(am) \in N$. Since N is a primary submodule of M and $am \notin N$, we get that $b \in \sqrt{(N :_R M)}$. This shows that $(N :_R M)$ is a primary ideal of R.

Conversely, assume that $(N :_R M)$ is a primary ideal of R. Let $a \in R$ and $m \in M$ be such that $am \in N = (N :_R M)M$. Suppose that $m \notin N$. Since M is a multiplication module, $Rm = JM$ for some ideal J of R. Now, $am \in N$. Thus $aJM \subseteq (N :_R M)M$. So, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $aJ \subseteq (N :_R M)$. By assumption, $m \notin (N :_R M)M$ and so, $J \not\subseteq (N :_R M)$. Now, $aJ \subseteq (N :_R M)$ and $J \not\subseteq (N :_R M)$. As $(N :_R M)$ is a primary ideal of R, we obtain that $a^k \in (N :_R M)$ for some $k \geq 1$. That is, $a \in \sqrt{(N :_R M)}$. This proves that N is a primary submodule of M. $\square$

Proposition 2.6. *Let M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is a maximal non-primary submodule of M if and only if $(N :_R M)$ is a maximal non-primary ideal of R.*

Proof. Suppose that N is a maximal non-primary submodule of M. Now, $N = (N :_R M)M$. It follows from Lemma 2.5 that $(N :_R M)$ is not a primary ideal of R. Let J be any proper ideal of R with $(N :_R M) \subset J$. Let $P = JM$. It follows from $(i) \Rightarrow (iii)$ of Proposition 2.1 that $N = (N :_R M)M \subset JM = P$. Since N is a maximal non-primary submodule of M, we obtain that P is a primary submodule of M. Hence, it follows from Lemma 2.5 that J is a primary ideal of R. So, $(N :_R M)$ is a maximal non-primary ideal of R.

Conversely, assume that $(N :_R M)$ is a maximal non-primary ideal of R. As $N = (N :_R M)M$, we get from Lemma 2.5 that N is not a primary submodule of M. Let P be any proper submodule of M with $N \subset P$. As $N = (N :_R M)M$ and $P = (P :_R M)M$, it follows from $N \subset P$ that $(N :_R M) \subset (P :_R M)$. Since $(N :_R M)$ is a maximal non-primary ideal of R, we obtain that $(P :_R M)$ is a primary ideal of R. So, it follows from Lemma 2.5 that P is a primary submodule of M. Therefore, N is a maximal non-primary submodule of M. □

Proposition 2.7. *Let R be a commutative ring with identity and M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is a radical submodule if and only if $(N :_R M)$ is a radical ideal of R.*

Proof. We know from [2, Theorem 2.12] that for a proper submodule N of M, $rad(N) = \sqrt{(N :_R M)}M$. Thus a proper submodule N of M is a radical submodule of M if and only if $N = rad(N)$ if and only if $(N :_R M)M = \sqrt{(N :_R M)}M$. Now, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $(N :_R M)M = \sqrt{(N :_R M)}M$ if and only if $(N :_R M) = \sqrt{(N :_R M)}$. This shows that a proper submodule N of M is a radical submodule of M if and only if $(N :_R M) = \sqrt{(N :_R M)}$. That is, if and only if $(N :_R M)$ is a radical ideal of R. □

We use Propositions 2.3, 2.4 and 2.6 and some other results of this article in the proof of Theorem 2.8.

Theorem 2.8. *Let R be a commutative ring with identity and M be a faithful finitely generated multiplication R-module. Let N be a proper radical submodule of M. Then the following statements are equivalent:*
(i) *N is a maximal non-primary submodule of M.*
(ii) *$N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M.*
(iii) *N is a maximal non-maximal submodule of M.*
(iv) *N is a maximal non-prime submodule of M.*

Proof. By assumption, N is a radical submodule of M. So, we obtain from Lemma 2.7 that $(N :_R M)$ is a radical ideal of R.
$(i) \Rightarrow (ii)$ Suppose that N is a maximal non-primary submodule of M. It follows from Proposition 2.6 that $(N :_R M)$ is a maximal non-primary ideal of R. So, we obtain from $(i) \Rightarrow (ii)$ of Proposition [9, Theorem 3.2] that $(N :_R M) = \mathfrak{m}_1 \cap \mathfrak{m}_2$ for some distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ of R. Hence, it follows from [2, Theorem 1.6] that $N = (N :_R M)M = (\mathfrak{m}_1 \cap \mathfrak{m}_2)M = \mathfrak{m}_1 M \cap \mathfrak{m}_2 M = N_1 \cap N_2$, where $N_1 = \mathfrak{m}_1 M$ and $N_2 = \mathfrak{m}_2 M$. As $\mathfrak{m}_1 \neq \mathfrak{m}_2$, it follows from $(i) \Rightarrow (ii)$ of Proposition [9, Theorem 3.2] that $N_1 \neq N_2$. So, N_1 and N_2 are distinct submodules of M.

Since $\mathfrak{m}_1$, $\mathfrak{m}_2$ are maximal ideals of R, it follows from [2, Theorem 2.5(ii)] that N_1 and N_2 are maximal submodules of M. Thus $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M.

$(ii) \Rightarrow (iii)$ Suppose that $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M. It follows from [2, Theorem 2.5(ii)] that $N_1 = \mathfrak{m}_1 M$ and $N_2 = \mathfrak{m}_2 M$ for some distinct maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of R. So, by [2, Theorem 1.6], $(N :_R M)M = \mathfrak{m}_1 M \cap \mathfrak{m}_2 M = (\mathfrak{m}_1 \cap \mathfrak{m}_2)M$. Hence, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.6.11 that $(N :_R M) = \mathfrak{m}_1 \cap \mathfrak{m}_2$. So, it follows from $(ii) \Rightarrow (iii)$ of Proposition [9, Theorem 3.2] that $(N :_R M)$ is a maximal non-maximal ideal of R. Hence, we obtain from from Proposition 2.4 that $N = (N :_R M)M$ is a maximal non-maximal submodule of M.

$(iii) \Rightarrow (iv)$ Suppose that N is a maximal non-maximal submodule of M. So, we obtain from Proposition 2.4 that $(N :_R M)$ is a maximal non-maximal ideal of R. It follows from $(iii) \Rightarrow (iv)$ of Proposition [9, Theorem 3.2] that $(N :_R M)$ is a maximal non-prime ideal of R. Hence, we obtain from Proposition 2.3 that N is a maximal non-prime submodule of M.

$(iv) \Rightarrow (i)$ Suppose that N is a maximal non-prime submodule of M. So, we obtain from Proposition 2.3 that $(N :_R M)$ is a maximal non-prime ideal of R. It follows from $(iv) \Rightarrow (i)$ of Proposition [9, Theorem 3.2] that $(N :_R M)$ is a maximal non-primary ideal of R. Hence, we obtain form Proposition 2.6 that N is a maximal non-primary submodule of M. $\square$

Let R be a ring and let M be a faithful finitely generated multiplication R-module. We prove in Proposition 2.10 that a proper submodule N of M is a maximal non-irreducible submodule if and only if $(N :_R M)$ is a maximal non-irreducible ideal of R. We use Lemma 2.9 in the proof of Proposition 2.10.

Lemma 2.9. *Let M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is an irreducible submodule of M if and only if $(N :_R M)$ is an irreducible ideal of R.*

Proof. Suppose that N is an irreducible submodule of M. Let $(N :_R M) = I_1 \cap I_2$ for some ideals I_1, I_2 of R. It follows from [2, Theorem 1.6] that $(N :_R M)M = (I_1 \cap I_2)M = I_1 M \cap I_2 M$. Therefore, $N = N_1 \cap N_2$, where $N_1 = I_1 M$ and $N_2 = I_2 M$. Since N is an irreducible submodule of M, either $N_1 = N$ or $N_2 = N$. So, either $I_1 M = (N :_R M)M$ or $I_2 M = (N :_R M)M$. Hence, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.1 that either $I_1 = (N :_R M)$ or $I_2 = (N :_R M)$. Therefore, $(N :_R M)$ is an irreducible ideal of R.

Conversely, assume that $(N :_R M)$ is an irreducible ideal of R. Let $N = N_1 \cap N_2$ for some submodules N_1, N_2 of M. It follows from $(i) \Rightarrow (iii)$ of Proposition 2.1 that $N_1 = I_1 M$ and $N_2 = I_2 M$ for unique ideals I_1, I_2 of R. So, by [2, Theorem 1.6], $(N :_R M)M = I_1 M \cap I_2 M = (I_1 \cap I_2)M$. Hence, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $(N :_R M) = I_1 \cap I_2$. Since $(N :_R M)$ is an irreducible ideal of R, we get either $I_1 = (N :_R M)$ or $I_2 = (N :_R M)$. So, either $N_1 = N$ or $N_2 = N$. Hence, N is an irreducible submodule of M. $\square$

Proposition 2.10. *Let R be a ring and let M be a faithful finitely generated multiplication R-module. Then a proper submodule N of M is a maximal non-irreducible submodule of M if and only if $(N :_R M)$ is a maximal non-irreducible ideal of R.*

Proof. Suppose that N is a maximal non-irreducible submodule of M. As $N = (N :_R M)M$, it follows from Lemma 2.9 that $(N :_R M)$ is not an irreducible ideal of R. Let J be any proper ideal of R with $(N :_R M) \subset J$. Let $P = JM$. It follows from $(i) \Rightarrow (iii)$ of Proposition 2.1 that $N = (N :_R M)M \subset JM = P$. By assumption, N is a maximal non-irreducible submodule of M. So, we get P is an irreducible submodule of M. Hence, it follows from Lemma 2.9 that J is an irreducible ideal of R. Therefore, $(N :_R M)$ is a maximal non-irreducible ideal of R.

Conversely, assume that $(N :_R M)$ is a maximal non-irreducible ideal of R. As $N = (N :_R M)M$, it follows from Lemma 2.9 that N is not an irreducible submodule of M. Let P be any proper submodule of M with $N \subset P$. As $N = (N :_R M)M$ and $P = (P :_R M)M$, it follows from $N \subset P$ that $(N :_R M) \subset (P :_R M)$. Since $(N :_R M)$ is a maximal non-irreducible ideal of R, we get $(P :_R M)$ is an irreducible ideal of R. Hence, it follows from Lemma 2.9 that P is an irreducible submodule of M. Therefore, N is a maximal non-irreducible submodule of M. $\square$

Theorem 2.11. *Let R be a commutative ring with identity and M be a faithful finitely generated multiplication R-module. Let N be a proper radical submodule of M. Then the following statements are equivalent:*

(i) N is a maximal non-irreducible submodule of M.

(ii) $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M.

Proof. By assumption, N is a radical submodule of M. So, we obtain from Lemma 2.7 that $(N :_R M)$ is a radical ideal of R.

$(i) \Rightarrow (ii)$ Suppose that N is a maximal non-irreducible submodule of M. It follows from Proposition 2.10 that $(N :_R M)$ is a maximal non-irreducible ideal of R. Hence, it follows from $(i) \Rightarrow (ii)$ of Proposition [9, Theorem 5.1] that $(N :_R M) = \mathfrak{m}_1 \cap \mathfrak{m}_2$ for some distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ of R. So, by [2, Theorem 1.6], $N = (N :_R M)M = (\mathfrak{m}_1 \cap \mathfrak{m}_2)M = \mathfrak{m}_1 M \cap \mathfrak{m}_2 M = N_1 \cap N_2$, where $N_1 = \mathfrak{m}_1 M$ and $N_2 = \mathfrak{m}_2 M$. Since $\mathfrak{m}_1 \neq \mathfrak{m}_2$, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.11 that $\mathfrak{m}_1 M \neq \mathfrak{m}_2 M$. That is, $N_1 \neq N_2$. So, N_1 and N_2 are distinct submodules of M. Since $\mathfrak{m}_1, \mathfrak{m}_2$ are maximal ideals of R, it follows from [2, Theorem 2.5(ii)] that N_1 and N_2 are maximal submodules of M. Thus $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M.

$(ii) \Rightarrow (i)$ Suppose that $N = N_1 \cap N_2$ for some distinct maximal submodules N_1, N_2 of M. It follows from [2, Theorem 2.5(ii)] that $N_1 = \mathfrak{m}_1 M$ and $N_2 = \mathfrak{m}_2 M$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are distinct maximal ideals of R. So, by [2, Theorem 1.6], $(N :_R M)M = \mathfrak{m}_1 M \cap \mathfrak{m}_2 M = (\mathfrak{m}_1 \cap \mathfrak{m}_2)M$. Hence, it follows from $(i) \Rightarrow (iii)$ of Proposition 2.1 that $(N :_R M) = \mathfrak{m}_1 \cap \mathfrak{m}_2$. Therefore, it follows from $(ii) \Rightarrow (i)$ of Proposition [9, Theorem 5.1] that $(N :_R M)$ is a maximal non-irreducible ideal of R. Hence, we obtain form Proposition 2.10 that N is a maximal non-irreducible submodule of M. $\square$

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