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# SOME UNIQUE COMMON FIXED POINT RESULTS ON $C^*$-ALGEBRA VALUED METRIC SPACE USING $C_*$-CLASS FUNCTION

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ABSTRACT. Fixed point theorems have been establishing and developing in many metric spaces, Banach spaces, Uniform spaces etc. under many different type of expansive and non-expansive conditions. C. Park et al. [1], H. Massit et al. [3], Z. Ma et al. ([5], [6]), M. Erden et al. [14] and B. Moeini et al. [15] have established various fixed point results on $C^*$-algebra valued metric under different type of contractive conditions. In this paper we have establish some unique common fixed point results for six mappings on $C^*$-algebra valued metric spaces on $C_*$-class function, which is a generalization of theorems of C. Park et al. [1]. Here we have discussed about some basic definitions, properties and lemmas on $C^*$-algebra valued metric spaces in the introduction preliminaries section. Some examples and corollaries are also given on the basis of the results.

## 1. INTRODUCTION AND PRELIMINARIES

Let $X$ be a non-empty set and $T : X \to X$ be a map on $X$.A point $x \in X$ is said to be a fixed point on $X$ if $Tx = x$. In 2000, Branciary [11] first replacing the triangular inequality $d(u, v) \leq d(u, w) + d(w, v)$ by rectangular inequality $d(u, v) \leq d(u, w) + d(w, z) + d(z, v)$, for all $u, v, w, z \in X$ and $w, z$ are different from $u, v$, generalized the generalization of metric spaces.Fixed point theorem have many generalization in many directions, one of such direction is unique common fixed point theorem. Ma et al [6] introduced $C^*$-algebra valued metric space and developed some fixed point results.

A complete normed algebra $A$ is called a Banach algebra.A complex algebra with linear involution $*$ satisfying $(uv)^* = v^*u^*$ and $u^{**} = u$, for all $u, v \in A$ is said to Banach $*$-algebra if it is complete with respect to sub multiplicative norm such that for all $u \in A$, $\|u^*\| = \|u\|$.A Banach $*$- algebra satisfying $\|u^*u\| = \|u\|^2$ is called $C^*$-algebra.A complete unital normed algebra is called a unital Banach algebra.In this paper, we denote $A$ by an unital( unity element $I_A$) $C^*$-algebra with linear involution $*$.

We call an element $u \in A$ a positive element and denote it by $u \succeq \theta_A$, where $\theta_A$ is the zero element in $A$, if $u = u^*$ and $\rho(u) \subset [0, +\infty)$, where $\rho(u)$ is the spectrum of $u$.

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We define norm on $A$ by $\|u\| = (u^*u)^{\frac{1}{2}}$.Set $A_h = \{u \in A : u = u^*\}$.
We define a partial ordering $\preceq$ on $A_h$ if it satisfying $u \preceq v$ if and only if $v - u \succeq \theta_A$.
Here $A_+$ and $A'$ are defined by $A_+ = \{u \in A : \theta_A \preceq a\}$ and $A' = \{u \in A : uv = vu, \forall v \in A\}$ When $A$ is a unital $C^*$-algebra, then for any $u \in A_+$ we have

$$u \preceq I_A \Leftrightarrow \|u\| \leq 1 \text{ and } \|I_A\| = 1.$$

In this paper we establish some unique common fixed point theorems for six mappings in $C^*$-algebra valued metric spaces under different conditions,which is a generalization of theorems in [1] and [5].

**Definition 1.1.** [1] Let $X$ be a non-empty set and $d : X \times X \to A$ be a mapping such that for $u, v \in X$ and for $w \in X$, where $w$ is different from $u$ and $v$ such that

$(c_1)$ $d(u, v) = \theta_A$ if and only if $u = v$; and $\theta_A \preceq d(u, v)$, for all $u, v \in X$;
$(c_1)$ $d(u, v) = d(v, u)$ for all distinct points $u, v \in X$;
$(c_1)$ $d(u, w) \preceq d(u, v) + d(v, w)$, for all $u, v, w \in X$;

Then $(X, A, d)$ is called a $C^*$-algebra valued metric space.

**Note:** If $A = \mathbb{R}$, then $C^*$-algebra valued metric space becomes equivalent to the definition to the real metric space.

**Definition 1.2.** [1] Let $(X, A, d)$ be a $C^*$-algebra valued metric space with norm $\|, \|$. A sequence $\{x_n\} \subset X$ is said to be convergent to $x \in X$ with respect to $A$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\|d(x_n, x)\| < \epsilon$.

**Definition 1.3.** [3] Let $(X, A, d)$ be a $C^*$-algebra valued metric space with norm $\|, \|$. A sequence $\{x_n\} \subset X$ is said to be Cauchy sequence in $X$ with respect to $A$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $\|d(x_n, x_m)\| < \epsilon$.

The space $(X, A, d)$ is complete $C^*$-algebra valued metric space iff every Cauchy sequence in $X$ with respect to $A$ is convergent in $X$.

**Definition 1.4.** [5] Two mappings $f$ and $g$ on a $C^*$-algebra valued metric space $(X, A, d)$ is said to be compatible if for arbitrary sequence $\{u_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = z \in X \text{ implies } \lim_{n\to\infty} fgu_n = \lim_{n\to\infty} fgu_n.$$

**Definition 1.5.** [5] Let $f$ and $g$ be two maps on a set $X$ (i.e, $f, g : X \to X$). If $w = fx = gx$ for some $x \in X$, then $x$ is called a coincidence point of $f$ and $g$; and $w$ is called a point of coincidence of $f$ and $g$.

A pair of maps $f$ and $g$ is called weakly compatible if they commute at coincidence points.

**Lemma 1.1.** [5]

*(i) If $\{u_n\}_{n=1}^{\infty} \subset A$ and $\lim_{n\to\infty} u_n = \theta_A$, then for any $v \in A$, $\lim_{n\to\infty} v^*u_nv = \theta_A$.*
*(ii) If $u, v \in A_+$ and $w \in A'_+$, then $u \preceq v$ deduces $wu \preceq wv$, where $A'_+ = A_+ \cap A'$*
*(iii) Limit of a convergent sequence in a $C^*$ -algebra valued metric space is unique.*

**Definition 1.6.** [1] Let $A$ be a unit $C^*$-algebra and $C_*$ be the set of all continuous functions $F_* : A_+ \times A_+ \to A_+$ such that for any $P, Q \in A_+$, the following conditions hold:
(i) $F_*(P, Q) \preceq P$; (ii) $F_*(P, Q) = P$ implies either $P = \theta_A$, or, $Q = \theta_A$; (iii) $F_*(\theta_A, \theta_A) = \theta_A$;

**Definition 1.7.** [1] $\Psi$ be the set of all functions $\psi : A_+ \to A_+$ satisfying the following conditions:
(i) $\psi$ is continuous and non-decreasing; (ii) $\psi(T) = \theta_A \Leftrightarrow T = \theta_A$.

**Theorem 1.1.** [1] *Let $(X, A, d)$ be a complete $C^*$-algebra valued metric space and $T : X \to X$ be a self mapping such that for all $x, y \in X$, the following holds:*

$$\varphi(d(Tx, Ty)) \preceq F_*\{\varphi(d(x, y)), \phi(d(x, y))\};\ \textit{where } \phi, \varphi \in \Psi \textit{ and } F_* \in C_*.$$

*Then $T$ has a unique fixed point in $X$.*

## 2. Main results

**Definition 2.1.** A function $\psi : A_+ \to A_+$ is said to be monotonic non-decreasing if

$$u \preceq v \Rightarrow \psi(u) \preceq \psi(v),\ \forall u, v \in A_+.$$

**Definition 2.2.** A function $J_* : A_+ \times A_+ \to A_+$ is called a non-decreasing function if

$$U \subset W, V \subset Z \Rightarrow J_*(U, V) \preceq J_*(W, Z).$$

**Definition 2.3.** ($C^*$ -Class function) Let $A$ be a unit $C^*$-algebra. Then a continuous function $J_* : A_+ \times A_+ \to A_+$ is called a $C_*$-class function if for any $U, V \in A_+$, such that the following conditions hold:

(i) $J_*$ is non-decreasing;
(ii) $J_*(U, V) \preceq U$;
(iii) $J_*(U, V) = U$ implies either $U = \theta_A$ or $V = \theta_A$;
(iv) $J_*(\theta_A, \theta_A) = \theta_A$; Here $C_*$ will denote the class of all $C_*$-class function.

Let $\Phi$ be the set of all functions $\phi : A_+ \to A_+$ satisfying the following conditions:

(i) $\lim\limits_{T \to Y} \phi(T)$ exists, for all $Y \in A_+$;
(ii) $\phi$ is continuous at $\theta_A$ and $\phi(u) = \theta_A$ if and only if $u = \theta_A$.

**Theorem 2.1.** *Let $(X, A, d)$ be a complete $C^*$-algebra valued metric space with norm $\|,\|$ and $F, G, S, T, H, L$ are self maps on $X$ such that for all $x, y \in X$, the following conditions hold:*

$$\psi\{d(Hx, Ly)\} \preceq J_*\{\psi\{M(x, y)\}, \phi\{M(x, y)\}\},\ \psi, \phi \in \Phi \textit{ and } J_* \in C_*; \tag{2.1}$$

$$\begin{aligned} \textit{where, } M(x, y) &= a_1 d(FGx, Hx) + a_2 d(STy, Ly) + a_3 d(STy, Hx) + a_4 d(FGx, Ly) \\ &+ a_5 d(FGx, STy);\ \textit{where } a_i \in A',\ \forall\, i = 1, 2, 3, 4, 5. \end{aligned} \tag{2.2}$$

*with $a_1 + a_2 + 2(a_3 + a_4) + a_5 \preceq I_A;\ \theta_A \preceq a_i$, for all $i = 1, 2, 3, 4, 5$;*
*and $\|a_1\| + \|a_2\| + 2(\|a_3\| + \|a_4\|) + \|a_5\| \leq 1$.*
*Also*

*(i) $H(X) \subset ST(X), L(X) \subset FG(X)$;*

*(ii)* $FG = GF, ST = TS, HG = GH, LT = TL$;
*(iii)* $\{H, FG\}$ *is compatible and* $\{L, ST\}$ *is weakly compatible; either* $H$ *or* $FG$ *is continuous.*

*Then* $F, G, S, T, H, L$ *have a unique common fixed point in* $X$.

**Proof:** Let $x_0 \in X$. From condition (i), there exist $x_1, x_2 \in X$ such that $Hx_0 = STx_1 = y_0$ and $Lx_1 = FGx_2 = y_1$. Proceeding inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$y_{2n} = Hx_{2n} = STx_{2n+1} \text{ and } y_{2n+1} = Lx_{2n+1} = FGx_{2n+2}, \text{ for all } n = 0, 1, 2, ...$$

**Case-I:** If for some $n \in \mathbb{Z}^+, y_n = y_{n+1} \Rightarrow y_{n+1} = y_{n+2}$.
Now for $n$ is even i.e., for $n = 2m$, where $m \in \mathbb{Z}^+$, we have

$$y_{2m} = y_{2m+1} \tag{2.3}$$

From (2.2) we have

$$\begin{aligned}
&M(x_{2m+2}, x_{2m+1}) \\
&= a_1 d(FGx_{2m+2}, Hx_{2m+2}) + a_2 d(STx_{2m+1}, Lx_{2m+1}) + a_3 d(STx_{2m+1}, Hx_{2m+2}) + \\
&a_4 d(FGx_{2m+2}, Lx_{2m+1}) + a_5 d(FGx_{2m+2}, STx_{2m+1}) \\
&= a_1 d(y_{2m+1}, y_{2m+2}) + a_2 d(y_{2m}, y_{2m+1}) + a_3 d(y_{2m}, y_{2m+2}) + a_4 d(y_{2m+1}, y_{2m+1}) + \\
&a_5 d(y_{2m+1}, y_{2m}) \\
&= a_1 d(y_{2m+1}, y_{2m+2}) + a_2 d(y_{2m}, y_{2m+1}) + a_3 d(y_{2m+1}, y_{2m+2}) + a_5 d(y_{2m+1}, y_{2m+1}) \\
&= (a_1 + a_3) d(y_{2m+1}, y_{2m+2}) \\
&\preceq d(y_{2m+1}, y_{2m+2})
\end{aligned} \tag{2.4}$$

From (2.1) we have

$$\begin{aligned}
\psi\{d(y_{2m+2}, y_{2m+1})\} &\preceq J_*\{\psi\{M(x_{2m+2}, x_{2m+1})\}, \phi\{M(x_{2m+2}, x_{2m+1})\}\} \\
&\preceq J_*\{\psi\{d(y_{2m+1}, y_{2m+2})\}, \phi\{d(y_{2m+1}, y_{2m+2})\}\} \\
&\preceq \psi\{d(y_{2m+1}, y_{2m+2})\}, \text{ which gives}
\end{aligned}$$

$J_*\{\psi\{d(y_{2m+1}, y_{2m+2})\}, \phi\{d(y_{2m+1}, y_{2m+2})\}\} = \psi\{d(y_{2m+1}, y_{2m+2})\}$
Therefore, either $\psi\{d(y_{2m+1}, y_{2m+2})\} = \theta_A$, or, $\phi\{d(y_{2m+1}, y_{2m+2})\} = \theta_A$.
Hence,

$$d(y_{2m+1}, y_{2m+2}) = \theta_A \text{ implies } y_{2m+1} = y_{2m+2} \tag{2.5}$$

For $n$ is odd i.e., for $n = 2m + 1, m \in \mathbb{N}$, then

$$y_{2m+1} = y_{2m+2} \tag{2.6}$$

Now from (2.2)we have

$$
\begin{aligned}
&M(x_{2m+2},x_{2m+3})\\
&= a_1d(FGx_{2m+2},Hx_{2m+2})+a_2d(STx_{2m+3},Lx_{2m+3})+a_3d(STx_{2m+3},Hx_{2m+2})+\\
&a_4d(FGx_{2m+2},Lx_{2m+3})+a_5d(FGx_{2m+2},STx_{2m+3})\\
&= a_1d(y_{2m+1},y_{2m+2})+a_2d(y_{2m+2},y_{2m+3})+a_3d(y_{2m+2},y_{2m+2})+a_4d(y_{2m+1},y_{2m+3})\\
&+a_5d(y_{2m+1},y_{2m+2})\\
&= a_1d(y_{2m+1},y_{2m+2})+a_2d(y_{2m+2},y_{2m+3})+a_4d(y_{2m+2},y_{2m+3})+a_5d(y_{2m+1},y_{2m+2})\\
&= (a_2+a_4)d(y_{2m+1},y_{2m+2})\\
&\preceq d(y_{2m+2},y_{2m+3})
\end{aligned}
\tag{2.7}
$$

From (2.1) we have

$$
\begin{aligned}
\psi\{d(y_{2m+2},y_{2m+3})\} &\preceq J_*\{\psi\{M(x_{2m+2},x_{2m+3})\},\phi\{M(x_{2m+2},x_{2m+3})\}\}\\
&\preceq J_*\{\psi\{d(y_{2m+2},y_{2m+3})\},\phi\{d(y_{2m+2},y_{2m+3})\}\}\\
&\preceq \psi\{d(y_{2m+2},y_{2m+3})\},\ \text{which gives,}
\end{aligned}
$$

$J_*\{\psi\{d(y_{2m+2},y_{2m+3})\},\phi\{d(y_{2m+2},y_{2m+3})\}\}=\psi\{d(y_{2m+2},y_{2m+3})\}$
Therefore, either $\psi\{d(y_{2m+2},y_{2m+3})\}=\theta_A$, or, $\phi\{d(y_{2m+2},y_{2m+3})\}=\theta_A$.
Hence,

$$d(y_{2m+2},y_{2m+3})=\theta_A \text{ implies } y_{2m+2}=y_{2m+3}. \tag{2.8}$$

From (2.6) and (2.8) we have

$$y_n=y_{n+1} \text{ implies } y_{n+1}=y_{n+2},\ \text{for all}\ \ n=1,2,3,...$$

Proceeding in this manner we have,

$$y_n=y_{n+1} \text{ implies } y_n=y_{n+k},\ \text{for } k=1,2,3,...$$

Therefore, $\{y_n\}$ becomes a constant sequence and hence a Cauchy one in $X$.
**Case-II:** Assume $y_n\neq y_{n+1}$, for all $n=1,2,3,...$
For $n=2m$, we have from (2.2)

$$
\begin{aligned}
&M(x_{2m},x_{2m+1})\\
&= a_1d(FGx_{2m},Hx_{2m})+a_2d(STx_{2m+1},Lx_{2m+1})+a_3d(STx_{2m+1},Hx_{2m})+\\
&a_4d(FGx_{2m},Lx_{2m+1})+a_5d(FGx_{2m},STx_{2m+1})\\
&= a_1d(y_{2m-1},y_{2m})+a_2d(y_{2m},y_{2m+1})+a_3d(y_{2m},y_{2m})+a_4d(y_{2m-1},y_{2m+1})+\\
&a_5d(y_{2m-1},y_{2m})\\
&\preceq a_1d(y_{2m-1},y_{2m})+a_2d(y_{2m},y_{2m+1})+a_4d(y_{2m-1},y_{2m})+a_4d(y_{2m},y_{2m+1})+\\
&a_5d(y_{2m-1},y_{2m})
\end{aligned}
\tag{2.9}
$$

If $d(y_{2m},y_{2m-1})\preceq d(y_{2m+1},y_{2m})$, then from (2.9) we have

$$
\begin{aligned}
M(x_{2m},x_{2m+1}) &\preceq (a_1+a_2+2a_4+a_5)d(y_{2m+1},y_{2m})\\
&\preceq d(y_{2m+1},y_{2m})\ [\text{as}(a_1+a_2+2a_4+a_5)\preceq I_A]
\end{aligned}
\tag{2.10}
$$

From (2.1) we have

$$\begin{aligned}\psi\{d(Hx_{2m}, Lx_{2m+1})\} &\preceq J_*\{\psi\{M(x_{2m}, x_{2m+1})\}, \phi\{M(x_{2m}, x_{2m+1})\}\}\\ \text{or, } \psi\{d(y_{2m}, y_{2m+1})\} &\preceq J_*\{\psi\{d(y_{2m+1}, y_{2m})\}, \phi\{d(y_{2m+1}, y_{2m})\}\}\\ &\preceq \psi\{d(y_{2m+1}, y_{2m})\}, \text{ which gives,}\end{aligned}$$

$J_*\{\psi\{d(y_{2m+1}, y_{2m})\}, \phi\{d(y_{2m+1}, y_{2m})\}\} = \psi\{d(y_{2m+1}, y_{2m})\}$
So, either $\psi\{d(y_{2m+1}, y_{2m})\} = \theta_A$, or, $\phi\{d(y_{2m+1}, y_{2m})\} = \theta_A$.
Hence, $d(y_{2m+1}, y_{2m}) = \theta_A$ implies $y_{2m+1} = y_{2m}$, which is a contradiction.
Hence,

$$d(y_{2m+1}, y_{2m}) \prec d(y_{2m}, y_{2m-1}) \tag{2.11}$$

For $n$ is odd i.e., for $n = 2m+1, m \in \mathbb{N}$, then we have from (2.2)

$$\begin{aligned}&M(x_{2m+2}, x_{2m+1})\\ &= a_1 d(FGx_{2m+2}, Hx_{2m+2}) + a_2 d(STx_{2m+1}, Lx_{2m+1}) + a_3 d(STx_{2m+1}, Hx_{2m+2}) +\\ &a_4 d(FGx_{2m+2}, Lx_{2m+1}) + a_5 d(FGx_{2m+2}, STx_{2m+1})\\ &= a_1 d(y_{2m+1}, y_{2m+2}) + a_2 d(y_{2m}, y_{2m+1}) + a_3 d(y_{2m}, y_{2m+2}) + a_4 d(y_{2m+1}, y_{2m+1}) +\\ &a_5 d(y_{2m+1}, y_{2m})\\ &\preceq a_1 d(y_{2m+1}, y_{2m+2}) + a_2 d(y_{2m}, y_{2m+1}) + a_3 d(y_{2m}, y_{2m+1}) + a_3 d(y_{2m+1}, y_{2m+2}) +\\ &a_5 d(y_{2m+1}, y_{2m})\end{aligned} \tag{2.12}$$

If $d(y_{2m}, y_{2m+1}) \preceq d(y_{2m+1}, y_{2m+2})$, then from (2.12) we have

$$\begin{aligned}M(x_{2m+2}, x_{2m+1}) &\preceq (a_1 + a_2 + 2a_3 + a_5) d(y_{2m+1}, y_{2m+2})\\ &\preceq d(y_{2m+1}, y_{2m+2}) \;[\text{as}(a_1 + a_2 + 2a_2 + a_5) \preceq I_A]\end{aligned} \tag{2.13}$$

Now from (2.1) we have

$$\begin{aligned}\psi\{d(Hx_{2m+2}, Lx_{2m+1})\} &\preceq J_*\{\psi\{M(x_{2m+2}, x_{2m+1})\}, \phi\{M(x_{2m+2}, x_{2m+1})\}\}\\ or, \psi\{d(y_{2m+2}, y_{2m+1})\} &\preceq J_*\{\psi\{d(y_{2m+2}, y_{2m+1})\}, \phi\{d(y_{2m+2}, y_{2m+1})\}\}\\ &\preceq \psi\{d(y_{2m+2}, y_{2m+1})\}, \text{ which gives}\end{aligned}$$

$J_*\{\psi\{d(y_{2m+2}, y_{2m+1})\}, \phi\{d(y_{2m+2}, y_{2m+1})\}\} = \psi\{d(y_{2m+2}, y_{2m+1})\}$
So, either $\psi\{d(y_{2m+2}, y_{2m+1})\} = \theta_A$, or, $\phi\{d(y_{2m+2}, y_{2m+1})\} = \theta_A$.
Hence, $d(y_{2m+2}, y_{2m+1}) = \theta_A \Rightarrow y_{2m+2} = y_{2m+1}$, which is a contradiction.
Hence

$$d(y_{2m+1}, y_{2m+2}) \prec d(y_{2m}, y_{2m+1}) \tag{2.14}$$

From (2.11) and (2.14) we have

$$d(y_n, y_{n+1}) \prec d(y_{n-1}, y_n), \text{for all } n = 1, 2, 3, ... \tag{2.15}$$

Therefore, $\{d(y_n, y_{n+1})\}$ is monotonic decreasing sequence and bounded below and hence convergent in $X$.
Let $\lim\limits_{n\to\infty} d(y_n, y_{n+1}) = U$, $\lim\limits_{n\to\infty} \psi\{d(y_n, y_{n+1})\} = V$ and $\lim\limits_{n\to\infty} \phi\{d(y_n, y_{n+1})\} = W$, where $U, V, W \in A_+$, i.e., $U, V, W \succeq \theta_A$.

**Claim:** $U=\theta_A$ i.e., $\lim\limits_{n\to\infty} d(y_n,y_{n+1})=\theta_A$.
From (2.1) we have

$$\begin{aligned} \psi\{d(y_{2m},y_{2m+1})\} &\preceq J_*\{\psi\{M(x_{2m},x_{2m+1})\},\phi\{M(x_{2m},x_{2m+1})\}\} \\ \text{or, } \psi\{d(Hx_{2m},Lx_{2m+1})\} &\preceq J_*\{\psi\{d(y_{2m-1},y_{2m})\},\phi\{d(y_{2m-1},y_{2m})\}\} \\ &\preceq \psi\{d(y_{2m-1},y_{2m})\} \end{aligned} \tag{2.16}$$

Taking limit as $m\to\infty$ on both sides of (2.16) we have

$$\begin{aligned} \lim_{m\to\infty}\psi\{d(y_{2m},y_{2m+1})\} &\preceq \lim_{m\to\infty} J_*\{\psi\{M(x_{2m},x_{2m+1})\},\phi\{M(x_{2m},x_{2m+1})\}\} \\ &\preceq \lim_{m\to\infty} J_*\{\psi\{d(y_{2m-1},y_{2m})\},\phi\{d(y_{2m-1},y_{2m})\}\} \\ &= J_*\{\lim_{m\to\infty}\psi\{d(y_{2m-1},y_{2m})\},\lim_{m\to\infty}\phi\{d(y_{2m-1},y_{2m})\}\} \\ &\preceq \lim_{m\to\infty}\psi\{d(y_{2m-1},y_{2m})\} \end{aligned}$$

Therefore,

$$\begin{aligned} V \preceq J_*\{\lim_{m\to\infty}\psi\{d(y_{2m-1},y_{2m})\},\lim_{m\to\infty}\phi\{d(y_{2m-1},y_{2m})\}\} \preceq V \\ \text{or, } V\preceq J_*(V,W)\preceq V \\ \text{or, } J_*(V,W)=V \end{aligned} \tag{2.17}$$

Therefore, either $V=\theta_A$, or, $W=\theta_A$.
Thus, either $\lim\limits_{m\to\infty}\psi\{d(y_{2m-1},y_{2m})\}=\theta_A$, or, $\lim\limits_{m\to\infty}\phi\{d(y_{2m-1},y_{2m})\}=\theta_A$, which gives,

$$\lim_{m\to\infty} d(y_{2m-1},y_{2m})=\theta_A.$$

Hence,

$$\lim_{m\to\infty} d(y_{2m-1},y_{2m})=\theta_A \tag{2.18}$$

Similarly, for $n=2m+1$, we have $U=\theta_A$ i.e.,

$$\lim_{m\to\infty} d(y_{2m+1},y_{2m})=\theta_A \tag{2.19}$$

Hence, from (2.18) and (2.19) we have

$$\lim_{n\to\infty} d(y_{n+1},y_n)=\theta_A \tag{2.20}$$

and hence

$$\lim_{n\to\infty}\|d(y_{n+1},y_n)\|=0 \tag{2.21}$$

**Step 1:**
Now we prove that $\{y_n\}$ is Cauchy in $X$ with respect to $A$. For this it is sufficient to show that $\{y_{2n}\}$ is Cauchy in $X$.
If not then, for $\epsilon>0$, there exist integers $2n_k$ and $2m_k$ with $2m_k>2n_k>k$ such that

$$\|d(y_{2m_k},y_{2n_k})\|>\epsilon \text{ and } \|d(y_{2m_k-2},y_{2n_k})\|\le\epsilon \tag{2.22}$$

Now $\epsilon<\|d(y_{2m_k},y_{2n_k})\|\le\|d(y_{2n_k},y_{2m_k-2})\|+\|d(y_{2m_k-2},y_{2m_k-1})\|+\|d(y_{2m_k-1},y_{2m_k})\|$.
Taking limit as $k\to\infty$ we get

$$\epsilon\le\lim_{k\to\infty}\|d(y_{2m_k},y_{2n_k})\|\le\epsilon.$$

Hence,

$$\lim_{k\to\infty}\|d(y_{2m_k},y_{2n_k})\|=\epsilon \tag{2.23}$$

Again $\epsilon<\|d(y_{2m_k+1},y_{2n_k})\|\leq\|d(y_{2m_k},y_{2m_k+1})\|+\|d(y_{2m_k+1},y_{2n_k})\|$.
Taking limit as $k\to\infty$ we get

$$\epsilon\leq+\lim_{k\to\infty}\|d(y_{2m_k+1},y_{2n_k})\|\leq\epsilon,\ \text{which gives}$$

$$\lim_{k\to\infty}\|d(y_{2m_k+1},y_{2n_k})\|=\epsilon \tag{2.24}$$

By similar way we obtain

$$\lim_{k\to\infty}\|d(y_{2m_k},y_{2n_k+1})\|=\epsilon=\lim_{k\to\infty}\|d(y_{2n_k-1},y_{2m_k+1})\| \tag{2.25}$$

Therefore, there exists $U\in A_+$ with $\|U\|=\epsilon$ such that

$$\lim_{k\to\infty}d(y_{2m_k},y_{2n_k})=\lim_{k\to\infty}d(y_{2m_k+1},y_{2n_k})=\lim_{k\to\infty}d(y_{2m_k},y_{2n_k+1})=\lim_{k\to\infty}d(y_{2n_k-1},y_{2m_k+1})=U.$$

Furthermore, there exists $K\in\mathbb{N}$ and with $\epsilon>0$ such that

$$\|d(y_{2n_k-1},y_{2m_k})\|>\tfrac{\epsilon}{2},\ \|d(y_{2n_k-1},y_{2n_k})\|<\tfrac{\epsilon}{2},\ \|d(y_{2m_k+1},y_{2m_k})\|<\tfrac{\epsilon}{2},\ \|d(y_{2n_k},y_{2m_k+1})\|>\tfrac{\epsilon}{2},\ \text{for all } 2m_k,2n_k>K.$$

Now for $2m_k,2n_k>K$, from (2.2) we have

$$\begin{aligned}
&M(x_{2n_k},x_{2m_k+1})\\
&=a_1d(FGx_{2n_k},Hx_{2m_k})+a_2d(STx_{2m_k+1},Lx_{2m_k+1})+a_3d(STx_{2m_k+1},Hx_{2n_k})+\\
&\quad a_4d(FGx_{2n_k},Lx_{2m_k+1})+a_5d(FGx_{2n_k},STx_{2m_k+1})\\
&=a_1d(y_{2n_k-1},y_{2n_k})+a_2d(y_{2m_k},y_{2m_k+1})+a_3d(y_{2m_k},y_{2n_k})+a_4d(y_{2n_k-1},y_{2m_k+1})+\\
&\quad a_5d(y_{2n_k-1},y_{2m_k})
\end{aligned} \tag{2.26}$$

Therefore, from (2.26) we have

$$\begin{aligned}
\|M(x_{2n_k},x_{2m_k+1})\|\leq{}&\|a_1\|\|d(y_{2n_k-1},y_{2n_k})\|+\|a_2\|\|d(y_{2m_k},y_{2m_k+1})\|+\|a_3\|\|d(y_{2m_k},y_{2n_k})\|\\
&+\|a_4\|\|d(y_{2n_k-1},y_{2m_k+1})\|+\|a_5\|\|d(y_{2n_k-1},y_{2m_k})\|
\end{aligned} \tag{2.27}$$

Taking limit as $k\to\infty$ on both sides of (2.26) we have

$$\begin{aligned}
\lim_{k\to\infty}M(x_{2n_k},x_{2m_k+1})&=\lim_{k\to\infty}\{a_1d(y_{2n_k-1},y_{2n_k})+a_2d(y_{2m_k},y_{2m_k+1})+a_3d(y_{2m_k},y_{2n_k})+\\
&\quad a_4d(y_{2n_k-1},y_{2m_k+1})+a_5d(y_{2n_k-1},y_{2m_k})\}\\
&=(a_3+a_4+a_5)U\\
&\preceq U
\end{aligned} \tag{2.28}$$

from (2.1) we get

$$\begin{aligned}
\psi\{d(y_{2n_k},y_{2m_k+1})\}&=\psi\{d(Hx_{2n_k},Lx_{2m_k+1})\}\\
&\preceq J_*\{\psi\{M(x_{2n_k},x_{2m_k+1})\},\phi\{M(x_{2n_k},x_{2m_k+1})\}\}\\
&\preceq\psi\{M(x_{2n_k},x_{2m_k})\}
\end{aligned} \tag{2.29}$$

Taking limit as $k \to \infty$ on both sides of (2.29) we have

$$\begin{aligned}\lim_{k\to\infty} \psi\{d(y_{2n_k}, y_{2m_k+1})\} &\preceq \lim_{k\to\infty} J_*\{\psi\{M(x_{2n_k}, x_{2m_k+1})\}, \phi\{M(x_{2n_k}, x_{2m_k+1})\}\} \\ &\preceq \lim_{k\to\infty} \psi\{M(x_{2n_k}, x_{2m_k})\}\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{k\to\infty} \psi\{d(y_{2n_k}, y_{2m_k+1})\} &\preceq J_*\{\lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}, \lim_{k\to\infty} \phi\{d(x_{2n_k}, x_{2m_k+1})\}\} \\ &\preceq \lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}\end{aligned} \tag{2.30}$$

$$[\text{As } \lim_{k\to\infty} d(y_{2n_k}, y_{2m_k+1}) = \lim_{n\to\infty} M(x_{2n_k}, x_{2m_k+1}) = U]$$

Therefore, from (2.30) we have
$J_*\{\lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}, \lim_{k\to\infty} \phi\{d(x_{2n_k}, x_{2m_k+1})\}\} = \lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}$.
Hence, by the definition of $J_*$ function

$$\text{either } \lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\} = \theta_A, \text{ or, } \lim_{k\to\infty} \phi\{d(x_{2n_k}, x_{2m_k+1})\} = \theta_A,$$

which implies that

$\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}) = \theta_A$, which is a contradiction.
Hence $\{y_n\}$ is a Cauchy sequence in $X$.
Since $(X, A_+, d)$ is complete so, there exists $z \in X$ such that

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Hx_{2n} = \lim_{n\to\infty} STx_{2n+1} = z. \tag{2.31}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Lx_{2n+1} = \lim_{n\to\infty} FGx_{2n+2} = z. \tag{2.32}$$

Assume $FG$ is continuous.
Then $\lim_{n\to\infty} (FG)^2 x_{2n} = FGz$ and $\lim_{n\to\infty} (FG)Hx_{2n} = FGz$
As $(H, FG)$ is compatible, so we have $\lim_{n\to\infty} H(FGx_{2n}) = FGz$
**Step 2:**
**Claim:** $FGz = z$.
Now from (2.2) we have

$$\begin{aligned}M(FGx_{2n}, x_{2n+1}) = a_1 d((FG)^2 x_{2n}, H(FGx_{2n})) + a_2 d(STx_{2n+1}, Lx_{2n+1}) + \\ a_3 d(STx_{2n+1}, H(FGx_{2n})) + a_4 d((FG)^2 x_{2n}, Lx_{2n+1}) + \\ a_5 d((FG)^2 x_{2n}, STx_{2n+1})\end{aligned} \tag{2.33}$$

Taking limit as $n \to \infty$ on both sides of (2.33) we have

$$\begin{aligned}\lim_{n\to\infty} M(FGx_{2n}, x_{2n+1}) &= a_1 d(FGz, FGz) + a_2 d(z,z) + a_3 d(z, FGz) + a_4 d(FGz, z) + \\ &\quad a_5 d(FGz, z) \\ &= (a_3 + a_4 + a_5) d(FGz, z) \\ &\preceq d(FGz, z)\end{aligned} \tag{2.34}$$

Now from (2.1) and using (2.34) we have

$$\psi\{d(Hx_{2n}, Lx_{2n+1})\} \preceq J_*\{\psi\{M(x_{2n}, x_{2n+1})\}, \phi\{M(x_{2n}, x_{2n+1})\}\} \tag{2.35}$$

Taking limit as $n \to \infty$ on both sides of (2.35) we have

$$\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lx_{2n+1})\} \preceq \lim_{n\to\infty} J_*\{\psi\{M(x_{2n}, x_{2n+1})\}, \phi\{M(x_{2n}, x_{2n+1})\}\}$$

or, $\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{M(x_{2n}, x_{2n+1})\}, \lim_{n\to\infty} \phi\{M(x_{2n}, x_{2n+1})\}\}$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lx_{2n+1})\} &\preceq J_*\{\lim_{n\to\infty} \psi\{d(FGx_{2n}, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(FGx_{2n}, Lx_{2n+1})\}\} \\ &\preceq \lim_{n\to\infty} \psi\{d(FGx_{2n}, Lx_{2n+1})\}\end{aligned} \tag{2.36}$$

Therefore,
$J_*\{\lim_{n\to\infty} \psi\{d(FGx_{2n}, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(FGx_{2n}, Lx_{2n+1})\}\} = \lim_{n\to\infty} \psi\{d(FGx_{2n}, Lx_{2n+1})\}$,
which gives, either $\lim_{n\to\infty} \psi\{d(FGx_{2n}, Lx_{2n+1})\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(FGx_{2n}, Lx_{2n+1})\} = \theta_A$.
Therefore, in both cases

$$\lim_{n\to\infty} d(FGx_{2n}, Lx_{2n+1}) = \theta_A,\ or,\ d(FGz, z) = \theta_A$$

Hence,

$$FGz = z \tag{2.37}$$

**Step 3:**
**Claim:** $Hz = z$.
Now from (2) we have

$$\begin{aligned}&M(z, x_{2n+1}) \\ &= a_1 d(FGz, Hz) + a_2 d(STx_{2n+1}, Lx_{2n+1}) + a_3 d(STx_{2n+1}, Hz) + a_4 d(FGz, Lx_{2n+1}) \\ &\quad + a_5 d(FGz, STx_{2n+1})\end{aligned} \tag{2.38}$$

Taking limit as $n \to \infty$ on both sides of (2.38) we have

$$\begin{aligned}&\lim_{n\to\infty} M(z, x_{2n+1}) \\ &= a_1 d(FGz, Hz) + a_2 d(z,z) + a_3 d(z, Hz) + a_4 d(FGz, z) + a_5 d(FGz, z) \\ &= a_3 d(z, Hz) \\ &\preceq d(z, Hz)\end{aligned} \tag{2.39}$$

Now from (2.1) and using (2.39) we have

$$\psi\{d(Hz, Lx_{2n+1})\} \preceq J_*\{\psi\{M(z, x_{2n+1})\}, \phi\{M(z, x_{2n+1})\}\} \tag{2.40}$$

Taking limit as $n \to \infty$ on both sides of (2.40) we have

$$\begin{aligned}
&\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\} \preceq \lim_{n\to\infty} J_*\{\psi\{M(z, x_{2n+1})\}, \phi\{M(z, x_{2n+1})\}\}\\
\text{or, }&\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{M(z, x_{2n+1})\}, \lim_{n\to\infty} \phi\{M(z, x_{2n+1})\}\}\\
\text{or, }&\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Hz, Lx_{2n+1})\}\}\\
&\qquad \preceq \lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}
\end{aligned} \tag{2.41}$$

Hence,
$J_*\{\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Hz, Lx_{2n+1})\}\} = \lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}$,
which gives either $\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Hz, Lx_{2n+1})\} = \theta_A$.
Hence,

$$\lim_{n\to\infty} d(Hz, Lx_{2n+1}) = \theta_A \text{ or, } d(Hz, z) = \theta_A$$

Hence,

$$Hz = z \tag{2.42}$$

Hence, from (2.37) and (2.42) we have

$$FGz = Hz = z. \tag{2.43}$$

**Step 4:**
**Claim** $Gz = z$.
Again putting $x = z$ and $y = x_{2n+1}$ in (2.2) we have

$$\begin{aligned}
&M(Gz, x_{2n+1})\\
&= a_1 d(FGGz, Hz) + a_2 d(STx_{2n+1}, Lx_{2n+1}) + a_3 d(STx_{2n+1}, HGz)+\\
&a_4 d(FGGz, Lx_{2n+1}) + a_5 d(FGGz, STx_{2n+1})
\end{aligned} \tag{2.44}$$

Now as $FG = GF, HG = GH$, so

$$H(Gz) = G(Hz) = Gz \text{ and } FG(Gz) = GF(Gz) = G(FGz) = Gz$$

Taking limit as $n \to \infty$ on both sides of (2.44) we have

$$\begin{aligned}
\lim_{n\to\infty} M(Gz, x_{2n+1}) &= a_1 d(Gz, z) + a_2 d(z, z) + a_3 d(z, Gz) + a_4 d(Gz, z) + a_5 d(Gz, z)\\
&= (a_1 + a_3 + a_4 + a_5) d(Gz, z)\\
&\preceq d(Gz, z)
\end{aligned} \tag{2.45}$$

Now from (2.1) and using (2.45) we have

$$\psi\{d(HGz, Lx_{2n+1})\} \preceq J_*\{\psi\{M(Gz, x_{2n+1})\}, \phi\{M(Gz, x_{2n+1})\}\} \tag{2.46}$$

Taking limit as $n \to \infty$ on both sides of (2.46) we have

$$\begin{aligned}
\lim_{n\to\infty} \psi\{d(HGz, Lx_{2n+1})\} &\preceq \lim_{n\to\infty} J_*\{\psi\{M(Gz, x_{2n+1})\}, \phi\{M(Gz, x_{2n+1})\}\} \\
\text{or, } \lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} &\preceq J_*\{\lim_{n\to\infty} \psi\{M(Gz, x_{2n+1})\}, \lim_{n\to\infty} \phi\{M(Gz, x_{2n+1})\}\} \\
\text{or, } \lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} &\preceq J_*\{\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Gz, Lx_{2n+1})\}\} \\
&\preceq \lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\}
\end{aligned} \tag{2.47}$$

Therefore,
$J_*\{\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Gz, Lx_{2n+1})\}\} = \lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\}$.
Therefore, either $\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Gz, Lx_{2n+1})\} = \theta_A$.
Hence,

$$\lim_{n\to\infty} d(Gz, Lx_{2n+1}) = \theta_A \text{ or, } d(Gz, z) = \theta_A.$$

Hence,

$$Gz = z \tag{2.48}$$

Therefore,

$$Hz = Gz = Fz = z \tag{2.49}$$

**Step 5:**
As $H(X) \subset ST(X)$, there exists $u \in X$ such that $z = Hz = STu$.
**Claim:** $Lu = z$.
Again putting $x = x_{2n}$ and $y = u$ in (2.2) we have

$$\begin{aligned}
M(x_{2n}, u) = &a_1 d(FGx_{2n}, Hx_{2n}) + a_2 d(STu, Lu) + a_3 d(STu, Hx_{2n}) + a_4 d(FGx_{2n}, Lu) \\
&+ a_5 d(FGx_{2n}, STu)
\end{aligned} \tag{2.50}$$

Taking limit as $n \to \infty$ on both sides of (2.49) we have

$$\begin{aligned}
\lim_{n\to\infty} M(x_{2n}, u) &= a_1 d(z, z) + a_2 d(z, Lu) + a_3 d(z, z) + a_4 d(z, Lu) + a_5 d(z, z) \\
&= (a_2 + a_4) d(z, Lu) \\
&\preceq d(z, Lu)
\end{aligned} \tag{2.51}$$

Now from (2.1) and using (2.51) we have

$$\psi\{d(HGz, Lx_{2n+1})\} \preceq J_*\{\psi\{M(Gz, x_{2n+1})\}, \phi\{M(Gz, x_{2n+1})\}\} \tag{2.52}$$

Taking limit as $n \to \infty$ on both sides of (2.52) we have

$$\begin{aligned}
\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} &\preceq \lim_{n\to\infty} J_*\{\psi\{M(x_{2n}, u)\}, \phi\{M(x_{2n}, u)\}\} \\
\text{or, } \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} &\preceq J_*\{\lim_{n\to\infty} \psi\{M(x_{2n}, u)\}, \lim_{n\to\infty} \phi\{M(x_{2n}, u)\}\} \\
\text{or, } \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} &\preceq J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Lu)\}\} \\
&\preceq \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\}
\end{aligned} \tag{2.53}$$

Therefore, $J_*\{\lim_{n\to\infty}\psi\{d(Hx_{2n},Lu)\},\lim_{n\to\infty}\phi\{d(Hx_{2n},Lu)\}\}=\lim_{n\to\infty}\psi\{d(Hx_{2n},Lu)\}$.
Therefore, either $\lim_{n\to\infty}\psi\{d(Hx_{2n},Lu)\}=\theta_A$, or, $\lim_{n\to\infty}\phi\{d(Hx_{2n},Lu)\}=\theta_A$.
Hence,

$$\begin{aligned}\lim_{n\to\infty} d(Hx_{2n},Lu)&=\theta_A\\ \text{or, } d(z,Lu)&=\theta_A.\end{aligned}$$

Hence,

$$z=Lu \tag{2.54}$$

Hence,

$$STu=z=Lu \tag{2.55}$$

As $\{L,ST\}$ is weakly compatible, we have $STLu=LSTu$.
Thus

$$STz=Lz \tag{2.56}$$

**Step 6:**
**Claim:** $Lz=z$.
Now from (2) we have

$$\begin{aligned}M(x_{2n},z)&=a_1d(FGx_{2n},Hx_{2n})+a_2d(STz,Lz)+a_3d(STz,Hx_{2n})+a_4d(FGx_{2n},Lz)\\&\quad+a_5d(FGx_{2n},STz)\end{aligned} \tag{2.57}$$

Taking limit as $n\to\infty$ on both sides on (2.57) we have

$$\begin{aligned}\lim_{n\to\infty}M(x_{2n},z)&=a_1d(z,z)+a_2d(STz,Lz)+a_3d(Lz,z)+a_4d(z,Lz)+a_5d(z,Lz)\\&=(a_3+a_4+a_5)d(Lz,z)\\&\preceq d(Lz,z)\end{aligned} \tag{2.58}$$

Now from (2.1) and using (2.58) we have

$$\psi\{d(Hx_{2n},Lz)\}\preceq J_*\{\psi\{M(x_{2n},z)\},\phi\{M(x_{2n},z)\}\} \tag{2.59}$$

Taking limit as $n\to\infty$ on both sides of (2.59) we have

$$\begin{aligned}\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\}&\preceq\lim_{n\to\infty}J_*\{\psi\{M(x_{2n},z)\},\phi\{M(x_{2n},z)\}\}\\ \text{or, }\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\}&\preceq J_*\{\lim_{n\to\infty}\psi\{M(x_{2n},z)\},\lim_{n\to\infty}\phi\{M(x_{2n},z)\}\}\\ \text{or, }\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\}&\preceq J_*\{\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\},\lim_{n\to\infty}\phi\{d(Hx_{2n},Lz)\}\}\\&\preceq\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\}\end{aligned} \tag{2.60}$$

Therefore, $J_*\{\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\},\lim_{n\to\infty}\phi\{d(Hx_{2n},Lz)\}\}=\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\}$.
Therefore, either $\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\}=\theta_A$, or, $\lim_{n\to\infty}\phi\{d(Hx_{2n},Lz)\}=\theta_A$.
Hence,

$$\lim_{n\to\infty}d(Hx_{2n},Lz)=\theta_A,\ \text{or, }\ d(Lz,z)=\theta_A.$$

Hence,

$$Lz = z. \tag{2.61}$$

As $LT = TL$ and $ST = TS$, we have

$$LTz = TLz = Tz \text{ and } ST(Tz) = T(STz) = Tz.$$

**Step 7:**
**Claim:** $Tz = z$.
Now from (2.2) we have

$$\begin{aligned}
&M(x_{2n}, Tz)\\
&= a_1 d(FGx_{2n}, Hx_{2n}) + a_2 d(STTz, LTz) + a_3 d(STTz, Hx_{2n}) + a_4 d(FGx_{2n}, LTz)+\\
&a_5 d(FGx_{2n}, STTz)
\end{aligned} \tag{2.62}$$

Taking limit as $n \to \infty$ on both sides on (2.62) we have

$$\begin{aligned}
\lim_{n\to\infty} M(x_{2n}, Tz) &= a_1 d(z,z) + a_2 d(Tz,Tz) + a_3 d(Tz,z) + a_4 d(z,Tz) + a_5 d(z,Tz)\\
&= (a_3 + a_4 + a_5) d(z, Tz)\\
&\preceq d(z, Tz)
\end{aligned} \tag{2.63}$$

Now from (2.1) and using (2.63) we have

$$\psi\{d(Hx_{2n}, LTz)\} \preceq J_*\{\psi\{M(x_{2n}, Tz)\}, \phi\{M(x_{2n}, Tz)\}\} \tag{2.64}$$

Taking limit as $n \to \infty$ on both sides of (2.64) we have

$$\begin{aligned}
\lim_{n\to\infty} \psi\{d(Hx_{2n}, LTz)\} &\preceq \lim_{n\to\infty} J_*\{\psi\{M(x_{2n}, Tz)\}, \phi\{M(x_{2n}, Tz)\}\}\\
\text{or, } \lim_{n\to\infty} \psi\{d(Hx_{2n}, LTz)\} &\preceq J_*\{\lim_{n\to\infty} \psi\{M(x_{2n}, Tz)\}, \lim_{n\to\infty} \phi\{M(x_{2n}, Tz)\}\}\\
\text{or, } \lim_{n\to\infty} \psi\{d(Hx_{2n}, LTz)\} &\preceq J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Tz)\}\}\\
&\preceq \lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}
\end{aligned} \tag{2.65}$$

Therefore, $J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Tz)\}\} = \lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}$.
Therefore, either $\lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Hx_{2n}, Tz)\} = \theta_A$, which gives

$$\lim_{n\to\infty} d(Hx_{2n}, Tz) = \theta_A, \text{ or, } d(Tz, z) = \theta_A.$$

Hence,

$$Tz = z. \tag{2.66}$$

Now $STz = Tz = z \Rightarrow Sz = z$.
Hence,

$$Sz = Tz = Lz = z. \tag{2.67}$$

Combining (2.49) and (2.67) we have

$$Hz = Gz = Fz = Sz = Tz = Lz = z. \tag{2.68}$$

Therefore, $F, G, S, T, H, L$ have a common fixed point in $X$.

**Step 8:**

Assume $H$ is continuous.

Then $\lim_{n\to\infty} H^2 x_{2n} = \lim_{n\to\infty} Hz$ and $\lim_{n\to\infty} H(FGx_{2n}) = Hz$.

As $\{H, FG\}$ is compatible, so we have $\lim_{n\to\infty} FG(Hx_{2n}) = Hz$.

**Claim:** $Hz = z$.

Now from (2.2) we have

$$M(Hx_{2n}, x_{2n+1}) = a_1 d(FGHx_{2n}, H^2x_{2n}) + a_2 d(STx_{2n+1}, Lx_{2n+1}) + a_3 d(STx_{2n+1}, H^2x_{2n}) + a_4 d(FGHx_{2n}, Lx_{2n+1}) + a_5 d(FGHx_{2n}, STx_{2n+1}) \tag{2.69}$$

Taking limit as $n \to \infty$ on both sides of (2.69) we have

$$\begin{aligned}\lim_{n\to\infty} M(Hx_{2n}, x_{2n+1}) &= a_1 d(Hz, Hz) + a_2 d(z, z) + a_3 d(z, Hz) + a_4 d(Hz, z) + a_5 d(Hz, z)\\ &= (a_3 + a_4 + a_5) d(Hz, z)\\ &\preceq d(Hz, z)\end{aligned} \tag{2.70}$$

Now from (2.1) and using (2.70) we have

$$\psi\{d(HHx_{2n}, Lx_{2n+1})\} \preceq J_*\{\psi\{M(Hx_{2n}, x_{2n+1})\}, \phi\{M(Hx_{2n}, x_{2n+1})\}\} \tag{2.71}$$

Taking limit as $n \to \infty$ on both sides of (2.71) we have

$$\begin{aligned}&\lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\} \preceq \lim_{n\to\infty} J_*\{\psi\{M(Hx_{2n}, x_{2n+1})\}, \phi\{M(Hx_{2n}, x_{2n+1})\}\}\\ \text{or, } &\lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{M(Hx_{2n}, x_{2n+1})\}, \lim_{n\to\infty} \phi\{M(Hx_{2n}, x_{2n+1})\}\}\\ \text{or, } &\lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(H^2x_{2n}, Lx_{2n+1})\}\}\\ &\qquad \preceq \lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\}\end{aligned} \tag{2.72}$$

Therefore,

$J_*\{\lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(H^2x_{2n}, Lx_{2n+1})\}\} = \lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\}$.

Therefore, either $\lim_{n\to\infty} \psi\{d(H^2x_{2n}, Lx_{2n+1})\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(H^2x_{2n}, Lx_{2n+1})\} = \theta_A$, which gives

$$\lim_{n\to\infty} d(H^2x_{2n}, Lx_{2n+1}) = \theta_A, \text{ or, } d(Hz, z) = \theta_A.$$

Hence,

$$Hz = z \tag{2.73}$$

Proceeding same as in **Step 2-Step 4** we have

$$Lz = STz = Sz = Tz = z. \tag{2.74}$$

**Step 9:**

As $L(X) \subset FG(X)$, there exists $v \in X$ such that $z = Lz = FGv$.

**Claim:** $Lu = z$.
Again putting $x = v$ and $y = x_{2n+1}$ in (2.2) we have

$$\begin{aligned} & M(v, x_{2n+1}) \\ & = a_1 d(FGv, Hv) + a_2 d(STx_{2n+1}, Lx_{2n+1}) + a_3 d(STx_{2n+1}, Hv) + a_4 d(FGv, Lx_{2n+1}) \\ & \ + a_5 d(FGv, STx_{2n+1}) \end{aligned} \tag{2.75}$$

Taking limit as $n \to \infty$ on both sides of (2.75) we have

$$\begin{aligned} \lim_{n\to\infty} M(v, x_{2n+1}) &= a_1 d(z, Hv) + a_2 d(z, z) + a_3 d(z, Hv) + a_4 d(z, z) + a_5 d(z, z) \\ &= (a_1 + a_3) d(z, Hv) \\ &\preceq d(z, Hv) \end{aligned} \tag{2.76}$$

Now from (2.1) and using (2.76) we have

$$\psi\{d(Hv, Lx_{2n+1})\} \preceq J_*\{\psi\{M(v, x_{2n+1})\}, \phi\{M(v, x_{2n+1})\}\} \tag{2.77}$$

Taking limit as $n \to \infty$ on both sides of (2.77) we have

$$\begin{aligned} & \lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\} \preceq \lim_{n\to\infty} J_*\{\psi\{M(v, x_{2n+1})\}, \phi\{M(v, x_{2n+1})\}\} \\ \text{or, } & \lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{M(v, x_{2n+1})\}, \lim_{n\to\infty} \phi\{M(v, x_{2n+1})\}\} \\ \text{or, } & \lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Hv, Lx_{2n+1})\}\} \\ & \qquad \preceq \lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\} \end{aligned} \tag{2.78}$$

Therefore, $J_*\{\lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Hv, Lx_{2n+1})\}\} = \lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\}$.
Therefore either $\lim_{n\to\infty} \psi\{d(Hv, Lx_{2n+1})\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Hv, Lx_{2n+1})\} = \theta_A$, which gives,

$$\lim_{n\to\infty} d(Hv, Lx_{2n+1}) = \theta_A, \text{ or, } d(Hv, z) = \theta_A.$$

Hence,

$$Hv = z \tag{2.79}$$

Therefore,

$$FGv = Hv = Lz = z. \tag{2.80}$$

As $\{H, FG)\}$ is compatible so, we have $H(FGv) = FG(Hv)$.
Therefore, $Hz = FGz$ and $Gz = z$.
Proceeding similarly as in **Step 6** we have

$$Fz = Gz = Hz = z \tag{2.81}$$

Hence, by (2.80) and (2.81) we have

$$Fz = Gz = Hz = Lz = Sz = Tz = z. \tag{2.82}$$

Therefore, $z$ is a common fixed point of $F, G, H, L, S, T$.
**Step 10:**
**Uniqueness:** If possible let, $u$ and $z$ be two distinct fixed points of $F, G, H, L, S, T$

i.e., $Fu = Gu = Hu = Lu = Su = Tu = u$ and $Fz = Gz = Hz = Lz = Sz = Tz = z$ with $u \neq z$. Now from (2.2) we have

$$\begin{aligned} M(u,z) &= a_1 d(FGu, Hu) + a_2 d(STz, Lz) + a_3 d(STz, Hu) + a_4 d(FGu, Lz) + a_5 d(FGu, STz) \\ &= (a_3 + a_4 + a_5) d(u,z) \\ &\preceq d(u,z) \end{aligned} \tag{2.83}$$

Now from (2.1) and using (2.83) we have

$$\begin{aligned} \psi\{d(Hu, Lz)\} &\preceq J_*\{\psi\{M(u,z)\}, \phi\{M(u,z)\}\} \\ &\preceq J_*\{\psi\{d(u,z)\}, \phi\{d(u,z)\}\} \\ &\preceq \psi\{d(u,z)\} \end{aligned}$$

Therefore, $J_*\{\psi\{d(u,z)\}, \phi\{d(u,z)\}\} = \psi\{d(u,z)\}$, which gives either $\psi\{d(u,z)\} = \theta_A$, or, $\phi\{d(u,z)\} = \theta_A$.
Hence, $d(u,z) = \theta_A$, which is a contradiction.
Hence, the common fixed point of $F, G, H, L, S, T$ is unique.
This completes the proof.
**Note:** If we replace the condition (iii) by if $\{L, ST\}$ is compatible and $\{H, FG\}$ is weakly compatible and either $L$, or, $ST$ is continuous, then **Theorem 2.1** also holds.

**Example 2.1.** *Assume $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$ be the set of all bounded linear operators on a Hilbert space $\mathbb{R}^2$.*
*Define $d(x,y) = \begin{pmatrix} k|x-y| & 0 \\ 0 & |x-y| \end{pmatrix}$, where $k > 0$ is a constant.Then $(X, A_+, d)$ be a complete $C^*$-algebra valued metric space.*
*Assume $S(x) = 18x, T(x) = \frac{x}{3}, H(x) = x, L(x) = 2x, F(x) = 6x, G(x) = \frac{x}{2}$, then $ST(x) = 6x, FG(x) = 3x$ and $H(X) \subset ST(X), L(X) \subset FG(X)$; and $FG = GF, ST = TS, HG = GH, LT = TL$.*
*Now*

$$d(FGx, Hx) = \begin{pmatrix} k|3x-x| & 0 \\ 0 & |3x-x| \end{pmatrix}, d(STy, Ly) = \begin{pmatrix} k|12y-2y| & 0 \\ 0 & |12y-2y| \end{pmatrix},$$

$$d(STy, Hx) = \begin{pmatrix} k|12y-x| & 0 \\ 0 & |12y-x| \end{pmatrix}, d(FGx, Ly) = \begin{pmatrix} k|3x-2y| & 0 \\ 0 & |3x-2y| \end{pmatrix},$$

$$d(FGx, STy) = \begin{pmatrix} k|3x-12y| & 0 \\ 0 & |3x-12y| \end{pmatrix}, d(Hx, Ly) = \begin{pmatrix} k|x-2y| & 0 \\ 0 & |x-2y| \end{pmatrix}$$

*Here*

$$\begin{aligned} &M(x,y) \\ &= a_1 \begin{pmatrix} k|3x-x| & 0 \\ 0 & |3x-x| \end{pmatrix} + a_2 \begin{pmatrix} k|12y-2y| & 0 \\ 0 & |12y-2y| \end{pmatrix} + a_3 \begin{pmatrix} k|12y-x| & 0 \\ 0 & |12y-x| \end{pmatrix} \\ &+ a_4 \begin{pmatrix} k|3x-2y| & 0 \\ 0 & |3x-2y| \end{pmatrix} + a_5 \begin{pmatrix} k|3x-12y| & 0 \\ 0 & |3x-12y| \end{pmatrix} \end{aligned}$$

*We take $\psi(P) = 2P$ and $\phi(P) = \frac{P}{2}$, $\forall P \in A_+$. Then*

$\psi\{d(Hx, Ly)\} = \begin{pmatrix} k|2x-4y| & 0 \\ 0 & |2x-4y| \end{pmatrix}$

*Assume* $a_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a_5 = \begin{pmatrix} \frac{5}{6} & 0 \\ 0 & \frac{5}{6} \end{pmatrix}$

*Now*

$\psi\{M(x,y)\} = \begin{pmatrix} \frac{5k}{2}|x-2y| & 0 \\ 0 & \frac{5}{2}|x-2y| \end{pmatrix}$ *and* $\phi\{M(x,y)\} = \begin{pmatrix} \frac{5k}{6}|x-2y| & 0 \\ 0 & \frac{5k}{6}|x-2y| \end{pmatrix}$

*Therefore the condition* $\psi\{d(Hx, Ly)\} \preceq J_*\{\psi\{M(x,y)\}, \phi\{M(x,y)\}\}$ *holds.*
*Here* $x = 0$ *is the unique common fixed point of* $F, G, S, T, H, L$.

**Corollary 2.1.** *Let* $(X, A, d)$ *be a complete* $C^*$*-algebra valued metric space and* $F, G, S, T, H, L$ *are self maps on* $X$ *such that for all* $x, y \in X$*, the following conditions hold:*

$$\psi\{d(Hx, Ly)\} \preceq J_*\{\psi\{M(x,y\}), \phi\{M(x,y)\}\},\ \psi, \phi \in \Phi \text{ and } J_* \in C_*; \tag{2.84}$$

*where,* 
$$M(x,y) = a_1 d(FGx, Hx) + a_2 d(STy, Ly) + a_3 d(STy, Hx) + a_4 d(FGx, Ly) + a_5 d(FGx, STy); \tag{2.85}$$

*with* $a_1 + a_2 + 2(a_3 + a_4) + a_5 \leq 1, a_i \geq 0$ *for all* $i = 1, 2, 3, 4, 5.$
*and*

*(i)* $H(X) \subset ST(X), L(X) \subset FG(X)$*;*
*(ii)* $FG = GF, ST = TS, HG = GH, LT = TL$;
*(iii)* $\{H, FG\}$ *is compatible and* $\{L, ST\}$ *is weakly compatible; either* $H$ *or* $FG$ *is continuous.*

*Then* $F, G, S, T, H, L$ *have a unique common fixed point in* $X$.

**Corollary 2.2.** *Let* $(X, A, d)$ *be a complete* $C^*$*-algebra valued metric space and* $F, G, S, T, H, L$ *are self maps on* $X$ *such that for all* $x, y \in X$*, the following conditions hold:*

$$\psi\{d(Hx, Ly)\} \preceq \psi\{M(x,y\} - \phi\{M(x,y)\},\ \psi, \phi \in \Phi; \tag{2.86}$$

*where,* 
$$M(x,y) = a_1 d(FGx, Hx) + a_2 d(STy, Ly) + a_3 d(STy, Hx) + a_4 d(FGx, Ly) + a_5 d(FGx, STy);\ \text{where } a_i \in A',\ \text{for all } i = 1, 2, 3, 4, 5. \tag{2.87}$$

*with* $a_1 + a_2 + 2(a_3 + a_4) + a_5 \preceq I_A;\ \theta_A \preceq a_i,\ \forall\, i = 1, 2, 3, 4, 5;$
$\|a_1\| + \|a_2\| + 2(\|a_3\| + \|a_4\|) + \|a_5\| \leq 1.$
*and*

*(i)* $H(X) \subset ST(X), L(X) \subset FG(X)$*;*
*(ii)* $FG = GF, ST = TS, HG = GH, LT = TL$;
*(iii)* $\{H, FG\}$ *is compatible and* $\{L, ST\}$ *is weakly compatible; either* $H$ *or* $FG$ *is continuous.*

*Then* $F, G, S, T, H, L$ *have a unique common fixed point in* $X$.

**Corollary 2.3.** *Let $(X, A, d)$ be a complete $C^*$-algebra valued metric space and $F, G, S, T$, $H, L$ are self maps on $X$ such that for all $x, y \in X$, the following conditions hold:*

$$\psi\{d(Hx, Ly)\} \preceq \psi\{M(x, y)\} - \phi\{M(x, y)\},\ \psi, \phi \in \Phi; \tag{2.88}$$

*where,* $M(x, y) = a_1 d(FGx, Hx) + a_2 d(STy, Ly) + a_3 d(STy, Hx) + a_4 d(FGx, Ly) + a_5 d(FGx, STy);$

$$\tag{2.89}$$

*with* $a_1 + a_2 + 2(a_3 + a_4) + a_5 \leq 1;\ a_i \geq 0$ *for all* $i = 1, 2, 3, 4, 5;$
*and*

*(i)* $H(X) \subset ST(X), L(X) \subset FG(X)$*;*
*(ii)* $FG = GF, ST = TS, HG = GH, LT = TL;$
*(iii)* $\{H, FG\}$ *is compatible and* $\{L, ST\}$ *is weakly compatible; either* $H$ *or* $FG$ *is continuous.*

*Then $F, G, S, T, H, L$ have a unique common fixed point in $X$.*

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