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CONVERGENCE OF AKUTSAH ITERATION SCHEME FOR MEAN NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACE WITH APPLICATION

KIRAN DEWANGAN[1*] AND NIYATI GURUDWAN[2]

Abstract. In this paper, we deal with strong convergence and stability of three-step iteration scheme given by Akutsah et al., for mean nonexpansive mappings in uniformly convex Banach space. We also show the fastness of this iteration scheme with other well-known iteration schemes with the help of some numerical examples. Also we discuss here an application of fixed point theory in solution of integral equation.

1. Introduction

Let $(X, ||.||)$ be a Banach space and K be a non-empty subset of X. Let $T : K \to K$ be a mapping. Then a point $x \in K$ is called fixed point of T, if $Tx = x$. Through-out the literature, $F(T)$ denotes set of fixed points of T. Note that a mapping $T : K \to K$ is called nonexpansive if

$$||Tx - Ty|| \leq ||x - y||,$$

for all $x, y \in K$.

In 1965, Browder [5] and Gohde [6] independently proved that " every nonexpansive self mapping of a closed covex and bounded subset of a uniformly convex Banach space has a fixed point". In the same year, Kirk [13] proved that if "X is a reflexive Banach space with normal structure, then X has the fixed point property".

The concept of mean nonexpansive mapping was introduces by Zhang [25] in 1975 as a generalization of nonexpansive mapping in Banach space. A mapping $T : K \to K$ is called mean nonexpansive if

$$||Tx - Ty|| \leq a||x - y|| + b||x - Ty||,$$

where $a, b \geq 0$ and $a + b \leq 1$. Note that a nonexpansive mapping is mean nonexpansive for $a = 1$ and $b = 0$, but a mean nonexpansive mapping is not necessarily

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* Corresponding author.
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nonexpansive mapping.

Zhang also proved existence and uniqueness of fixed points of mean nonexpansive mapping in Banach space along with normal structure. After that Wu and Zhang [24] investigated some properties of mean nonexpansive mappings and they proved that if $a+b<1$, then mean nonexpansive mapping has a unique fixed point. In 2012, Ouahab [16] proved fixed point theorem for strong semigroup of mean nonexpansive mappings in uniformly convex Banach space. In 2014, Zuo [26] proved that a mean nonexpansive mapping has approximate fixed point sequence and under some suitable conditions, he got some existence and uniqueness theorems of fixed point. In 2022, Ezeora et al. [7] studied approximation of common fixed points of mean nonexpansive mappings in hyperbolic spaces.

Fixed point theory plays an important role in different fields of mathematics. The solution of fixed point problem is difficult analytically, and hence there is need of approximate solution. Several iterative schemes have been developed by many researchers for solving fixed point problems for different operators over different spaces and still it is ongoing. Some of the well known iterations are- Abbas [1], Agrawal [3], Ishikawa [9], Karakaya [12], Mann [14], Noor [15], Normal $S-$iteration [18], Thakur [22], etc.

Recently, Akutsah et al. [4] introduced the following iteration scheme in the framework of Banach space. Let $(X,||.||)$ be a Banach space and K be a nonempty closed convex subset of X. For $x_1\in K$, the sequence $\{x_k\}$ in K is defined by

$$\begin{cases} z_k=(1-\beta_k)x_k+\beta_kTx_k, \\ y_k=Tz_k, \\ x_{k+1}=T((1-\alpha_k)y_k+\alpha_kTy_k),\ k\geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in $(0,1)$.

Akutsah et al. [4] proved that their iteration scheme is faster than the teration scheme of Karakaya [12], Thakur [22] and M iteration scheme [23] and also established convergence results of the iteration scheme defined by (1.1) for contractive-like mappings. In this paper, we proved strong convergence and the stability of the iteration scheme defined by (1.1) for mean nonexpansive mapping in uniformly convex Banach space and an application of the fixed point theory to obtain solution of an integral equation.

2. Preliminaries

This section starts some necessary concepts and contains some useful results to get main results.

Definition 2.1. [19] Let X be a Banach space with the norm $||.||$ and K a convex subset of X. A mapping $T:K\to K$ with non-empty fixed point set $F(T)$ in K will be said to satisfy Condition (I), if there is a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$ such that $||x - Tx|| \geq f(d(x, F(T))$ for all $x \in K$, where $d(x, F(T)) = \inf\{||x - z|| : z \in F(T)\}$.

Definition 2.2. [11] A Banach space X is called uniformly convex Banach space if and only if for every choice of $r \in (0, 2]$, one has a $s > 0$ such that for every $p, q \in X$, $\frac{1}{2}||p + q|| \leq (1 - s)$, whenever $||p|| \leq 1$, $||q|| \leq 1$ and $||p - q|| > r$.

Definition 2.3. [10] Let K be a non-empty subset of a uniformly convex Banach space X. A sequence $\{x_k\}$ in X is said to be Fejer monotone with respect to subset K, if

$$||x_{k+1} - p|| \leq ||x_k - p||,$$

for all $p \in K$, $k \geq 1$.

Proposition 2.4. [10] *Let K be a non-empty subset of a uniformly convex Banach space X. Suppose that $\{x_k\}$ is Fejer monotone sequence with respect to K. Then the following holds:*

(a) *Sequence $\{x_k\}$ is bounded.*
(b) *For every $x \in K$, $\{||x_k - x||\}$ converges.*

Lemma 2.5. [17] *Let X be a uniformly convex Banach space, and $\{\alpha_k\}$ be a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that $\{x_k\}$ and $\{y_k\}$ are in X such that $\limsup_{k\to\infty} ||x_k|| \leq c$, $\limsup_{k\to\infty} ||y_k|| \leq c$, and $\limsup_{k\to\infty} ||\alpha_k x_k + (1 - \alpha_k)y_k|| = c$ for some $c \geq 0$. Then $\lim_{k\to\infty} ||x_k - y_k|| = 0$.*

Definition 2.6. [8] Let $\{t_k\}$ be any arbitrary sequence in K. An iteration procedure $x_{k+1} = f(T, x_k)$ converging to fixed point z, is said to be $T-$ stable or stable with respect to T if for $\epsilon_k = ||t_{k+1} - f(T, t_k)||$, for all $n \in \mathbb{N}$, we have $\lim_{k\to\infty} \epsilon_k = 0$ if and only if $\lim_{k\to\infty} t_k = z$.

Lemma 2.7. [21] *Let $\{\psi_k\}$ and $\{\zeta_k\}$ be non-negative real sequences satisfying the following inequality:*

$$\psi_{k+1} \leq (1 - \phi_k)\psi_k + \zeta_k,$$

where $\phi_k \in (0, 1)$ for all $k \in \mathbb{N}$, $\sum_{k=0}^{\infty} \phi_k = \infty$ and $\lim_{k\to\infty} \frac{\zeta_k}{\phi_k} = 0$, then $\lim_{k\to\infty} \psi_k = 0$.

3. Main Results

The following lemma discuss the structure of fixed point set of mean nonexpansive mappings.

Lemma 3.1. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping. Then $F(T)$ is closed.*

Proof. Let $\{x_k\}$ be a sequence in $F(T)$ such that $x_k \to x$ as $k \to \infty$. Since T is mean nonexpansive mapping, we have

$$\begin{aligned} 0 \leq ||x - Tx|| &\leq ||x - x_k|| + ||x_k - Tx|| \\ &= ||x - x_k|| + ||Tx_k - Tx|| \\ &\leq ||x - x_k|| + a||x_k - x|| + b||x_k - Tx|| \\ &\leq (1 + a)||x - x_k|| + b||x_k - x|| + b||x - Tx|| \\ (1 - b)||x - Tx|| &\leq (1 + a + b)||x_k - x|| \ \to \ 0 \ as \ k \to \infty. \end{aligned}$$

This shows that $Tx = x$, i.e., $x \in F(T)$. Hence $F(T)$ is closed. □

Lemma 3.2. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1.1). Then $\lim_{k\to\infty} ||x_k - z||$ exists for all $z \in F(T)$.*

Proof. Let $z \in F(T)$. Note that

$$\begin{aligned} ||y_k - z|| &= ||Tz_k - z|| \\ &\leq ||Tz_k - Tz|| \\ &\leq a||z_k - z|| + b||z_k - Tz|| \\ &= (a + b)||z_k - z|| \\ &\leq ||z_k - z||, \end{aligned}$$

$$\begin{aligned} ||z_k - z|| &= ||(1 - \beta_k)x_k + \beta_k Tx_k - z|| \\ &\leq (1 - \beta_k)||x_k - z|| + \beta_k||Tx_k - z|| \\ &\leq (1 - \beta_k)||x_k - z|| + \beta_k(a||x_k - z|| + b||x_k - Tz||) \\ &= (1 - \beta_k)||x_k - z|| + \beta_k(a + b)||x_k - z|| \\ &\leq ||x_k - z||, \end{aligned}$$

and

$$\begin{aligned} ||x_{k+1} - z|| &= ||T((1 - \alpha_k)y_k + \alpha_k Ty_k) - z|| \\ &\leq a||(1 - \alpha_k y_k + \alpha_k Ty_k - z|| + b||(1 - \alpha_k)y_k + \alpha_k Ty_k - Tz|| \\ &\leq a(1 - \alpha_k)||y_k - z|| + a\alpha_k||Ty_k - z|| + b(1 - \alpha_k)||y_k - Tz|| \\ &+ b\alpha_k||Ty_k - Tz|| \\ &= (a + b)(1 - \alpha_k)||y_k - z|| + (a + b)\alpha_k||Ty_k - Tz|| \\ &\leq (1 - \alpha_k)||y_k - z|| + \alpha_k(a||y_k - z|| + b||y_k - Tz||) \\ &\leq ||y_k - z|| \\ &\leq ||x_k - z||. \end{aligned}$$

It follows that the sequence $\{x_k\}$ is Fejer monotone sequence with respect to $F(T)$. Hence from Proposition 2.4, sequence $\{x_k\}$ is bounded and $\{||x_k - z||\}$ converges, i.e., $\lim_{k\to\infty} ||x_k - z||$ exists for all $z \in F(T)$. □

Lemma 3.3. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1.1) and $\{\gamma_k\}$ be a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$.*

Proof. From Lemma 3.2, $\lim_{k\to\infty} ||x_k - z||$ exists for all $z \in F(T)$, so suppose that $\lim_{k\to\infty} ||x_k - z|| = p$, where $p \geq 0$. If $p = 0$, then

$$\begin{aligned} ||x_k - Tx_k|| &\leq ||x_k - z|| + ||z - Tx_k|| \\ &\leq ||x_k - z|| + a||x_k - z|| + b||x_k - Tz|| \\ &\leq (1 + a + b)||x_k - z|| \ \to \ 0 \ ask \to \infty. \end{aligned}$$

Let $p > 0$. Since $\lim_{k\to\infty} ||x_k - z|| = p \Rightarrow \limsup_{k\to\infty} ||x_k - z|| \leq p$. Also,

$$\begin{aligned} ||Tx_k - z|| &\leq (a + b)||x_k - z|| \\ \Rightarrow \limsup_{k\to\infty} ||Tx_k - z|| &\leq p. \end{aligned}$$

Let $\{\gamma_k\}$ be a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, then we have

$$\begin{aligned} \limsup_{k\to\infty} ||(1-\gamma_k)(x_k - z) + \gamma_k(Tx_k - z)|| &\leq (1-\gamma_k) \limsup_{k\to\infty} ||x_k - z|| \\ &\quad + \gamma_k \limsup_{k\to\infty} ||Tx_k - z|| \\ &\leq p. \end{aligned}$$

So from Lemma 2.5, we have $\lim_{k\to\infty} ||(x_k - z) - (Tx_k - z)|| = 0$, i.e., $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$. □

Theorem 3.4. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1.1). Then the sequence $\{x_k\}$ converges strongly to a fixed point of T if and only if $\lim_{k\to\infty} d(x_k, F(T)) = 0$, where $d(x_k, F(T)) = \inf\{||x_k - z|| : z \in F(T)\}$.*

Proof. If the sequence $\{x_k\}$ converges strongly to a fixed point of T, then it is obvious that $\lim_{k\to\infty} d(x_k, F(T)) = 0$.
For the converse part, suppose that $\lim_{k\to\infty} d(x_k, F(T)) = 0$. For $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$d(x_k, F(T)) < \frac{\varepsilon}{4}.$$

In particular there must be $p \in F(T)$ such that

$$||x_{k_0} - p|| < \frac{\varepsilon}{2}.$$

For $k, m \geq k_0$, we have

$$\begin{aligned} ||x_{k+m} - x_k|| &\leq ||x_{k+m} - p|| + ||p - x_k|| \\ &= 2||x_{k_0} - p|| < \varepsilon. \end{aligned}$$

It follows that $\{x_k\}$ is a Cauchy sequence in K. Since K is closed subset of uniformly convex Banach space X, so there exists a point say $x \in K$ such that

$||x_k - x|| \to 0$ as $k \to \infty$. By our assumption $\lim_{k\to\infty} d(x_k, F(T)) = 0$, it gives that

$$d(x, F(T)) = 0 \Rightarrow x \in F(T).$$

□

Theorem 3.5. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping such that it satisfy Condition (I). Let $F(T) \neq \emptyset$ and $\{x_k\}$ be a sequence in K defined by (1.1). Then $\{x_k\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 3.3, we have $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$ and T satisfy Condition (I), so we have $\lim_{k\to\infty} d(x_k, F(T)) = 0$. From the the Theorem 3.4, we conclude that $\{x_k\}$ converges strongly to a fixed point of T. □

Theorem 3.6. *Let K be a non-empty compact convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1.1). Then the sequence $\{x_k\}$ converges strongly to a fixed point of T.*

Proof. Let $z \in F(T)$. Since from Lemma 3.3, we have $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$, and K is compact, there must exists a subsequence $\{x_{k_p}\}$ of $\{x_k\}$ such that $x_{k_p} \to z$ for some $z \in K$. Note that

$$\begin{aligned} ||x_{k_p} - Tz|| &\leq ||x_{k_p} - Tx_{k_p}|| + ||Tx_{k_p} - Tz|| \\ &\leq ||x_{k_p} - Tx_{k_p}|| + a||x_{k_p} - z|| + b||x_{k_p} - Tz|| \\ &\leq ||x_{k_p} - Tx_{k_p}|| + ||x_{k_p} - z|| \to \ 0 \ as \ k \to \infty. \end{aligned}$$

This shows that $x_{k_p} \to Tz$ as $k \to \infty$, i.e., $z = Tz$. Also by Lemma 3.2, $\lim_{k\to\infty} ||x_k - z||$ exists, thus z is the strong limit of the sequence $\{x_k\}$ itself . □

Theorem 3.7. *Let K be a non-empty compact convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mean nonexpansive mapping such that $a+b < 1$ and $F(T) \neq \emptyset$. Let $\{x_k\}$ be a sequence in K defined by (1.1). Then the sequence $\{x_k\}$ is $T-$ stable.*

Proof. Let $\{t_k\} \subseteq X$ be any arbitrary sequence in K and suppose that the sequence $\{x_k\}$ generated by (1.1) is $x_{k+1} = f(T, x_k)$ converges to a point $z \in F(T)$ and $\epsilon_k = ||t_{k+1} - f(T, t_k)||$. To prove the stability of the sequence $\{x_k\}$ with respect to T, it is sufficient to prove that $\lim_{k\to\infty} \epsilon_k = 0$ if and only if $\lim_{k\to\infty} t_k = z$. Let $\lim_{k\to\infty} \epsilon_k = 0$. Note that

$$||y_k - z|| \leq (a + b)||z_k - z||,$$

$$||z_k - z|| \leq (1 - \beta_k + \beta_k(a + b))||x_k - z||,$$

and

$$\begin{aligned}
||t_{k+1}-z|| &\le ||t_{k+1}-f(T,t_k)|| + ||f(T,t_k)-z|| \\
&= \epsilon_k + ||f(T,t_k)-z|| \\
&\le \epsilon_k + (a+b)(1-\alpha_k+\alpha_k(a+b))(a+b)(1-\beta_k+\beta_k(a+b))||t_k-z|| \\
&= \epsilon_k + (a+b)^2[(1-\alpha_k)+\alpha_k(a+b)][(1-\beta_k)+\beta_k(a+b)]||t_k-z|| \\
&= \epsilon_k + (a+b)^2[1-((\alpha_k+\beta_k)(1-(a+b))-\alpha_k\beta_k(1-2(a+b)) \\
&- (a+b)^2)]||t_k-z||.
\end{aligned}$$

Let $\phi_k = (\alpha_k+\beta_k)(1-(a+b))-\alpha_k\beta_k(1-2(a+b))-(a+b)^2$. Since $\{\alpha_k\}, \{\beta_k\} \subset (0,1)$ and $a+b<1$, we conclude that $\alpha_k\beta_k \in (0,1)$, $\alpha_k+\beta_k \in (0,1)$. Therefore $\phi_k \in (0,1)$. By our assumption $\lim_{k\to\infty} \epsilon_k = 0$, we have $\lim_{k\to\infty} \frac{\epsilon_k}{\phi_k} = 0$. So from Lemma 2.7, we have $\lim_{k\to\infty} t_k = z$.
Conversely suppose that $\lim_{k\to\infty} t_k = z$. Since,

$$\begin{aligned}
\epsilon_k &= ||t_{k+1}-f(T,t_k)|| \\
&\le ||t_{k+1}-z|| + ||z-f(T,t_k)|| \\
&\le ||t_{k+1}-z|| + (a+b)^2[1-((\alpha_k+\beta_k)(1-(a+b)) \\
&- \alpha_k\beta_k(1-2(a+b))-(a+b)^2)]||t_k-z|| \ \to 0 \ as \ \lim k\to\infty.
\end{aligned}$$

It follows that $\lim_{k\to\infty} \epsilon_k = 0$. This shows that the iteration scheme (1.1) is stable with respect to T. $\square$

4. Numerical Example

Now, we show the fastness of the iteration scheme defined by (1.1) with other well-known iteration schemes by taking following example-

Example 4.1. Let X be a real uniformly convex Banach space and $K = [0,1]$ be a non-empty closed convex subset of X. Let $T : [0,1] \to [0,1]$ be a mapping defined by

$$\text{Tx}= \begin{cases} 0, \ 0 \le x < \frac{1}{2}, \\ \frac{x}{8}, \ \frac{1}{2} \le x \le 1. \end{cases}$$

Since T is not continuous at $x = \frac{1}{2}$, hence T is not a nonexpansive mapping. We claim that T is mean nonexpansive mapping. Consider the following cases:

Case I: Suppose that $0 \le x, y < \frac{1}{2}$. Then $||Tx-Ty|| = 0$. For $a = b = \frac{1}{3}$, we have $||Tx-Ty|| = 0 \le a||x-y|| + b||x-Ty||$.

Case II: Suppose that $\frac{1}{2} \leq x, y \leq 1$. Then $||Tx - Ty|| = \frac{1}{8}||x - y||$. Also

$$\begin{aligned} a||x-y|| + b||x-Ty|| &= \frac{1}{3}||x-y|| + \frac{1}{3}||x - \frac{y}{8}|| \\ &\geq \frac{1}{3}||x-y|| \\ &\geq \frac{1}{8}||x-y|| = ||Tx-Ty||. \end{aligned}$$

Case III: Let $0 \leq x < \frac{1}{2}$, $\frac{1}{2} \leq y \leq 1$. Then $||Tx - Ty|| = \frac{1}{8}||y||$, and

$$\begin{aligned} a||x-y|| + b||x-Ty|| &= \frac{1}{3}||x-y|| + \frac{1}{3}||x - \frac{y}{8}|| \\ &\geq \frac{1}{3}||x - y - (x - \frac{y}{8})|| \\ &= \frac{7}{24}||y|| \geq \frac{1}{8}||y|| = ||Tx-Ty||. \end{aligned}$$

Case IV: Let $\frac{1}{2} \leq x \leq 1$, $0 \leq y < \frac{1}{2}$, then $||Tx - Ty|| = \frac{1}{8}|||x||$, and

$$\begin{aligned} a||x-y|| + b||x-Ty|| &= \frac{1}{3}||x-y|| + \frac{1}{3}||x|| \\ &\geq \frac{1}{3}||x|| \geq \frac{1}{8}||x|| = ||Tx-Ty||. \end{aligned}$$

Thus we conclude that T is mean nonexpansive mapping.

Example 4.2. Let $X = R^2$ with the norm

$$||x-y|| = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$. Let K be a closed unit ball in R^2 such that $x, y \in K$ with $||x - y|| \leq \frac{1}{4}$. Now let $T : K \to K$ be a mapping defined by

$$Tx = \begin{cases} \frac{x}{5}, if\ 0 \leq x = (x_1, x_2) < \frac{1}{15}, \\ \frac{x}{6}, if\ \frac{1}{15} \leq x = (x_1, x_2) \leq 1. \end{cases} \tag{4.1}$$

Since T is not continuous at $x = \frac{1}{15}$, hence T is not a nonexpansive mapping. Althogh, it is mean nonexpansive mapping for $a = 1$ and $b = 0$.

TABLE 1. Comparison between convergence of (1.1) with Abbas [1], Karakaya [12], Thakur [22] iteration scheme to the fixed point $x = 0$ of T in Example 4.1

Iteration	Akutsah	Thakur	Abbas	Karakaya
0	0.50000000	0.50000000	0.50000000	0.50000000
1	0.00000000	-0.01953125	0.00781250	-0.00585938
2	0.00000000	0.00029564	-0.00002480	0.00002861
3	0.00000000	-0.00000257	0.00000011	-0.00000007
4	0.00000000	0.00000001	0.00000000	0.00000000
5	0.00000000	0.00000000	0.00000000	0.00000000
6	0.00000000	0.00000000	.00000000	0.00000000
7	0.00000000	0.00000000	0.00000000	0.00000000
8	0.00000000	0.00000000	0.00000000	0.00000000

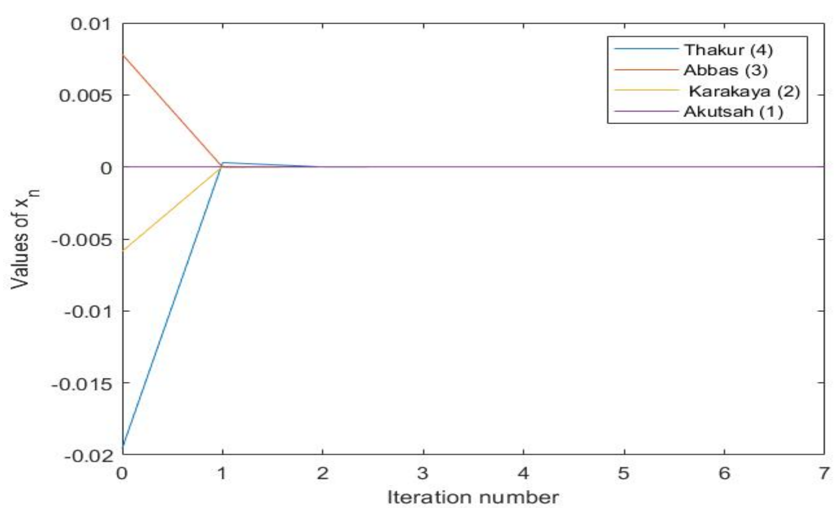

FIGURE 1. Convergence behaviour of iteration scheme (1.1) (purple), Thakur (cyan), Abbas (carrot orange) and Karakaya (yellow).

TABLE 2. (Comparison between convergence of (1.1) with MPIH [2], Agrawal [3], Sintunavarat [20] iterations to the fixed point $x = 0$ of T in Example 4.2.

Iteration	Akutsah (1)	MPIH (2)	Sintunavarat (3)	Agrawal (4)
0	0.20000000	0.20000000	0.20000000	0.20000000
1	0.00288000	0.01440000	-0.03840000	-0.08800000
2	0.00000461	0.00011520	0.00015360	0.01408000
3	0.00000000	0.00000010	0.00000086	-0.00118898
4	0.00000000	0.00000000	0.00000001	0.00005945
5	0.00000000	0.00000000	0.00000000	-0.00000181
6	0.00000000	0.00000000	0.00000000	0.00000003
7	0.00000000	0.00000000	0.00000000	0.00000000
8	0.00000000	0.00000000	0.00000000	0.00000000

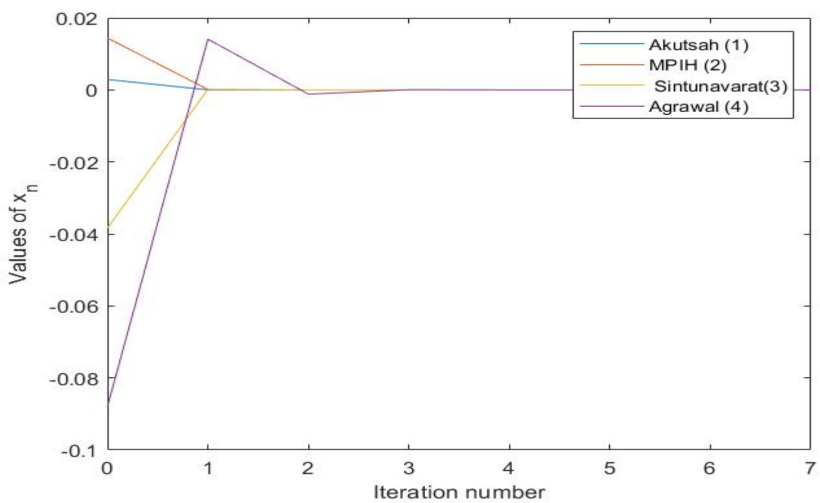

FIGURE 2. Behaviors of Akutsah iteration (cyan), Modified Picard Ishikawa hybrid iteration (carrot orange), Sintunavarat iteration (yellow), Agrawal iteration (magenta) to the fixed point $x = 0$ of the mapping T in Example 4.2.

5. APPLICATION IN SOLUTION OF NONLINEAR INTEGRAL EQUATION

Consider the following nonlinear integral equation-

$$x(t) = \lambda \int_a^1 f(t, s, x(s))ds + y(t),\ a \in [0, 1], \tag{5.1}$$

where $\lambda \in [0,1]$, $y : [0,1] \to \mathbb{R}$ and $f(t,s,x(s)) : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ all are continuous. Let $X = \mathbb{R}$. We define $||.|| : \mathbb{R} \to \mathbb{R}$ by

$$||x-y|| = \sup\{|x(s) - y(s)| : s \in [0,1]\},$$

for all $x \in \mathbb{R}$.

Theorem 5.1. *Let $X = \mathbb{R}$, $K = [0,1]$ and $T : K \to K$ defined by*

$$Tx(t) = \begin{cases} 0, \ x \in [0,a); \\ \lambda \int_a^1 f(t,s,x(s))ds + y(t), \ x \in [a,1]. \end{cases} \tag{5.2}$$

where $y : [0,1] \to \mathbb{R}$ and $f(t,s,x(s)) : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ all are continuous. suppose that the following conditions are satisfied-

(i) *There exists a continuous mapping $F : X \times X \to [0,\infty)$ such that*

$$|f(t,s,x(s)) - f(t,s,y(s))| \leq F(x,y)|x(s) - y(s)|,$$

for all $s \in [0,1]$, $x, y \in X$;

(ii) *there exists $\zeta \in (0,1]$ such that $\int_a^1 F(x,y) \leq \zeta$;*

(iii) $\lambda\zeta \leq 1$.

Let $\{x_k\}$ be a sequence in K defined by (1.1). Then the sequence $\{x_k\}$ converges to a point of $F(T)$ which is the solution of integral equation (5.2).

Proof. Let $x, y \in [0,1]$. Since T is discontinuous at $x = a$, hence T is not nonexpansive mapping. Now we show that T is mean nonexpansive mapping for $a' + b' \leq 1$. Consider the following cases:
Case I: when $x, y \in [0,a)$. Obviously T is mean nonexpansive for $a' = 1$, $b' = 0$.

Cae II: when $x, y \in [a,1]$. Then we have,

$$\begin{aligned} |Tx(s) - Ty(s)| &= |\lambda \int_a^1 f(t,s,x(s))dt - \lambda \int_a^1 f(t,s,y(s))dt| \\ &= |\lambda \int_a^1 (f(t,s,x(s)) - f(t,s,y(s)))dt| \\ &\leq \lambda \int_a^1 F(x,y)|x(s) - y(s)|dt \\ \sup_{s\in[0,1]} |Tx(s) - Ty(s)| &\leq \sup_{s\in[0,1]} |x(s) - y(s)|\lambda \int_a^1 F(x,y)dt \\ &\leq ||x-y||\lambda\zeta \\ &\leq ||x-y|| \\ &\Rightarrow ||Tx - Ty|| \leq a'||x-y|| + b'||x - Ty||. \end{aligned}$$

Hence, we conclude that T is mean nonexpansive mapping for $a^{'} = 1$, $b^{'} = 0$. Also

$$\begin{aligned}
||x - Tx|| &= \sup\{|x(s) - Tx(s)| : s \in [0,1]\} \\
&= \sup\{|x(s) - \lambda \int_a^1 f(t,s,x(s))dt - y(s)||\} \\
&= \sup\{|x(s) - x(s)| : s \in [0,1]\} \\
&\Rightarrow ||x - Tx|| = 0.
\end{aligned}$$

From Theorem 3.6, we have $F(T) \neq \emptyset$ and the sequence $\{x_k\}$ converges strongly to a fixed point of T. This point will be solution of integral equation (5.2). □

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[1] Department of Mathematics, Government Dudhadhari Bajrang Girls Postgraduate Autonomous College, Raipur (Chhattisgarh)-492001, India.
Email address: dewangan.kiran@gmail.com

[2] Department of Mathematics, Government J. Yoganandam Chhattisgarh College, Raipur (Chhattisgarh)-492001, India.
Email address: second@amcs.org